Solution of the Navier Stokes model in 1D using finite differences schemes

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Introduction

The Navier Stokes equations are ones that describe the behavior of fluids. The computational solution of these allows for a way of understanding and predicting them while being cost effective. The fundamental equations arise from the principles of conservation of energy, momentum and mass described in Newton's second law, the first law of thermodynamics and the continuity equation respectively. The obtained system of equations can be used for different fluid simulations under different circumstances such as Newtonian, compressible or isothermal flow fluids. The objectives of this project are to describe the problem and the origin of the equations; to approximate the solution to the Navier Stokes system in one dimension through a finite differences discretization scheme used in numerical analysis to solve PDE; to mathematically analyse the selected approach in terms of error and convergence; and to present examples using different boundaries conditions.

Key Words

Navier-Stokes in 1 Dimension, fluid modeling, finite differences, Partial differential equations.

1 Problem description

Navier Stokes equations derive from the mass, momentum and energy conservation principles [3], the first one results in the equation

$$\frac{d\rho}{dt} + \rho\nabla \cdot \vec{u} = 0 \tag{1}$$

In the case of mass conservation on an incompressible fluid (constant density independent of space and time) the continuity equation is

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

where $\vec{u}$ is the velocity vector. Newton's second law ensures momentum conservation, if ρ denotes the fluids density, ν is the fluids viscosity and p is the pressure over the fluid related to a stress factor [9], the second equation is

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\nabla \cdot \vec{u}\right) = -\nabla p + \nu\nabla^2\vec{u} + f \tag{3}$$

this can be derive from the initial approach in Newton's second law $\sum F = ma$ by analysing the momentum conservation over each of the fluids particles, that is to say, taking $\frac{m}{v}$ instead of m, we obtain a convection term of the equation

$$\frac{m}{v}a = \rho\frac{d\vec{u}}{dt} = \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u}\right) \tag{4}$$

and the sum of forces is defined by pressure gradient ∇p that makes reference to stress, friction related to viscosity $\mu\nabla^2\vec{u}$ where $\mu = \nu/\rho$ is the dynamic viscosity and it defines the dispersion term and other possible external forces f, for example, if gravity where taken into account, the previous reasoning would transform mg into ρg The last equation is given by energy conservation as stated in the first law of thermodynamics, where e is the specific internal energy, q is the heat-flux vector.

In the case of isothermal incompressible fluid with constant viscosity and none external forces, given that the analysis is all on one dimension, the equations can be expressed as

$$\frac{\partial u}{\partial x} = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} \tag{6}$$

1.1 Relevant characteristics of the equations

1.1.1 Reynolds number

The Reynolds number represents the relative importance of the viscous stress, mathematically, $Re = \rho \cdot \hat{u} \cdot diameter\ /\nu$, where $\hat{u}$ indicates the mean velocity. The Reynolds number indicates whether the flow is completely governed by viscous effects (at low Reynolds number) or effectively inviscid (at high Reynolds number). One of the advantages in using nonlinear equations can be found in the turbulence analysis. The response of laminar flow to small perturbations is expected to vary according to the influence of the fluids viscosity, at larger values of Re, a small perturbation should be amplified. Sufficiently small perturbations will decay at sufficiently high Reynolds numbers in a linear analysis [4], but since turbulence can only be studied through large perturbations, only nonlinear Navier stokes equations can simulate it.

1.1.2 Convection-Dispersion ratio

In equation 6, the second term on the left and the second term on the right are representations of convection and diffusion parts of the equation respectively. Their ratio determines the general behavior of the Navier Stokes equation, if the dispersion term is much larger than the convection term, the equation will have a parabolic PDE behavior, similar to the heat equation. However if the convection term is much bigger than the dispersion term, the Navier Stokes equation will have a first order hyperbolic PDE behavior. Even further, if the viscous effect is not considered at all, the Navier stokes momentum equation becomes Euler's equation which is a hyperbolic equation [5].

2 Finite differences discretization scheme

To reach the final final differences scheme, the equations must be manipulated in order to include the first one as a constrain where ρ evolves in a way that the rate of expansion of $\vec{u}$ is zero at all points. This is because there is no obvious way to couple velocity and pressure in an incompressible fluid, if it where compressible, a relation between ρ and p would be derived from the continuity equation. First, partially deriving equation (6)

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) \tag{7}$$

an expression for the Laplacian is obtained

$$\frac{\partial^2 p}{\partial x^2} = -\rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - \rho\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) \tag{8}$$

$$\frac{\partial^2 p}{\partial x^2} = -\rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - \rho\left(\frac{\partial u}{\partial x}\right)^2 - \rho u\left(\frac{\partial^2 u}{\partial x^2}\right) + \nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) \tag{9}$$

which has the same form as a Poisson equation and gives a relation of pressure p in terms of the velocity u that can be used to couple the equations

2.1 Deriving an expression for pressure

The momentum conservation given by Eq.(3) semi-discrete scheme using a forward in time approach is

$$\vec{u}^{\,n+1} = \vec{u}^{\,n} + \Delta t\left[\mu\nabla^2\vec{u}^{\,n} - \frac{1}{\rho}\nabla p^n - \vec{u}^{\,n}\nabla\vec{u}^{\,n}\right] \tag{10}$$

Velocity in the temporary step $\vec{u}^{\,n+1}$ is calculated using a finite differences scheme and corrected by imposing a pressure condition that guarantees continuity. Applying divergence to equation (10) gives:

$$\nabla\cdot\vec{u}^{\,n+1} = \nabla\cdot\vec{u}^{\,n} + \Delta t\left[\mu\nabla^2(\nabla\vec{u}^{\,n}) - \frac{1}{\rho}\nabla^2 p^n - \nabla(\vec{u}^{\,n}\nabla\vec{u}^{\,n})\right] \tag{11}$$

given that $\nabla\vec{u}^{\,n+1}$ and $\nabla\vec{u}^{\,n}$ should be zero, the following expression can be derived

$$\nabla^2 p^n = -\rho\nabla(\vec{u}^{\,n}\nabla\vec{u}^{\,n}) \tag{12}$$

where $\mu = \nu/\rho$ is the dynamic viscosity [6]. AS we can see this is a similar scheme to the one found by Cebeci *et al.* [2]. Now, applying a one dimensional approach

$$\frac{\partial^2 p^n}{\partial x^2} = -\rho\frac{\partial}{\partial x}\left(u^n\frac{\partial u^n}{\partial x}\right) \tag{13}$$

$$\frac{\partial^2 p^n}{\partial x^2} = -\rho\left(\frac{\partial u^n}{\partial x}\right)^2 - \rho u^n\left(\frac{\partial^2 u^n}{\partial x^2}\right) \tag{14}$$

A centered differences in space discretization for the second derivative of the pressure p at a time n is

$$\frac{\partial^2 p^2}{\partial x^2} \approx \frac{p^n_{j+1} - 2p^n_j + p^n_{j-1}}{(\Delta x)^2}$$

and centered in space of order 2 schemes for $\partial u^n/\partial x$, $\partial^2 u^n/\partial x^2$ and $\partial^3 u^n/\partial x^3$ are

$$\frac{\partial u^n}{\partial x} \approx \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x}$$

$$\frac{\partial^2 u^n}{\partial x^2} \approx \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2}$$

$$\frac{\partial^3 u^n}{\partial x^3} \approx \frac{u^n_{j+2} - 2u^n_{j+1} + 2u^n_{j-1} - u^n_{j-2}}{2(\Delta x)^3}$$

Now the final pressure scheme is

$$\frac{p^n_{j+1} - 2p^n_j + p^n_{j-1}}{(\Delta x)^2} = -\rho\left(\frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x}\right)^2 - \rho u^n_j\left(\frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2}\right) \quad (15)$$

2.2 Final discretized schemes for pressure and velocity

For the velocity scheme in Eq.(6) apply a forward in time discretization in $\partial u/\partial t$, backward in space for $\partial u/\partial x$ and centered differences for $\partial p/\partial x$ and $\partial^2 u/\partial x^2$ to obtain

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} + u^n_j\left(\frac{u^n_j - u^n_{j-1}}{\Delta x}\right) = -\frac{1}{\rho}\frac{p^n_{j+1} - p^n_{j-1}}{2\Delta x} + \mu\frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2} \quad (16)$$

Finally, obtaining the expressions for u^{n+1}_j and p^n_j from equations (15) and (16)

$$u^{n+1}_j = u^n_j + \Delta t\left[\mu\frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2} - u^n_j\frac{u^n_j - u^n_{j-1}}{\Delta x} - \frac{1}{\rho}\frac{p^n_{j+1} - p^n_{j-1}}{2\Delta x}\right] \quad (17)$$

$$\begin{aligned} p^n_j = {} & \frac{p^n_{j+1} + p^n_{j-1}}{2} \\ & + \frac{(\Delta x)^2\rho}{2}\left[\left(\frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x}\right)^2 + u^n_j\left(\frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2}\right)\right] \end{aligned} \quad (18)$$

2.3 Relevant mathematical characteristics: consistency, stability, convergence

It is clear that our method is consistent since the method used for the discretization of the derivatives in our problem is proven to be a consistent method. Therefore all we have to check in order to be sure that our method converges to the correct answers is if it is stable, whether conditionally or not, or if it is unstable.

Based on the Von Neumann Analysis made by Konangi *et al.* in *von Neumann stability analysis of first-order accurate discretization schemes for one-dimensional (1D) and two-dimensional (2D) fluid flow equations, Computers and Mathematics with Applications*[7] the stability analysis starts by defining the equations in perturbation form, using the definition given by the mentioned author, the perturbations of each variable are defined as

$$u_j^n + \delta u_j^n \qquad\qquad p_j^n + \delta p_j^n$$

Now, in order to obtain the perturbation form of both Eq.(17) and Eq.(15), it is necessary to linearize their convection term. Beginning with the term

$$u_j^n u_{j+1}^n \tag{19}$$

and replacing the perturbations form of each gives

$$(u_j^n + \delta u_j^n)(u_{j+1}^n + \delta u_{j+1}^n) \tag{20}$$

now substracting Eq.(19) from Eq.(20) we have:

$$u_j^n + \delta u_j^n)(u_{j+1}^n + \delta u_{j+1}^n) - u_j^n u_{j+1}^n$$

$$u_j^n u_{j+1}^n + u_j^n \delta u_{j+1}^n + u_{j+1}^n \delta u_j^n + \delta u_j^n \delta u_{j+1}^n - u_j^n u_{j+1}^n$$

$$u_j^n \delta u_{j+1}^n + u_{j+1}^n \delta u_j^n + \delta u_j^n \delta u_{j+1}^n \tag{21}$$

and finally, neglecting the second order term leads to:

$$u_j^n \delta u_{j+1}^n + u_{j+1}^n \delta u_j^n \tag{22}$$

Using the same procedure used for finding Eq.(22) we find the perturbation form for $(u_j^n)^2$, $(u_{j+1}^n)^2$, $(u_{j-1}^n)^2$, $u_j^n u_{j-1}^n$ and $u_{j+1}^n u_{j-1}^n$ that will be used in order to

obtain the complete perturbation form of Eq.(17) and Eq.(15)
Now the momentum perturbation equation found by replacing in Eq.(17) is:

$$\delta u_j^{n+1} = \delta u_j^n + \frac{\Delta t \mu}{(\Delta x)^2}(\delta u_{j+1}^n - 2\delta u_j^n + \delta u_{j-1}^n) \\ - \frac{\Delta t}{\Delta x}\left(2u_j^n * \delta u_j^n - u_{j-1}^n \delta u_j^n - u_j^n \delta u_{j-i}^n\right) \\ - \frac{\Delta t}{2\rho \Delta x}\left(\delta p_{j+1}^n - \delta p_{j-1}^n\right) \quad (23)$$

And the Poisson form of the pressure perturbation equation obtained from Eq.(15) is:

$$\frac{1}{(\Delta x)^2}(\delta p_{j+1}^n - 2\delta p_j^n + \delta p_{j-1}^n) = \\ - \frac{\rho}{4(\Delta x)^2}\left[2u_{j+1}^n \delta u_{j+1}^n - 2(u_{j+1}^n \delta u_{j-1}^n + u_{j-1}^n \delta u_{j+1}^n) + 2u_{j-1}^n \delta u_{j-1}^n\right] \\ - \frac{\rho}{(\Delta x)^2}(u_j^n \delta u_{j+1}^n + u_{j+1}^n \delta u_j^n - 4u_j^n \delta u_j^n + u_j^n \delta u_{j-1}^n + u_{j-1}^n \delta u_j^n) \quad (24)$$

The coefficients u_x^n in Equations Eq.(23) and Eq.(24) are frozen to give:

$$\delta u_j^{n+1} = \delta u_j^n + \frac{\Delta t \mu}{(\Delta x)^2}(\delta u_{j+1}^n - 2\delta u_j^n + \delta u_{j-1}^n) \\ - \frac{\Delta t}{\Delta x}\left(2U * \delta u_j^n - U \delta u_j^n - U \delta u_{j-i}^n\right) \\ - \frac{\Delta t}{2\rho \Delta x}\left(\delta p_{j+1}^n - \delta p_{j-1}^n\right)$$

$$\delta u_j^{n+1} = \delta u_j^n + \frac{\Delta t \mu}{(\Delta x)^2}(\delta u_{j+1}^n - 2\delta u_j^n + \delta u_{j-1}^n) \\ - \frac{U \Delta t}{\Delta x}\left(\delta u_j^n - \delta u_{j-i}^n\right) - \frac{\Delta t}{2\rho \Delta x}\left(\delta p_{j+1}^n - \delta p_{j-1}^n\right) \quad (25)$$

$$\frac{1}{(\Delta x)^2}(\delta p_{j+1}^n - 2\delta p_j^n + \delta p_{j-1}^n) =$$
$$- \frac{\rho}{4(\Delta x)^2}\left[2U\delta u_{j+1}^n - 2U\delta u_{j-1}^n - 2U\delta u_{j+1}^n + 2U\delta u_{j-1}^n\right]$$
$$- \frac{\rho}{(\Delta x)^2}(U\delta u_{j+1}^n + U\delta u_j^n - 4U\delta u_j^n + U\delta u_{j-1}^n + U\delta u_j^n)$$

$$\frac{1}{(\Delta x)^2}(\delta p_{j+1}^n - 2\delta p_j^n + \delta p_{j-1}^n) = -\frac{U\rho}{(\Delta x)^2}(\delta u_{j+1}^n - 2\delta u_j^n + \delta u_{j-1}^n) \quad (26)$$

Then the increment notation is dropped for convenience and the equations are written as:

$$u_j^{n+1} = u_j^n + \frac{\Delta t \mu}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) - \frac{U\Delta t}{\Delta x}\left(u_j^n - u_{j-i}^n\right) - \frac{\Delta t}{2\rho\Delta x}\left(p_{j+1}^n - p_{j-1}^n\right) \quad (27)$$

$$\frac{1}{(\Delta x)^2}(p_{j+1}^n - 2p_j^n + p_{j-1}^n) = -\frac{U\rho}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \quad (28)$$

Now that we have our equations in a much easier way to work with, we proceed to do the Von Neumann Stability Analysis, where the perturbations are represented as

$$u_j^n = \zeta_u^n e^{i\theta j} \qquad\qquad p_j^n = \zeta_p^n e^{i\theta j}$$

therefore, substituting these expression for the perturbations into Eq.(27) and Eq.(28) and dividing by $e^{i\theta j}$ gives:

$$\zeta_u^{n+1} = \zeta_u^n \left[1 + \frac{\Delta t \mu}{(\Delta x)^2}(e^{i\theta} - 2 + e^{-i\theta}) - \frac{U\Delta t}{\Delta x}\left(1 - e^{-i\theta}\right)\right] - \zeta_p^n \frac{\Delta t}{2\rho\Delta x}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\zeta_u^{n+1} = \zeta_u^n \left[1 + \frac{\Delta t \mu}{(\Delta x)^2}(2\cos(\theta) - 2) - \frac{U\Delta t}{\Delta x}\left(1 - \cos(\theta) + i\sin(\theta)\right)\right] - \zeta_p^n \frac{\Delta t}{2\rho\Delta x} 2i\sin(\theta)$$

$$\zeta_u^{n+1} = \zeta_u^n \left[1 - \frac{4\Delta t \mu}{(\Delta x)^2}\sin^2(\theta/2) - \frac{U\Delta t}{\Delta x}\left(1 - \cos(\theta) + i\sin(\theta)\right)\right] - \zeta_p^n \frac{\Delta t}{\rho\Delta x} i\sin(\theta) \quad (29)$$

$$\zeta_p^{n+1}\frac{1}{(\Delta x)^2}(e^{i\theta} - 2 + e^{-i\theta}) + \zeta_u^{n+1}\frac{U\rho}{(\Delta x)^2}(e^{i\theta} - 2 + e^{-i\theta}) = 0$$

$$-\zeta_p^{n+1}\frac{4}{(\Delta x)^2}\sin^2(\theta/2) - \zeta_u^{n+1}\frac{4U\rho}{(\Delta x)^2}\sin^2(\theta/2) = 0 \quad (30)$$

Now, it is easy to see that both Eq.(29) and Eq.(30) can be combined into the next system of equations in matrix-vetor form:

$$\begin{bmatrix} 1 & 0 \\ A & B \end{bmatrix} \begin{bmatrix} \zeta_u^{n+1} \\ \zeta_p^{n+1} \end{bmatrix} = \begin{bmatrix} C & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_u^n \\ \zeta_p^n \end{bmatrix}$$

Where

$$A = -\frac{4U\rho}{(\Delta x)^2}\sin^2(\theta/2)$$

$$B = -\frac{4}{(\Delta x)^2}\sin^2(\theta/2)$$

$$C = 1 - \frac{4\Delta t\mu}{(\Delta x)^2}\sin^2(\theta/2) - \frac{U\Delta t}{\Delta x}\left(1 - \cos(\theta) + i\sin(\theta)\right)$$

$$D = -\frac{\Delta t}{\rho\Delta x} i\sin(\theta)$$

This system of Equations gives

$$\begin{bmatrix} \zeta_u^{n+1} \\ \zeta_p^{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ A & B \end{bmatrix}^{-1} \begin{bmatrix} C & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_u^{n} \\ \zeta_p^{n} \end{bmatrix} \tag{31}$$

then, the amplification matrix is defined as

$$Z = \begin{bmatrix} 1 & 0 \\ A & B \end{bmatrix}^{-1} \begin{bmatrix} C & D \\ 0 & 0 \end{bmatrix} \tag{32}$$

Now, the scheme is stable if all of Z's eigenvalues remain bounded by unity, or

$$|\lambda_{1,2}(Z)| < 1$$

When calculating the eigenvalues of Z we find:

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{a_2 - a_2 a_3 k_2 - a_2 c k_3 + 4 a_1 a_4 c k_1}{a_2}$$

where

$$a_1 = \frac{\rho}{\Delta x \Delta t} \quad a_2 = \frac{4}{(\Delta x)^2} \quad a_3 = 4\frac{\Delta t \mu}{(\Delta x)^2} \quad a_4 = \frac{\Delta t}{\rho \Delta x}$$
$$c = \frac{U \Delta t}{\Delta x}$$
$$k_1 = i\sin(\theta) \quad k_2 = \sin^2(\theta/2) \quad k_3 = 1 - \cos(\theta) + i\sin(\theta)$$

Now, replacing and simplifying:

$$\lambda_1 = 0$$

$$\lambda_2 = 1 - 4\frac{\mu \Delta t}{(\Delta x)^2}\sin^2(\theta/2) - c(1 - \cos(\theta))$$

$$\lambda_2 = 1 - 2c\sin^2(\theta/2) - 4\frac{\mu \Delta t}{(\Delta x)^2}\sin^2(\theta/2)$$

therefore, as $\lambda_1 = 0$, it is only necessary to check for which values $|\lambda_2| < 1$ to conclude that the scheme is stable

$$|\lambda_2| < 1$$

$$|1 - 2c\sin^2(\theta/2) - 4\frac{\mu \Delta t}{(\Delta x)^2}\sin^2(\theta/2)| < 1$$

$$-1 < 1 - 2c\sin^2(\theta/2) - 4\frac{\mu\Delta t}{(\Delta x)^2}\sin^2(\theta/2) < 1$$

$$-2 < -2c\sin^2(\theta/2) - 4\frac{\mu\Delta t}{(\Delta x)^2}\sin^2(\theta/2) < 0$$

$$0 < 2c\sin^2(\theta/2) + 4\frac{\mu\Delta t}{(\Delta x)^2}\sin^2(\theta/2) < 2$$

$$0 < \sin^2(\theta/2)\left(2c + 4\frac{\mu\Delta t}{(\Delta x)^2}\right) < 2$$

Now since $\sin^2(\theta/2)$ is always between 0 and 1, we just have to check for $\sin^2(\theta/2) = 1$

$$0 < 2c + 4\frac{\mu\Delta t}{(\Delta x)^2} < 2$$

$$-2c < 4\frac{\mu\Delta t}{(\Delta x)^2} < 2 - 2c$$

$$-\frac{c}{2} < \frac{\mu\Delta t}{(\Delta x)^2} < \frac{1-c}{2}$$

As we know, theoretically $0 < c < 1$ and $\frac{\mu\Delta t}{(\Delta x)^2} > 0$ from which it is concluded

$$0 < \frac{\mu\Delta t}{(\Delta x)^2} < \frac{1-c}{2}$$

Now we have our stability conditions, we can say that, working in this interval, this scheme will converge to the correct answer.

3 Implementation of the method

This implementation was programmed in MATLAB, and it follows the flowchart in Figure 1

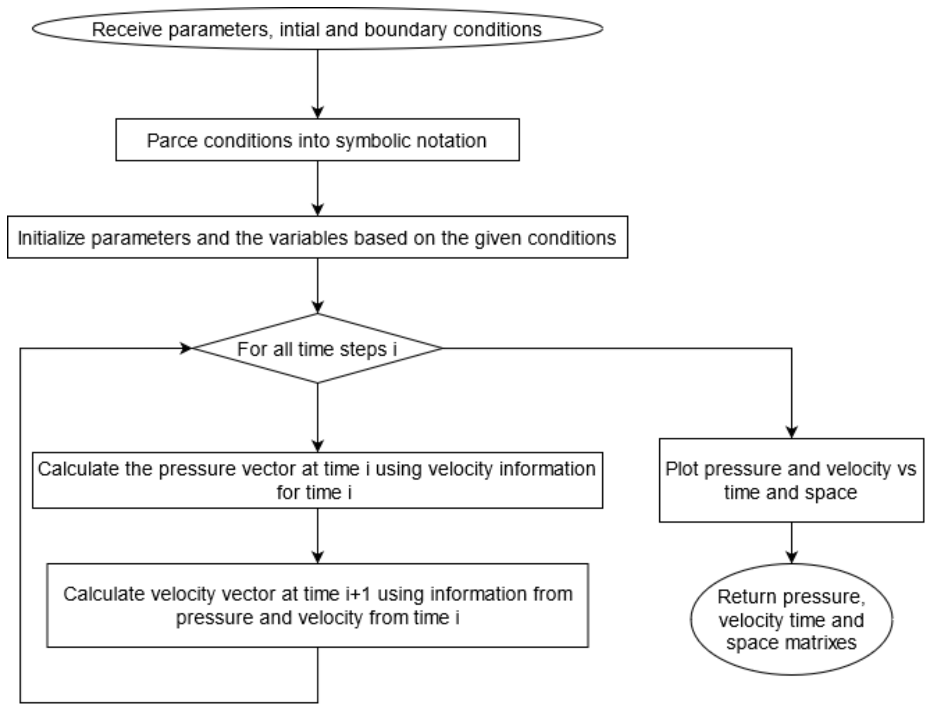


Figure 1: Flowchart implementation

To clarify, the parameters the implementation receives, in addition to initial and boundary conditions, are μ, ρ, L, N_x and c, being L the maximum distance considered in the problem, N_x the number of intervals in which our spatial domain is divided and c is an arbitrary constant between 0 and 1 that will act as the Courant number. The rest can be calculated as

$$\Delta x = \frac{L}{Nx} \qquad \nu = \mu\rho \qquad U = \frac{c\mu}{\Delta x(0.9-c)/2} \qquad \Delta t = c\frac{\Delta x}{U} \tag{33}$$

in order to guarantee stability.
Next, the results of using our implementation for solving three different problems with different types of boundary conditions were not possible to compare to results from previous implementations since the majority of research made for this field and literature with applications is based in two and three dimensional problems.

3.1 Example 1: Dirichlet Boundary Conditions

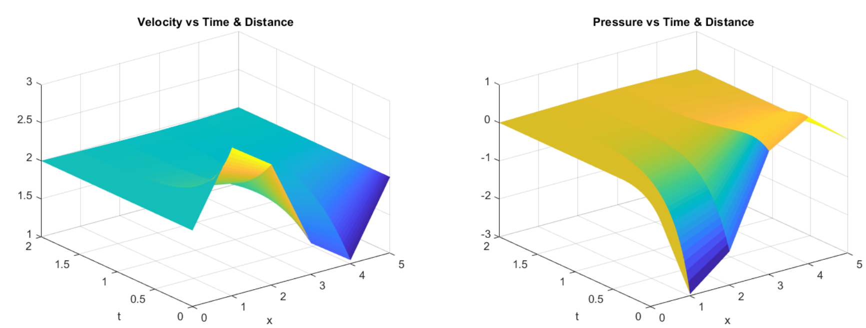
For these example the problem is defined as
solve

$$\begin{cases} \frac{\partial u}{\partial x} = 0 & [x,t] \in [0,5] \times \mathbb{R}^+ \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} & [x,t] \in [0,5] \times \mathbb{R}^+ \end{cases}$$

subject to

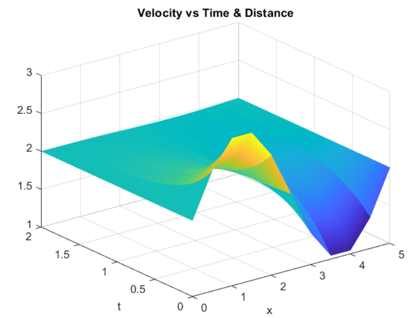
$$\begin{cases} u(0,t) = u(5,t) = 2 & t \in \mathbb{R}^+ \\ p(0,t) = p(5,t) = 0 & t \in \mathbb{R}^+ \\ u(x,0) = \sin\left(\frac{2\pi x}{5}\right) + 2 & x \in [0,5] \end{cases}$$

with $\rho = 1$ and $\mu = 1$ In the discrete scheme, it was decided to bound time to the interval $[0,2]$. Now, figures 2 to 6 are the graphics of the solutions given multiples Δx values

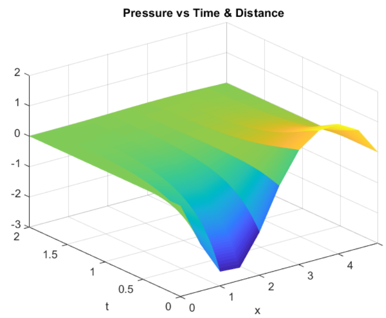


(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 2: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 1$

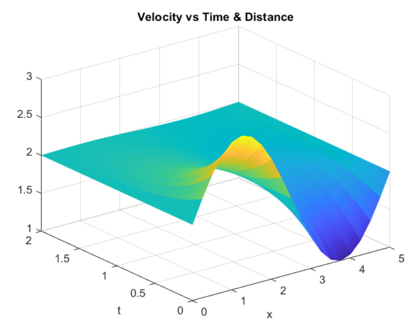


(a) $\hat{u}(x,t)$

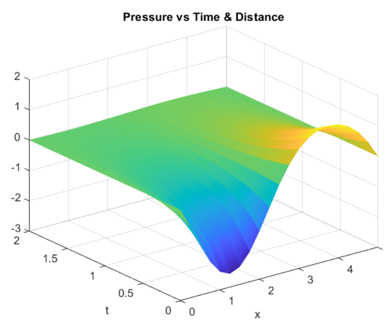


(b) $\hat{p}(x,t)$

Figure 3: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.5$

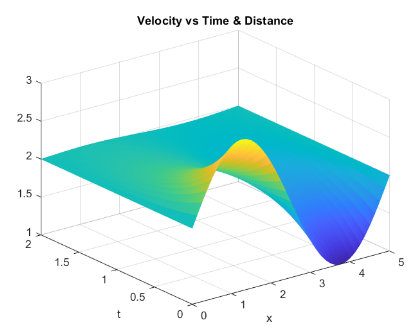


(a) $\hat{u}(x,t)$

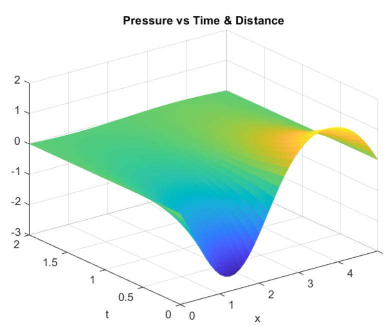


(b) $\hat{p}(x,t)$

Figure 4: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.25$

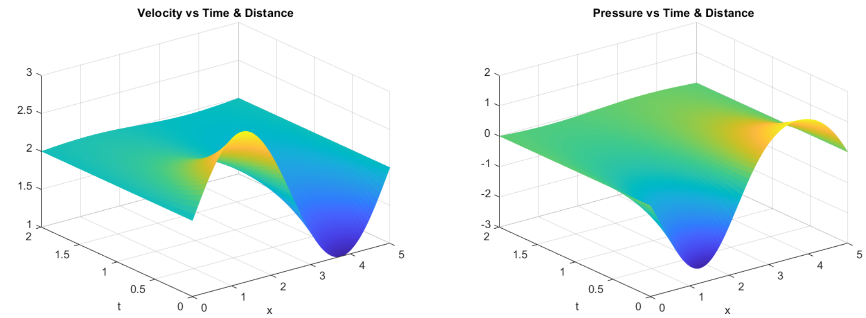


(a) $\hat{u}(x,t)$



(b) $\hat{p}(x,t)$

Figure 5: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.125$



(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 6: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.0625$

It is seen that the fluid has the expected behaviour, because, as the time passes, velocity and pressure tend to flatten

3.2 Example 2: Mixed Boundary Conditions

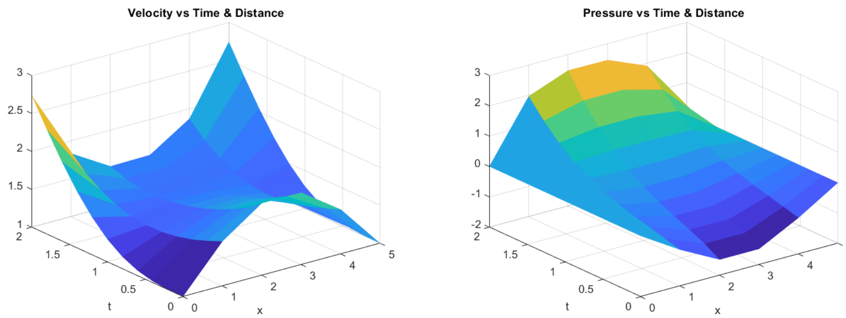
For these example the problem is defined as
solve

$$\begin{cases} \frac{\partial u}{\partial x} = 0 & [x,t] \in [0,5] \times \mathbb{R}^+ \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} & [x,t] \in [0,5] \times \mathbb{R}^+ \end{cases}$$

subject to

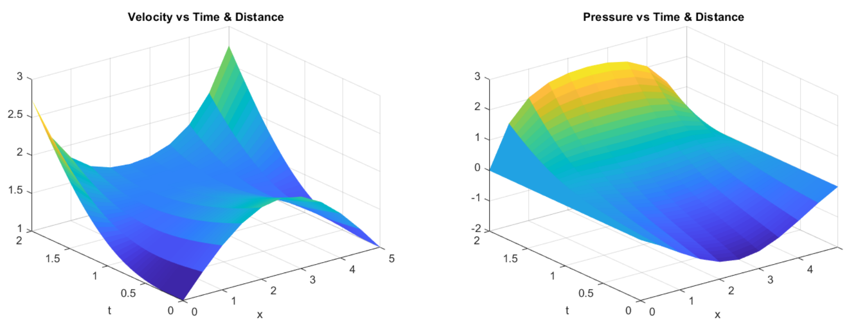
$$\begin{cases} u_t(0,t) = u_t(5,t) = t\sin(t) & t \in \mathbb{R}^+ \\ p(0,t) = p(5,t) = 0 & t \in \mathbb{R}^+ \\ u(x,0) = \sin\left(\frac{\pi x}{5}\right) + 1 & x \in [0,5] \end{cases}$$

with $\rho = 1$ and $\mu = 1$ In the discrete scheme, time is bounded to the interval $[0,2]$. Now, figures 7 to 11 present the graphical solutions given multiples Δx values



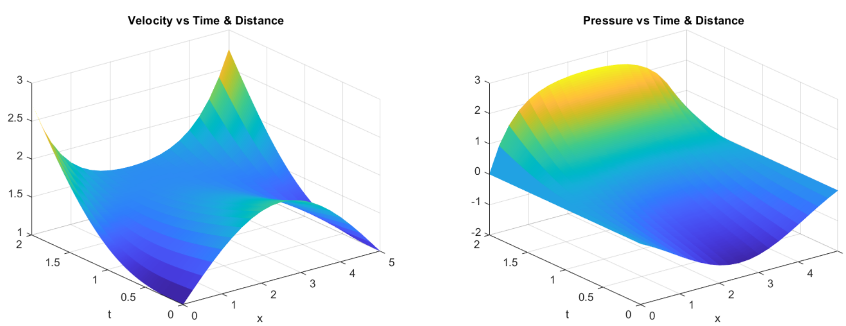
(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 7: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 1$



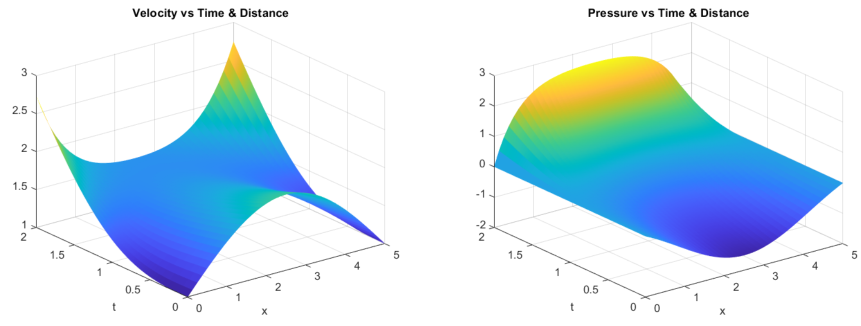
(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 8: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.5$



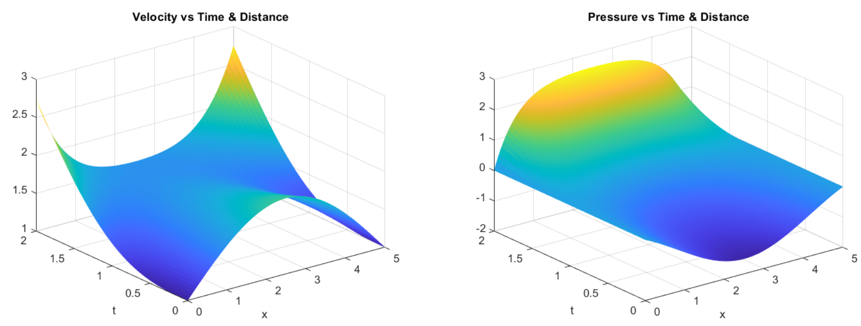
(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 9: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.25$



(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 10: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.125$



(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 11: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.0625$

3.3 Example 3: Robin Boundary Conditions

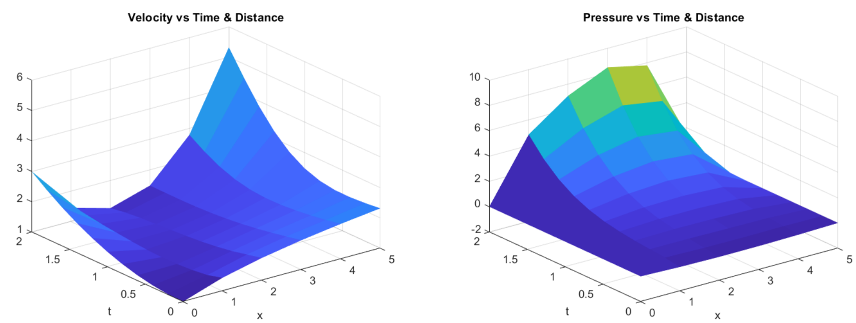
For these example the problem is defined as
solve

$$\begin{cases} \frac{\partial u}{\partial x} = 0 & [x,t] \in [0,5] \times \mathbb{R}^+ \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} & [x,t] \in [0,5] \times \mathbb{R}^+ \end{cases}$$

subject to

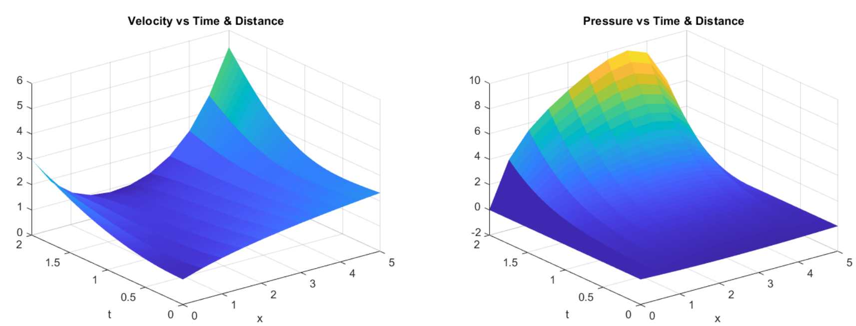
$$
\begin{cases}
u_t(0,t) = 1 & t \in \mathbb{R}^+ \\
0.5u(5,t) + 0.5u_t(5,t) = 1 + t^2 \sin(t) & t \in \mathbb{R}^+ \\
p(0,t) = p(5,t) = 0 & t \in \mathbb{R}^+ \\
u(x,0) = \log\left(\frac{x}{L}e + 1\right) + 1 & x \in [0,5]
\end{cases}
$$

with $\rho = 1$ and $\mu = 1$ In the discrete scheme, time is bounded to the interval $[0, 2]$. Now, figures to present the graphical solutions, given multiples Δx values



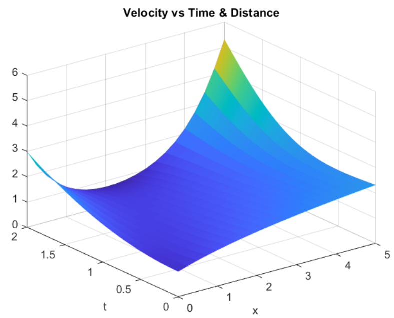
(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 12: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 1$

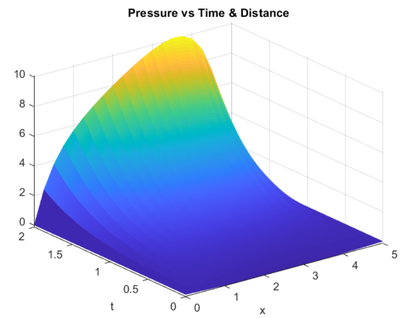


(a) $\hat{u}(x,t)$ (b) $\hat{p}(x,t)$

Figure 13: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.5$

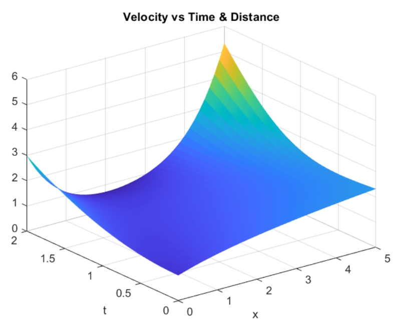


(a) $\hat{u}(x,t)$

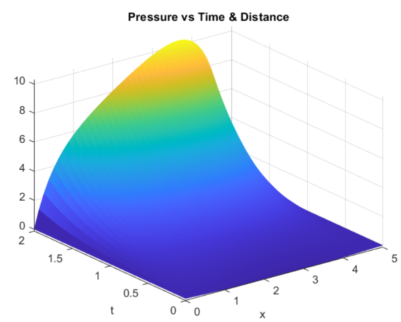


(b) $\hat{p}(x,t)$

Figure 14: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.25$

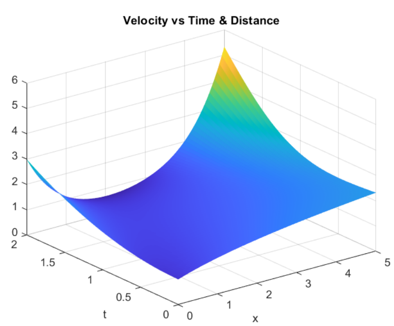


(a) $\hat{u}(x,t)$

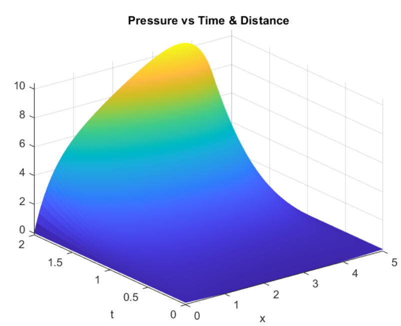


(b) $\hat{p}(x,t)$

Figure 15: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.125$



(a) $\hat{u}(x,t)$



(b) $\hat{p}(x,t)$

Figure 16: $\hat{u}(x,t)$ and $\hat{p}(x,t)$ for $\Delta x = 0.0625$

Conclusions

Proving there is a smooth and unique solution to the Navier Stokes equations is one of the seven millennium problems listed by the Clay Mathematics Institute, and it is definitely not an easy task. As it could be seen in this work, even for a one dimensional reduction of the equations it is hard to find a suitable method for solving and approximating the numerical solution, it makes sense that for a 3 dimensional approach, there has to be a very rigorous work in order to solve them.

These equations have, as seen at the beginning of this work, a theoretical basis in physics that undertakes many possible types of fluids, therefore they are widely used in the fluids dynamics to simulate different types of fluid flow through different computationally efficient numerical solutions. This solutions have applications in many areas such as aircraft design, weather forecast, vascular networks and arterial flow, oil and gas pipelines design and, as expected, it is easy to find multiple and different methods in the literature [9] [8] [1].

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