OPTIMAL STATIC HEDGING OF ENERGY PRICE AND VOLUME RISK: CLOSED-FORM RESULTS

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Optimal Static Hedging of Energy Price and Volume Risk: Closed-Form Results

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Abstract

Considering the problem of optimal designing a hedging claim by an economic agent facing both price and volume risk. This paper cover the typical case of an energy retailer procuring power from the wholesale market at the standing spot price and reselling it to industrial consumers exhibiting variable demand figures. The paper follows the line traced in a benchmark article by Oum and Oren (2008), and proposes the issue of determining the optimal derivative pay-off written on both electricity price and a weather-linked index. The latter aims to improve the performance of the hedging claim due to the link between demanded volume and weather-linked index. Operational results are derived under the assumption of statistical independence between price and the index and the Gaussian distribution of the underlying variables. An experiment shows the gain of the proposed strategy over the best performing claim derived by Oum and Oren.

Keywords: Static Hedging, Energy Risk Mitigation,Volumetric Hedging, Incomplete Markets.

JEL Classification: G0, G13, C32.

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1 Introduction

Electric power markets are going through an infancy period compared to other more developed markets such as fixed income securities, stocks and currencies. In addition, the energy market is a special case given that it has some added complexities. Electric power needs real time balancing between supply and demand because electricity is consumed at the same time as it is produced; inventories cannot be held to compensate price and quantity fluctuations. Electricity is the commodity with special condition unlike other kinds of financial products, the technological inability to store it efficiently and high marginal production costs create jumps in the spot price, so that arbitrage arguments have been difficult to deal with. All these specifications make classical dynamic hedging theory impossible to apply.

Furthermore, the market participants, (i.e. generators, marketers or load serving entities (LSE), who are not the end-users of electricity) have to sell or buy electricity at a price set by the supply and demand equilibrium when the final users consume the electricity at a fixed regulated price. In addition, the regulated demand is inelastic; a LSE unit has the obligation to deliver electricity on demand at a fixed price without fail, independent to the costs. This paper cover the typical case of an energy retailer procuring power from the wholesale market at the standing spot price and reselling it to industrial consumers exhibiting variable demand shapes. The paper follows the line traced in a seminal article by Oum and Oren (2008), and proposes the issue of determining the optimal derivative pay-off written on both electricity price and a weather-linked index.

The difficulty of storing electric power efficiently does not allow mitigation of volume risk. Weather derivatives can be used in order to hedge unexpected changes in weather. Weather derivatives are based on indexes of temperature, such as Chicago Mercantile Exchange (CME) indexes, Cooling-Degree-Days (CDD), or Heating-degree-Days (HDD). Sometimes insurance companies trying to transfer their climate-related risk to capital markets need to transform non-tradable risk into tradable financial securities such as weather derivatives, due to weather indexes allow to value the index variations.

Weather derivatives were first launched in 1996 in the United States as a mechanism of protection against weather anomalies. The purpose of weather derivatives is to smooth out the temporal fluctuations in the company's revenues. There are a number of financial and commercial reasons why this is beneficial (Jewson (2004)). Companies hedge their portfolios against unexpected weather variations using contracts that are not correlated with classical financial assets. For instance, the Niño phenomenon was responsible for weather anomalies that took place over thirteen months between April 1997 and May 1998 and over one year between April 2002 and April 2003 in South and North America. Chicago Mercantile Exchange Anon CME. (2005) started offering the first standardized weather derivatives in September 1999, with the purpose of increasing liquidity and accessibility on this kind of contract. The market was accepted this and grew quickly.

Anon CME. (2005) offers weather futures and options. Contract specifications include: type, contract size, product description, tick size, period and the settlement procedure (Anon CME. (2005)

2010). The daily average temperature T_j is defined as the arithmetic average of the maximum and minimum temperature recorded between 12:01 a.m. and 12:00 a.m. midnight as reported by MacDonald Dettwiler and Associates (MDA) information System, Inc.

$$T_j = \frac{T_j^{max} + T_j^{min}}{2} \tag{1}$$

For each day during winter, Heating-Degree-Days (HDD) is the maximum between zero and 65 degrees Fahrenheit (18 degrees Celsius) minus the daily average temperature T_j. For each day during summer, Cooling-Degree-Days (CDD) is the maximum between the daily average temperature T_j minus 65 degrees Fahrenheit (18 degrees Celsius) and zero (Anon CME. (2005)). Weather derivatives are basically a speculative security because those indexes are not a tradable commodity or a delivery asset. Due incomplete characterization, the weather derivatives market still does not have an effective pricing model.

Several authors have proposed pricing models for weather derivatives in continuous time framework. Richards et al (2004) presented an equilibrium pricing model based on temperature processes of a mean-reverting Brownian motion. Chaumont et al (2005) considered that under an equilibrium condition, the market price of risk is uniquely determined by a backward stochastic differential equation, and they translate these stochastic equations into semi-linear partial differential equations. They then choose two simple models for sea surface temperature. Lee and Oren (2009) derived an equilibrium pricing model for weather derivatives and measured risk hedging, including weather derivatives, in a volumetric hedging strategy.

Volume risk in electric power markets has significant dimensions when quantity is affected by weather conditions; in countries with seasons, random movements in temperature affect electric power demand. Some tropical countries are also affected by hydrological conditions and the correlation between the load volatility and the weather variable. In general, power generation is affected by hydrological variables when production system uses hydro generation. It has been empirically shown that the most important factor affecting the quantity of power generation is the climatic conditions, and load is correlated to the weather. Economic earnings obtained by industries which are weather-sensitive are affected by weather anomalies which is the case of energy industries (Dutton (2002)). The volumetric risk faced by electric power companies is correlated with unexpected changes in weather or hydrology which cause demand and price fluctuations. As an extension of the VaR-constrained hedging introduced in Oum and Oren (2008), this chapter proposes a new way to hedge a LSE's profit based on the constitution of an optimal portfolio composed by two claims: standard contracts on price and weather derivatives. The most important risks faced by the market participants are price risk and quantity risk. Variations in weather conditions affect both price and quantity; price risk is caused by extreme high volatility, and the volumetric risk is determined by the uncertainty of final consumption.

The main purpose of this paper is to derive the hedging portfolio model based on two claims: price and volumetric hedging instruments. We derive the optimal portfolio from the expected utility maximization problem using vanilla and weather derivatives whose payoffs will minimize losses.

This proposal is supported in the independence assumption that implied that both claims price and weather are not correlated, in this case we derived the optimal payoff functions, and found evidence that the inclusion of two payoffs generates incremental improvements over agent's revenues and minimizes risk measures.

LSE has to provide electric power on demand at a fixed price but faces uncertainty about the quantity of electric power to supply and the price it will pay. Hedging strategy allows the agents throughout the derivatives-contracts payoffs to mitigate losses caused by unexpected changes in price and quantity. We derive the hedging portfolio including the weather derivatives whose payoffs will minimize these losses. The portfolio construction problem follows Markowitz's (1952) model, where an investor's goal defines the portfolio construction in order to maximize expected future returns given a certain level of risk. The Markowitz model establishes that the volatility of portfolio returns measures the risk. Campbell et al (2001) introduced a similar portfolio allocation problem using VaR as a risk measure. In the electric power literature, several authors follow Markowitz's methodology to address hedging strategy using vanilla derivatives. Nasakkala and Keppo (2005), and Woo et al (2004),studied the interaction between stochastic consumption volumes and electricity prices, and proposed a mean-variance type model to determine optimal hedging strategies. Vehvilainen and Keppo (2006)) optimized hedging strategies taking into account the Value at Risk as risk measure. Huisman et al (2007) introduced a one-period framework to determine optimal positions in peak and off-peak contracts in order to purchase future consumption volume. In this framework, hedging strategy is assumed to minimize expected costs relating to an ex-ante risk limit defined in terms of Value at Risk.

The Markowitz (1952) concept of efficient frontier also applies to electricity, but previous authors did not consider the effect of volume risk exposure in their optimization solutions. Volume risk exposure can be a potent component of portfolio losses due to adverse movements in quantity in the electric power market. Authors cited above have tried to solve the Markowitz problem, but the portfolio is only composed in order to hedge price risk exposure. Oum and Oren (2008) developed a self-financed hedging portfolio consisting of derivatives contracts, and they obtained the optimal hedging strategy in order to hedge price, and volume risk maximizing the expected utility of hedge profit for the LSE.

This paper is organized as follows. In Section 2 we derive closed-form results for the hedging portfolio problem. In Section 3 we illustrate these results, and Section 4 concludes.

2   Closed-form model

Let $y(p,q)$ be the LSE's profit from serving the customers' demand q at the fixed retail rate r at time T, $x(p)$ is a function of the Spot price at time T, $z(\iota)$ is a function of the weather at time T and Y is the overall profit.
The hedged profit

$$Y(p,q,x(p),z(\iota)) = y(p,q) + x(p) + z(\iota) \qquad (2)$$

$$\text{Where, } y(p,q) = (r-p)q$$

This portfolio considers to buy a forward contracts for an amount q at the forward price F_p and a forward contract over the weather at the forward price F_ι in order to hedge a part the uncertainty on demand and spot price.

Then the payoffs function will be:

$$f(p,q) = (r-p)q + x(p) + z(\iota),$$

$$x(p) = x(F_p).1 + x'(F_p)(p-F_p) + \int_0^F x''(k)(K-p)^+ dk + \int_0^\infty x''(k)(p-k)^+ dk \qquad (3)$$

$$z(i) = z'(F_\iota)(\iota - F_\iota)$$

Then the problem is to know how many forwards and options and for which strike the LSE should purchase. Note that the hedging portfolio also includes money market accounts, letting the LSEs borrow money to finance hedging instruments. It is a one-period model where the hedging portfolio is built at time 0 for a delivery at time 1.

The LSE's preference utility is characterized by a concave utility function U defined over the total profit $Y(p,q,x(p),z(\iota))$ at time 1. Let $f(p,q)$ be the joint density function for positive p and q defined on the probability measure **P** which represents the beliefs on the realization of p and q. let **Q** be a risk neutral probability measure which is not unique since the electric power market is incomplete and $g(p)$ the density function of p under **Q**. Then it can formulate the problem as follows:

$$\max_{x(p),z(\iota)} E[U(y(p,q),x(p),z(\iota))] \qquad (4)$$

$$s.t \; E^Q[x(p)] = 0$$

$$E^Q[z(\iota)] = 0$$

VaR constraint could be expressed such as:

$$VaR_y(Y(x^*(p),z^*(\text{-}))) \leq V_0$$

It costs zero to construct a portfolio at time 0, where $E[.]$ and $E^Q[.]$ denote expectations under the probability measure **P** and **Q**, respectively.

2.1 Optimal pay-offs of the hedging strategy

Here we give an explicit solution to the optimization problem showed in (4), in order to improve the performance of the hedging claim due to the link between demanded volume and weather-linked index. We obtain an optimal pay-off of the hedging strategy which depends on the utility

function that describes the LSE's preferences. The LSE's hedging problem of price and volume risk under VaR criteria has been considered by Oum and Oren (2008), Kleindorfer and Li (2005), Woo et al (2004), and Wagner et al (2003). VaR defined as a maximum possible loss with $(1-\gamma)$ percent confidence, is considered such as risk measure in practice. Furthermore, the optimization problems with the VaR risk measure are hard to solve analytically without very restrictive assumptions more in the case of both price and volume are volatile.

2.1.1 Optimality condition

Let $x(p)$ the pay-off of the hedging strategy against price risk, $z(\iota)$ the pay-off of the hedging strategy against volumetric risk, and U is the utility function that describes the LSE's preferences. Thus, the optimal pay-offs of the hedging strategy against price and volumetric risk is the solution of the following optimization problem:

$$\max_{x(p),z(\iota)} E[U(y(p,q),x(p),z(\iota))]$$

$$s.t \ E^Q[x(p)] = 0$$

$$E^Q[z(\iota)] = 0$$

$$VaR_y(Y(x^*(p),z^*(\iota))) \le V_0$$

The optimal pay-offs $x^*(p)$ and $z^*(\iota)$ are:

$$E\left(U'(Y(p,q,x^*(p),z(\iota))|p)\right) = \lambda_x^* \frac{g_x(p)}{f_x(p)}$$

$$E\left(U'(Y(p,q,x^*(p),z(\iota))|\iota)\right) = \lambda_z^* \frac{g_z(\iota)}{f_z(\iota)}$$

Where λ is the Lagrange multiplier, and for an agent who maximizes mean-variance expected utility of profit,

$$U(Y) = Y - \frac{1}{2}a(Y^* - E[Y^*]^2)$$

Proof. The Langrangian function for the constrained optimal problem is given by,

$$\partial(x,z) = \iint_{R^2} U(Y|p,\iota) f_{p,\iota}(p,\iota) dp d\iota - \lambda_x \int_{R^2} x(p) g_x(p) dp - \lambda_x \int_{R^2} z(\iota) g_z(\iota) d\iota$$

$$\overrightarrow{grad}(L(x,z)) = \vec{0}$$

With the Lagrange multipliers λ_x, λ_z and the marginal density functions $f_x(p)$ of p and $f_z(\iota)$ of ι under P, by differentiation of $L(x(p))$ with respect to $x(p)$ and $L(z(\iota))$ with respect to $z(\iota)$ results in

$$\frac{\partial L}{\partial x} = E\left[\frac{\partial Y}{\partial x} U'(Y)|p\right] f_x(p) - \lambda_x g_x(p) = 0 \tag{1}$$

$$\frac{\partial L}{\partial z} = E\left[\frac{\partial Y}{\partial z} U'(Y)|\iota\right] f_z(\iota) - \lambda_z g_z(\iota) = 0$$

By the Euler equation from (1) and substituting $\frac{\partial Y}{\partial x} = 1$ and $\frac{\partial Y}{\partial z} = 1$ from (1) yields the first order conditions for the optimal solutions $x^*(p)$ and $z^*(\iota)$ as follows:

$$E\left[U'(Y(p,q,x^*(p),z^*(\iota))|p\right] = \lambda_x^* \frac{g_x(p)}{f_x(p)}$$

$$E\left[U'(Y(p,q,x^*(p),z^*(\iota))|\iota\right] = \lambda_z^* \frac{g_z(\iota)}{f_z(\iota)}$$

$\square$

Theorem 1. *Based on Kleindorfer and Li (2005) and Oum and Oren (2008), the assumption in this part is that $VaR(Y(x,z))$ is determined by $PrX_t \geq -VaR = \gamma$, where X_t denotes the typical daily cash flow. Therefore, $VaR_t = z(\gamma)\sigma_t - \mu_t$, where $z(\gamma)$ is the z-score of a standardized normal random variable. There exists a continuous function $\eta : (E, \Sigma, \gamma) \to \mathfrak{R}$, and that the function is strictly increasing in σ and where $Var_\gamma(\mu, \sigma, \gamma) = \eta(\mu, \sigma, \gamma) - \mu$ is non-increasing in σ, then:*

$$P\left(Y(x,z) \leq \mu_{\}} - \eta(\mu, \sigma, \gamma)\right) \equiv 1 - \gamma$$

If $Y(x,z)$ is normally distributed, then the risk aversion assumption is satisfied with $\eta(\mu, \sigma, \gamma) = Z(\gamma)\sigma$, where $Z(\iota)$ is the standard z-score at the confidence level. Where $\eta(\mu, \sigma, \gamma)$ is continuous and increasing in σ and the VaR function $VaR_\gamma(\mu, \sigma, \gamma)$ is non-increasing in μ for $\mu = E[Y(x,z)]$ and $\sigma^2 = var(Y(x,z))$. Therefore, if $x^*(p) + z^*(\iota)$ solves the problem (4), then it can hold that:

    i. *If $(x^*(p), z^*(i))$ is on efficient frontier of the $(E-V)$ space, then it can hold that any feasible pair $(x(p), z(i))$ is mapped to a corresponding point $(V(Y(x,z)), E[Y(x,z)])$.*

    ii. *I can assume that fixed $a \geq 0$, let $Y(x,z) = Y(x^a, z^a)$ be the portfolio obtained by maximizing $(E-aV)$, therefore $Y(x^a, z^a)$ is on the border of the feasible set in $(E-VaR_\gamma)$ space, and for any feasible portfolio $Y'(x,z)$ for which $E[Y'(x,z)] = E[Y(x^a, z^a)]$ and $VaR[Y'(x,z)] \geq E[Y(x^a, z^a)]$, there exists $a \geq 0$ such that $(x^*(p), z^*(\iota))$ solves $\max_{x(p) \in x(p), z(\iota)} E[Y(x,z)] - \frac{1}{2} a * var(Y(x,z))$.*

The proof of Theorem 1, will be provided in pag. 22 appendix.

Proposition 1. *Based on Oum and Oren (2008) We will show how the solution to the mean-variance problem can be used to approximate the solution to the VaR-constrained problem*

$$(x^a(p), z^a(\iota)) = argmax_{x(p) \in X(p), z(\iota) \in Z(\iota)} E[Y(x,z)] - \frac{1}{2} a * var(Y(x,z))$$

$$s.t. \, E^Q[x(p)] = 0$$

$$E^Q[z(\iota)] = 0$$

Then $E[Y(x^a(p) + z^a(\iota))]$ and $var(Y(x^a(p) + z^a(\iota)))$ are monotonically non-increasing in a

The proof of Proposition 1, will be provided in pag. 22 appendix.
Theorem 1 and Proposition 1 state that the feasible set of the VaR-constrained problem is restricted to the solution of mean-variance problem for varying a. Thus, the solution to expression (4) of the Theorem 1 can be obtained in the next algorithm:

  i. We can obtain $(x^a(p), z^a(\iota))$ that maximizes:

$$E[Y(x,z)] - \frac{1}{2} a * var(Y(x,z))$$

  ii. For each a, calculate associated $VaR(a) \equiv VaR(Y(x^a, z^a))$ such that

$$P\{Y(x^a, z^a) \geq -VaR(a)\} = \gamma$$

  iii. By Theorem 1, find smallest a such that $VaR_\gamma(Y(x^a, z^a)) \leq V_0$

Proposition 2 (Closed-Form Results). *Under independence assumption.*
Maximizing the mean-variance utility function on profit,

$$E[U(Y)] = E[Y(x,z)] - \frac{1}{2} a * var(Y(x,z))$$

For maximizing mean-variance expected utility the optimal solution $x^*(p)$ and $z^*(\iota)$ to problem

$$\max_{x(p), z(\iota)} E[U(y(p,q), x(p), z(\iota))]$$

$$s.t. \, E^Q[x(p)] = 0$$

$$E^Q[z(\iota)] = 0$$

That is given by:

$$x^* = \frac{1}{a} - E[y(p,q)|p] - E[z^*|p] + \left(E[y(p,q)] - \frac{1}{a}\right) \frac{g_x(p)}{f_x(p)} + (E[x^*] + E[z^*]) \frac{g_x(p)}{f_x(p)}$$

$$z^* = \frac{1}{a} - E[y(p,q)|\iota] - E[x^*|\iota] + \left(E[y(p,q)] - \frac{1}{a}\right) \frac{g_z(\iota)}{f_z(\iota)} + (E[x^*] + E[z^*]) \frac{g_z(\iota)}{f_z(\iota)}$$

7

Under the Independence assumption that p, and ι are uncorrelated then we can establish that:

$$E[z^*|p] = E[z^*]$$

$$E[x^*|\iota] = E[x^*]$$

And, finally we have:

$$x^* = \left[\frac{1}{a} - E[y(p,q)|p] + \left(E[y(p,q)] - \frac{1}{2}\right)\frac{g_x(p)}{f_x(p)}\right] + E[x^*]\left[\frac{g_x(p)}{f_x(p)}\right] + E[z^*]\left[\frac{g_x(p)}{f_x(p)} - 1\right] \quad (2)$$

$$z^* = \left[\frac{1}{a} - E[y(p,q)|\iota] + \left(E[y(p,q)] - \frac{1}{2}\right)\frac{g_x(\iota)}{f_x(\iota)}\right] + E[x^*]\left[\frac{g_x(\iota)}{f_x(\iota)}\right] + E[z^*]\left[\frac{g_x(\iota)}{f_x(\iota)} - 1\right] \quad (3)$$

Where,

$$E[x^*] = \frac{\left[\frac{1}{a} - E[y(p,q)|p] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_x(p)}{f_x(p)}\right]\frac{g_z(\iota)}{f_z(\iota)}}{\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right] - \left[\frac{g_x(p)}{f_x(p)}\right]\left[\frac{g_z(\iota)}{f_z(\iota)}\right]}$$
$$- \frac{\left[\frac{1}{a} - E[y(p,q)|\iota] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)}\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right]}{\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right] - \left[\frac{g_x(p)}{f_x(p)}\right]\left[\frac{g_z(\iota)}{f_z(\iota)}\right]} \quad (4)$$

$$E[z^*] = \frac{\left[\frac{1}{a} - E[y(p,q)|\iota] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)}\right] + \left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]}{\left[\frac{g_z(\iota)}{f_z(\iota)}\right]\left(\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right] - \left[\frac{g_x(p)}{f_x(p)}\right]\left[\frac{g_z(\iota)}{f_z(\iota)}\right]\right)}$$
$$\left(\left[\frac{1}{a} - E[y(p,q)|p] + \frac{\left(E[y(p,q)] - \frac{1}{a}\right)g_x(p)}{f_x(p)}\right]E^Q\left[\frac{g_z(\iota)}{f_z(\iota)}\right]\right.$$
$$\left. - \left[\frac{1}{a} - E[y(p,q)|\iota] + \frac{\left(E[y(p,q)] - \frac{1}{a}\right)g_z(\iota)}{f_z(\iota)}\right]\left[\frac{g_x(\iota)}{f_x(\iota)} - 1\right]\right). \quad (5)$$

The proof of Proposition 2, will be provided in pag. 22 appendix.

Proposition 3. *Let $(p;q)$ and $(\iota;q)$ be each a 2-dimensional random vector. p is the price the LSE pays when it buys electricity and ι is the weather index used to optimize hedging. q is the quantity of electricity purchased, if $(p;q)$ and $(\iota;q)$ follow a log-normal / normal distribution where,*

$$(\log p, q) \sim N(\mu_{p,q}, \Sigma_{p,q})$$

The proof of Proposition 3, will be provided in pag. 22 appendix.
We are assuming that (p, q) are correlated, by which, the density function of q, is given by:

$$\mu_{pq} = \begin{pmatrix} \mu_p \\ \mu_q \end{pmatrix},$$

$$\sum_{pq} = \begin{bmatrix} \sigma_p^2 & \rho_{p,q}\sigma_p\sigma_q \\ \rho_{p,q}\sigma_p\sigma_q & \sigma_q^2 \end{bmatrix}$$

$$(\log p, q) \sim N(\mu_p, \mu_q, \sigma_p^2, \sigma_q^2, \rho_{p,q})$$

$$q|p \sim N\left(\mu_q + \rho_{p,q}\frac{\sigma_q}{\sigma_p}(\ln p - \mu_p), \sigma_q^2(1 - \rho_{p,q}^2)\right)$$

The Independence case is special case of this expression and we can establish that when (p, q) are independent the density function of q, is given by:

$$q \sim N(\mu_q, \sigma_p)$$

And in the case of (ι, q) they are correlated so that

$$\mu_{\iota q} = \begin{pmatrix} \mu_\iota \\ \mu_q \end{pmatrix},$$

$$\sum_{\iota q} = \begin{bmatrix} \sigma_\iota^2 & \rho_{\iota,q}\sigma_\iota\sigma_q \\ \rho_{\iota,q}\sigma_\iota\sigma_q & \sigma_q^2 \end{bmatrix}$$

$$(\log\iota, q) \sim N(\mu_\iota, \mu_q, \sigma_\iota^2, \sigma_q^2, \rho_{\iota,q})$$

$$q|p \sim N\left(\mu_q + \rho_{\iota,q}\frac{\sigma_q}{\sigma_\iota}(\ln\iota - \mu_\iota), \sigma_q^2(1 - \rho_{\iota,q}^2)\right)$$

Then the density function of q knowing ι is given by:
Finally the marginal distribution of p, ι and q are as follows:

Under P:

$$\ln p \sim N(\mu 1_p, \sigma_P^2)$$
$$q \sim N(\mu_q, \sigma_q^2)$$
$$\ln\iota \sim N(\mu 1_\iota, \sigma_\iota^2)$$
$$Corr(\ln p, q) = \rho_{p,q}$$
$$Corr(\ln\iota, q) = \rho_{\iota,q}$$

Under Q:

$$\ln p \sim N(\mu 2_p, \sigma_P^2)$$
$$\ln \iota \sim N(\mu 2_\iota, \sigma_\iota^2)$$

From a density function of lognormal distribution, we have:

$$\frac{g_x(p)}{f_x(p)} = e^{\frac{\mu 2_p - \mu 1_p}{\sigma_p^2} \ln p + \frac{\mu 1_p^2 - \mu 2_p^2}{\sigma_p^2}}$$

$$\frac{g_z(\iota)}{f_z(\iota)} = e^{\frac{\mu 2_\iota - \mu 1_\iota}{\sigma_\iota^2} \ln p + \frac{\mu 2_i^2 - \mu 1_\iota^2}{\sigma_\iota^2}}$$

$$E^Q \left[\frac{g_x(p)}{f_x(p)} \right] = e^{\left(\frac{\mu 2_p - \mu 1_p}{\sigma_p} \right)^2}$$

$$E^Q \left[\frac{g_z(\iota)}{f_z(\iota)} \right] = e^{\left(\frac{\mu 2_i - \mu 1_\iota}{\sigma_\iota} \right)^2}$$

where,

$$\frac{g_x(p)}{f_x(p)} = \frac{\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_p} e^{-\frac{1}{2} \left(\frac{\log p - \mu 2_p}{\sigma_p} \right)^2}}{\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_p} e^{-\frac{1}{2} \left(\frac{\log p - \mu 1_p}{\sigma_p} \right)^2}}$$

$$= e^{\frac{1}{2} \left(\frac{\log p - \mu 1_p}{\sigma_p} \right)^2 - \frac{1}{2} \left(\frac{\log p - \mu 2_p}{\sigma_p} \right)^2}$$

$$= e^{\frac{\mu 2_p - \mu 1_p}{\sigma_p} \log p - \frac{1}{2} \frac{(\mu 2_p)^2 - (\mu 1_p)^2}{\sigma^2 p}}$$

Under Q,

$$\frac{\mu 2_p - \mu 1_p}{\sigma_p} \log p - \frac{1}{2} \frac{(\mu 1_p)^2 - (\mu 2_p)^2}{\sigma^2 p} \sim N \left(\frac{\mu 2_p - \mu 1_p}{\sigma_p} \mu 2_p + \frac{1}{2} \frac{(\mu 1_p)^2 - (\mu 2_p)^2}{\sigma^2 p}, \left(\frac{\mu 2_p - \mu 1_p}{\sigma_p^2} \right)^2 \sigma_p^2 \right)$$

Then

$$E^Q \left[\frac{g_x(p)}{f_x(p)} \right] = e^{\frac{\mu 2_p - \mu 1_p}{\sigma_p} \mu 2_p + \frac{1}{2} \frac{(\mu 1_p)^2 - (\mu 2_p)^2}{\sigma^2 p} + \frac{1}{2} \left(\frac{\mu 2_p - \mu 1_p}{\sigma_p^2} \right)^2 \sigma_p^2}$$

$$= e^{\left(\frac{\mu 2_p - \mu 1_p}{\sigma_p} \right)^2}$$

Then, under a bivariate log-normal distribution, we can compute the next mathematical means:

$$E[y(p,q)|\iota] = (r-p)E(q|\iota) = (r-p)\left(\mu_q + \rho_{\iota,q}\frac{\sigma_q}{\sigma_\iota}(\ln\iota - \mu_\iota)\right)$$

$$
\begin{aligned}
E[y(p,q)] &= E[(r-p)q] \\
&= r\mu_q - E[pq] \\
&= r\mu_q - \mu_q e^{\mu_p + \frac{1}{2}\sigma_p^2}
\end{aligned}
$$

Hence,

$$E^Q[E[y(p,q)]] = \mu_q\left(r - e^{\mu_{2p}+\frac{1}{2}\sigma_p^2}\right)$$

$$
\begin{aligned}
E^Q[E[y(p,q)|\iota]] &= \left(r - \exp\left(\mu_{2\iota}+\frac{1}{2}\sigma_\iota^2\right)\right)\left(\mu_q + \rho_{\iota,q}\frac{\sigma_q}{\sigma_\iota}\mu_{1\iota}\right) \\
&\quad + \rho_{\iota,q}\frac{\sigma_q}{\sigma_\iota}\left(r\mu_{2\iota} - (\mu_{2\iota}+\sigma_\iota^2)\exp\left(\mu_{2\iota}+\frac{1}{2}\sigma_\iota^2\right)\right)
\end{aligned}
$$

2.2 The replication of pay-offs

Carr and Madan (2001) showed that any continuously differentiable functions $x(p)$ and $z(\iota)$ can be written in the following form: for an arbitrary positive s,

$$x(p) = [x(s) - x'(s)s] + x'(s)p + \int_0^s x''(K)(K-p)^+ dK + \int_s^\infty x''(K)(p-K)^+ dK$$

$$z(\iota) = [z(s) - z'(s)s] + z'(s)\iota$$

In this case, if F_p is the forward price of electricity and F_ι is the forward weather-related claim, the property proved by Carr and Madan (2001) has the next interpretation:

$$x(p) = x(F_p).1 + x'(F_p)(p - F_p) + \int_0^{F_p} x''(K)(K-p)^+ dK + \int_{F_p}^\infty x''(K)(p-K)^+ dK$$

$$z(\iota) = z'(F_\iota)(\iota - F_\iota) + \int_0^{F_p} x''(K)(K-p)^+ dK + \int_{F_p}^\infty x''(K)(p-K)^+ dK$$

To replicate in continuous time a hedging strategy against price risk and quantity risk, the LSE should have a position on:

- $x(F_p)$ units of bonds

- $x'(F_p)$ units of forward price

11

- $z'(F_1)$ units of forward weather-related claim

- $x''(K)dK$ units of put options with strike K for $K < F_p$

- $x''(K)dK$ units of call options with strike K for $K > F_p$

In practice, we do not have a continuous set of strike prices and we need to work in discrete time. Thus, by assuming we have n strike prices for put options and m strike prices for call options such that $0 < K_1 < K_n < F_p < K'_1 < K'_2 < \cdots < K'_m$, replicating the hedging strategy should require a position on:

- $x(F_p)$ units of bonds

- $x'(F_p)$ units of forward price

- $z'(F_i)$ units of forward weather-related claim

- $\frac{1}{2}(x''(K_{i+1}) - x''(K_{i-1}))$ units of put options with strike $(K_i, i = 1, \cdots n)$

- $\frac{1}{2}(x''(K'_{i+1}) - x''(K'_{i-1}))$ units of call options with strike $(K'_i, i = 1, \cdots n)$

In this approximation scheme, the error will be small if $x''(p)$ is a constant in each interval between two consecutive strike prices, and when price realizations p are close to the discrete strike prices.

3 Empirical Results

3.1 Implementation Algorithm

The problem is to know how many forwards and options and for which strike the LSE should purchase. Note that the hedging portfolio also includes money market accounts, letting the LSEs borrow money to finance hedging instruments. It is a one-period model where the hedging portfolio is built at time 0 for a delivery at time 1. The feasible set of the VaR-constrained problem is restricted to the solution of mean-variance problem for varying a (see Theorem 1 and Proposition 2). Thus, the solution to VaR-constrained optimization problem can be obtained in the next algorithm:

i Fix parameters including range for a (min, max and steps).

ii Fix number of simulations $num_t rab$ "large".

iii Generate random price p, load q and weather variable w, using a multivariate normal distribution.

iv Compute the payoff $x^*(p)$ and $z^*(\iota)$ (Equations (4) and (5)).

12

v We can obtain $(x^a(p),\, z^a(\iota))$ that maximizes:

$$E[Y(x,z)] - \frac{1}{2}a * var(Y(x,z))$$

vi For each a, calculate associated $VaR(a) \equiv VaR(Y(x^a, z^a))$ such that $P\{Y(x^a, z^a) \geq -VaR(a)\} = \gamma$

vii Find smallest $a(a_{opt})$ such that $VaR_\gamma(Y(x^a, z^a)) \leq V_0$

viii Using a_{opt} in order to find $Y(x^{(}a_{opt}),\, z^{(}a_{opt}))$. $Y(\cdot)$, be the profit distribution of the expected utility maximizing solution, under Optimal Static Hedging including the weather claim (see Figure 3)

ix Using the payoff functions $x(a_{opt})$, and $z^{(}a_{opt})$, and based on Carr and Madan (2001) we can define the replication of payoff (Equation (6)), (see Figure 4).

The algorithm above permits us obtain the optimal static hedging profit distribution using two claims and the concerning replication payoff function.

3.2 Empirical Result

Computing an approximate optimal VaR-constrained volumetric hedging problem according to the above development, we will show two groups of results: results under the independence assumption, and also under the general case. In both we will present the comparison of different possibilities, which are: without-hedge, hedging using $x^*(p)$ following Oum and Oren's model, and our proposal using $[x^*(p) + z'^*(\iota)]$; note that $x^*(p) \neq x'^*(p)$, because $x^*(p)$ corresponds to Oum and Oren model. Following the same application made by Oum and Oren (2008), the hedging strategy for an LSE that maximizes the expected pay-off with VaR constraint of -\$60.000 is composed by a hypothetical LSE that charges a flat retail rate of \$120 per MWh. The spot price p at which the agent has to buy electric power, the weather-index ι and the quantity q is the load at which the LSE supplies in a fixed interval; the three variables, price, temperature and quantity are volatile and these variations affect the agents' revenues; that is the problem that agents will try to solve using an optimal static hedging solution. In order to obtain the solution of the mean-variance problem for varying a we assume that P and Q distributions are different. All of three variables are distributed according to a bivariate distribution in log price and quantity, and the log weather-index and quantity, as follows:

Under Independence assumption:

Under P: $lnp \sim N(4, 0.7^2)$ $q \sim N(3000, 650^2)$ $logi \sim N(2.2, 0.0821^2)$
$Corr(lnp, q) = 0$ $Corr(ln\iota, q) = 0.5$
Under Q: $ln\iota \sim N(4, 0.7^2)$ $log\iota \sim N(2.1, 0.0821^2)$
General Case: $Corr(lnp, q) = 0.5 Corr(ln\iota, q) = 0.4$

13

Taking in account the last parameters, and the normal bivariate probability distribution, I fitted Monte-Carlo simulation technique to generate spot price, load and weather index patterns. Figure 1 shows the spot price, load and weather index patterns.

Figure 2 shows the basis of the problem; profit distribution without hedging, considering afore-mentioned distribution of parameters. The profit without hedging only considers the LSE fixed rate, the spot price and quantity denoted by $y(p \ q) = (r \quad p)q$
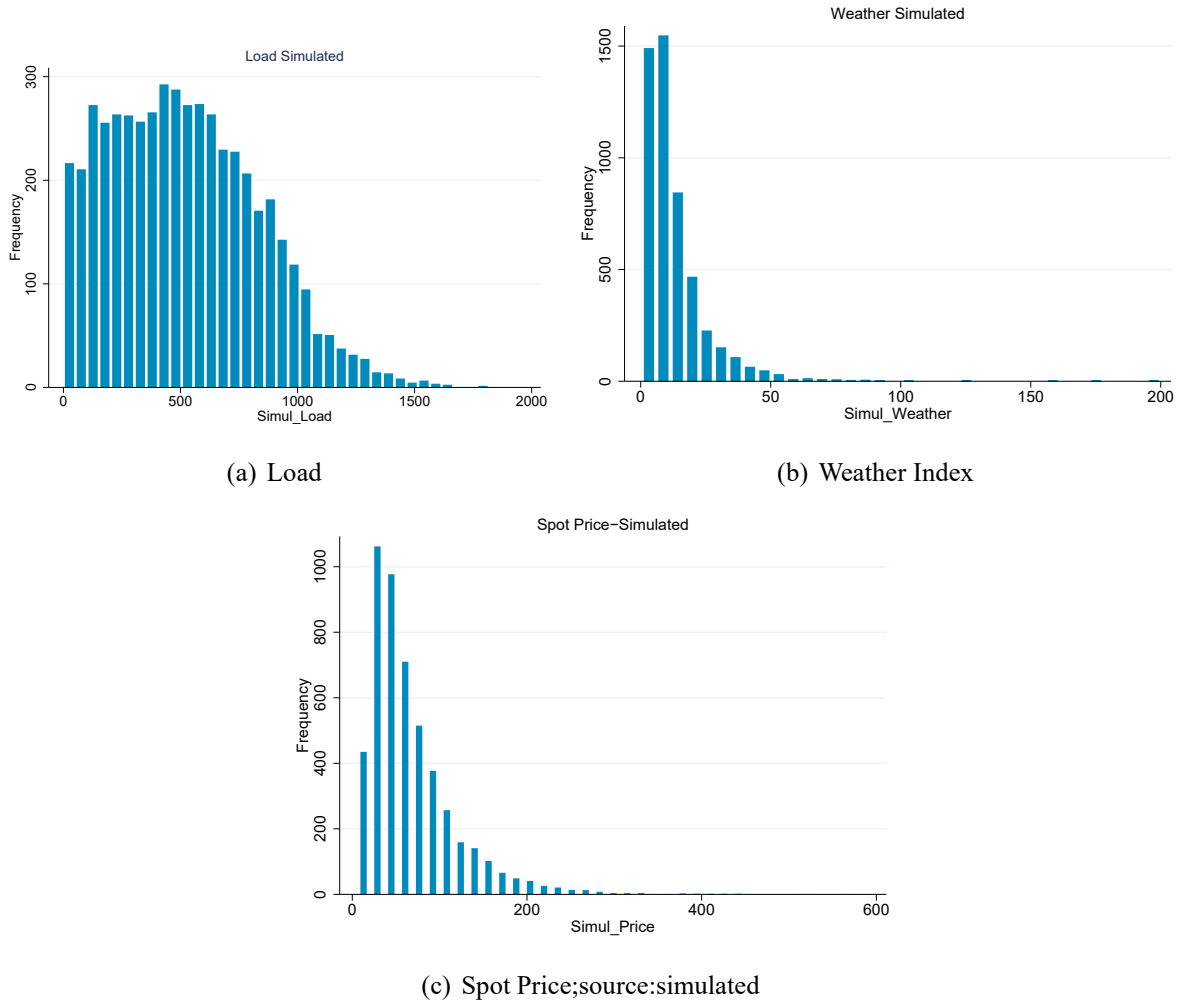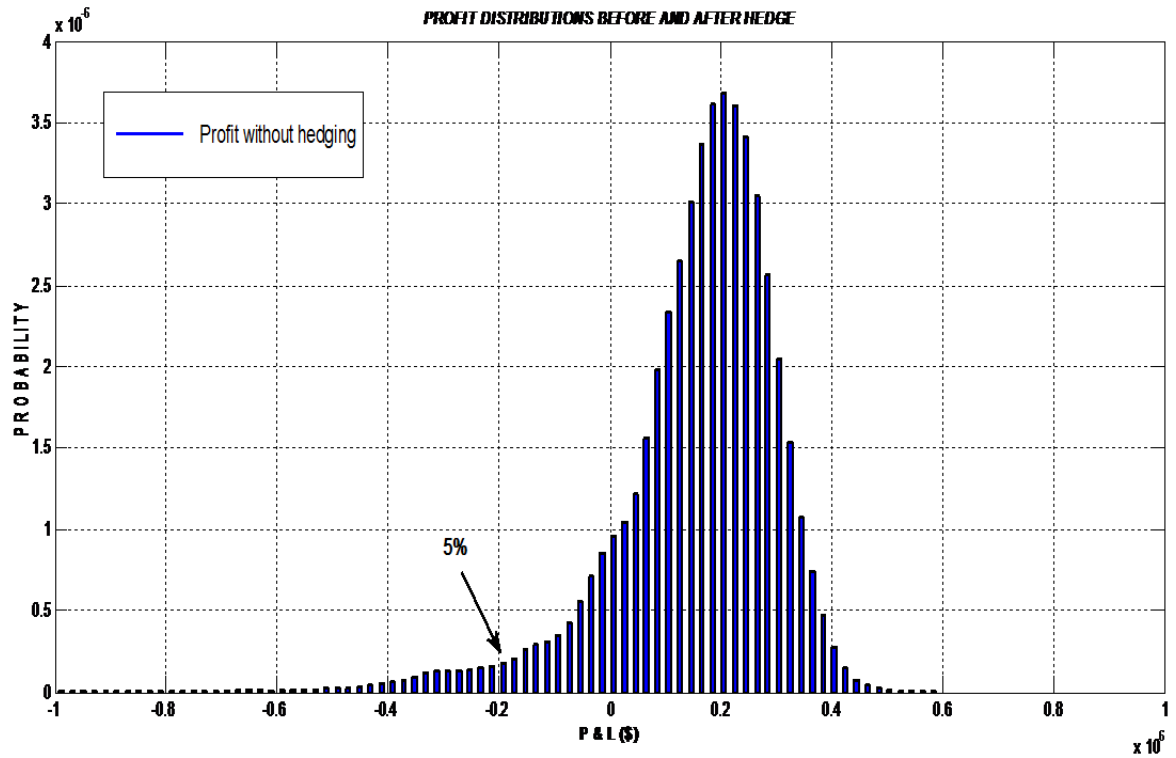

(a) Load

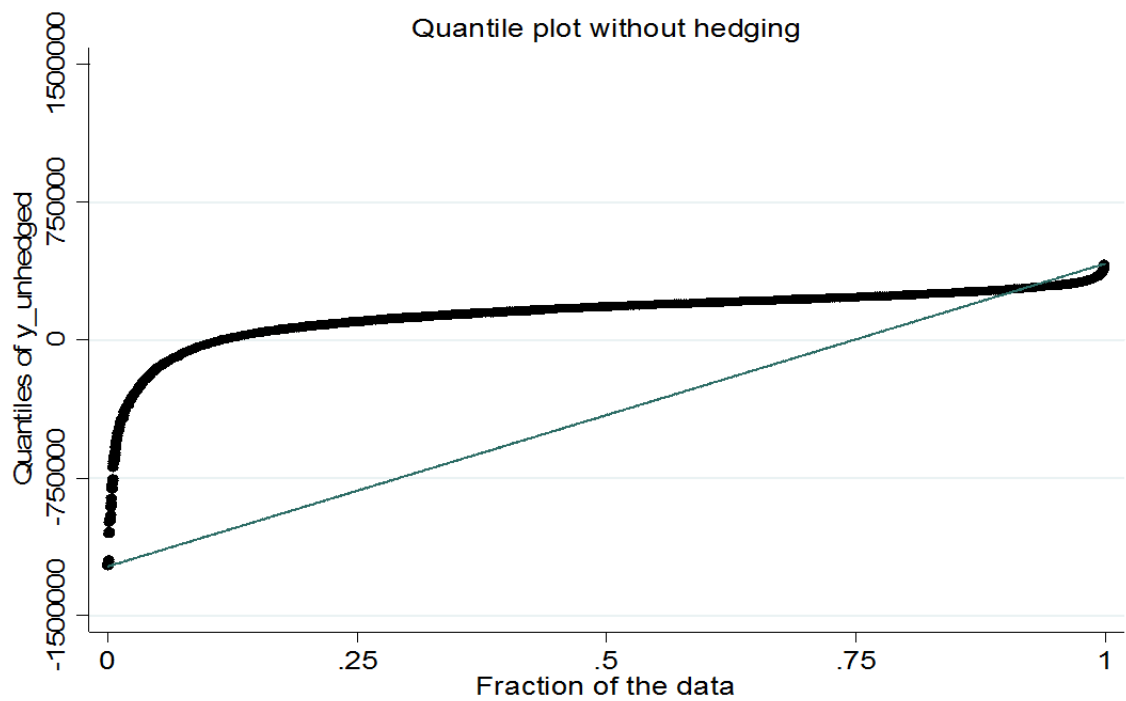(b) Weather Index

(c) Spot Price;source:simulated

Figure 1: Simulated patterns using Oum and Oren Parameters.

(a) Normal bivariate distribution of profit

(b) Quantile plot without hedging

Figure 2: Distribution of profit without hedging y(p,q)=(r-p)q assuming r=$120/MWh.

Due to P distribution being different from Q, for various levels of risk aversion a there exists a mean-variance problem solution. We restrict the set of solutions using the VaR-constrained problem (see Theorem 1) in order to find the optimal one.

Figure 3 shows the optimal mean-variance hedging strategy corresponding to optimal a^*. I show the optimal payoff function obtained as an approximation for VaR-constrained problem.
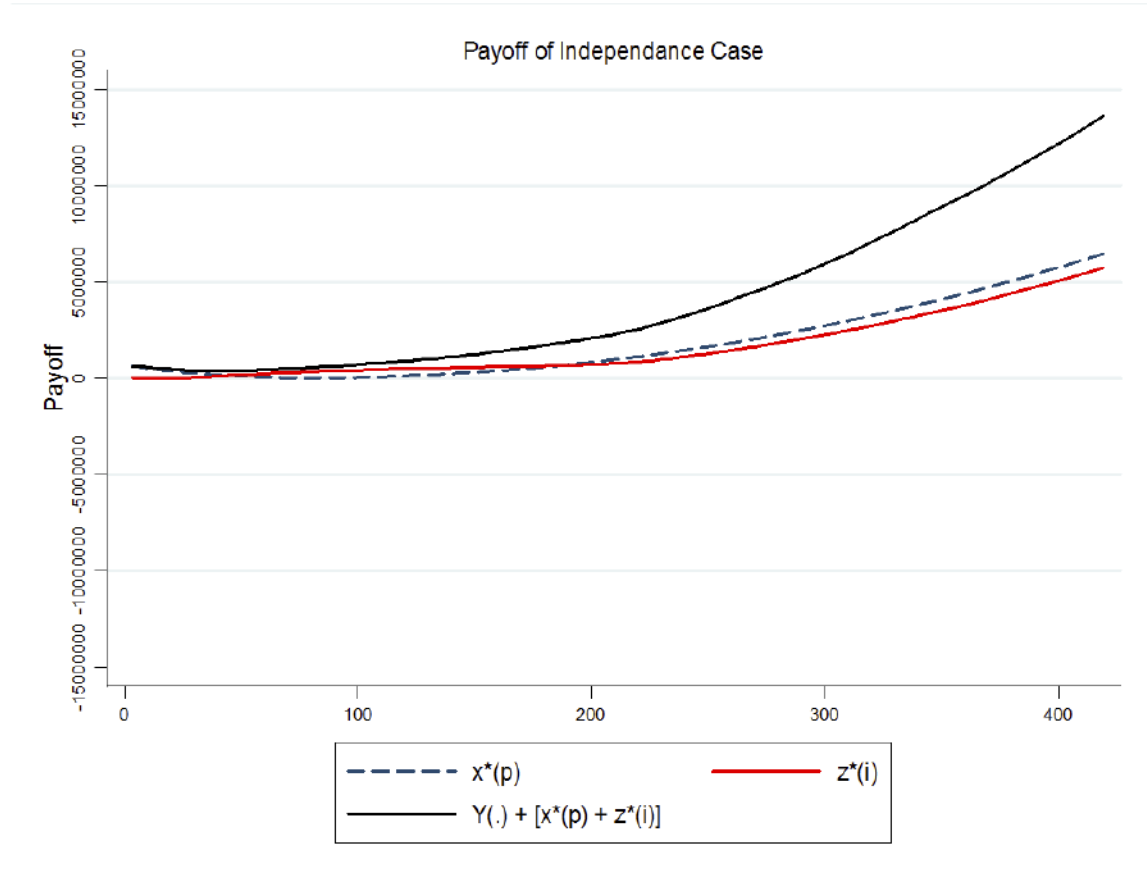

Figure 3: : Hedging Strategy for an LSE that maximizes the expected payoff with VaR constraint. Black line represents the hedging position; dashed line represents the payoff linear in price, and the red line exhibit the weather payoff.

Figure 4 shows the comparison of different possibilities, which are: without-hedge, hedging using $x^*(p)$ following Oum and Oren model, and our proposal using $[x'^*(p) + z'^*(\iota)]$, assumption.
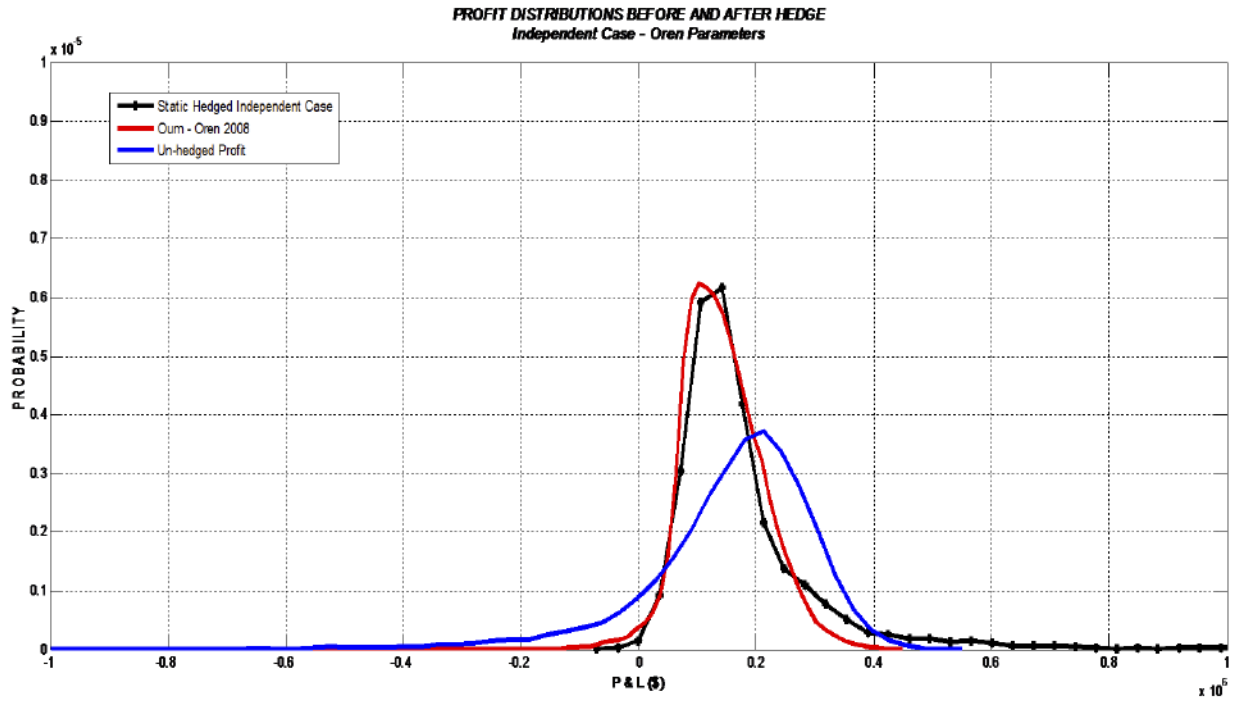
Figure 4: Profit distributions under three cases: without-hedge, Oum and Oren results and our proposal $[x'^*(p) + z'^*(\iota)]$.

Table 1 shows the percentiles for the cases shown in the Figure 4.

Table 1: Percentiles when fewer than three cases occur: without-hedge, Oum and Oren results and our proposal $[x'^*(p) + z'^*(\iota)]$ for independence assumption.

Percentiles			
	Without Hedging	**Oum-Oren Case**	**Independence Case**
1%	-518395	-48111.5	27427.5
5%	-158000	46809	57007
10%	-22051	58726	76734.5
25%	100755	84408	106565
50%	180310	126775	144225
75%	232070	174930	203805
90%	272505	219820	327205
95%	295805	248770	442905
99%	340210	303990	778085
Mean	140582.8	126351.9	182955.1
Std. Dev	162090.6	69343.86	76437.6
Skewness	-2.990513	-0.1061136	4.653612
Kurtosis	17.11238	4.751921	43.2694

4 Conclusions

Transfer of climatic risk exposure to capital markets allows transforming of non-tradable risk into financial assets which are, of course, tradable. Using forward contracts over weather offers to agents the chance to hedge their volumetric risk exposure in electric power markets. While the optimal electric power portfolio is an open problem in stating specific conditions to define the payoff structure of portfolios according to the agents' exposure, this paper presents closed-form results that permit the second claim to complete the market.

This paper develops along the lines traced in a benchmark article by Oum and Oren and put forward the issue of determining the optimal derivative pay-off written on both electricity price and a weather-linked index. This latter aims at improving the performance of the hedging claim due to the link between demanded volume and weather-linked index. Operational results are derived under the assumption of 1) statistical independence between price and the index and 2) Gaussian distribution of the underlying variables. We developed the optimization problem of portfolios composed of two claims, price and weather, according factors featured in electric power markets such as price volatility, price spikes, and climatic conditions that influence quantity volatility. Our results arose due to the inclusion of the weather variable, and the hedging position was improved by minimizing the risk and increasing mean according to positive correlation among price, quantity, and the weather variable. For the electric power market, wholesale spot price and quantity are volatile, and the latter is correlated with weather conditions. Results confirm that the weather payoff allows adjustment of hedge strategy with the price payoff in order to hedge the double exposure of the agents. Table 1 shows statistics of all of the cases and the experiment shows the gain of the proposed strategy over the best performing claim derived by Oum and Oren. Limiting the problem using a VaR-constrained solution permits to address the solution against the non-linearity condition of the hedging strategy. The hedging portfolio is solved using the price and weather payoff functions that represent the payoff of electric power derivatives and the payoff of the forward weather-related index, solving those payoffs we obtain a hedging portfolio in realistic conditions.

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Appendix

Proof of Theorem 1

Proof. i. $(x^*(p), z^*(\iota))$ is the optimal solution to (4) and is on the efficient frontier of $(E - VaR_\gamma)$ plane. Then considering the alternative $(x^*(p), z^*(\iota)) \in X(p), Z(\iota)$ that reduce the variance without reducing the mean of the $Y(x(p), z(\iota))$ distribution, then $\mu \geq \mu^*$ where $\mu = E[Y(x(p), z(\iota))]$ and $\mu^* = E[Y(x^*(p), z^*(\iota))]$ and $\sigma^2 < \sigma^{*2}$ where $\sigma^2 = V[Y(x(p), z(\iota))]$ and $\sigma^{*2} = V[Y(x^*(p), z^*(\iota))]$ then $\eta(\mu, \sigma, \gamma)$ which is increasing in σ and non-increasing in μ.

$$VaR_\gamma(Y(x(p), z(\iota))) = \eta(\mu, \sigma, \gamma)$$
$$\leq \eta(\mu^*, \sigma, \gamma) < \eta(\mu^*, \sigma^*, \gamma)$$
$$= VaR_\gamma(Y(x^*(p), z^*(\iota)))$$

Thus, the statement shows before contradicts the assumption that $(x^*(p), z^*(\iota))$ is on the efficient frontier in the $(E - VaR_\gamma)$ plane. This implies that for a fixed γ a feasible perturbation on $(x^*(p), z^*(\iota))$ that solves (4) cannot reduce the variance of the $Y(x(p), z(\iota))$ distribution without increasing the mean. Hence, $(x^*(p), z^*(\iota))$ is also on the efficient frontier in the $(E - V)$ plane.

ii. Let $Y(x(p), z(i))$ be an electric power portfolio on the efficient frontier in $(E - V)$ space; the equation $(E = aV + c)$ defines a straight line for any constant c.

Thus, maximizing $E\{Y(x(p), z(\iota))\} - aV\{Y(x(p), z(\iota))\}$ is equivalent to maximizing $(E - aV)$. Then, any $Y(x(p), z(\iota))$ maximizing $E\{Y(x(p), z(\iota))\} - aV\{Y(x(p), z(\iota))\}$ must be on the efficient frontier in $(E - V)$ space. This same $Y(x(p), z(\iota))$ must clearly also be on the efficient frontier in $(E - \sigma)$ space, due to any portfolio $Y'(x(p), z(\iota))$ with the same or equal expected payoff and smaller variance, having smaller standard deviation: if $Y(x(p), z(\iota))$ has expected profit μ_1 and standard deviation μ_1. Whether there is a portfolio with expected profit and VaR, say μ_2, and VaR_2 such that $\mu_1 = \mu_2$ and $VaR_2 < VaR_1$, thus $-(\mu_2, \sigma_2, \gamma) - \mu_2 < -(\mu_1, \sigma_1, \gamma) - \mu_1$ and hence we have $\eta(\mu_2, \sigma_2, \gamma) < \eta(\mu_1, \sigma_1, \gamma)$ by the which the monotonicity of η in σ implies $\sigma_2 < \sigma_1$. Which is impossible since $Y(x(p), z(\iota))$ was assumed to be on the $E - \sigma$ frontier. Then, $Y(x(p), z(\iota))$ be on the left border of the feasible set in $(E - VaR)$ space.

Sharpe (2000) establishes that taking in account the linear constraints, the efficient frontier in $(E - \sigma)$ space is concave. Furthermore, if for any portfolio ι we have $(E_i, \sigma_i), (E_{i+1}, \sigma_{i+1})$ and (E_{i+2}, σ_{i+2}) are on the efficient frontier and $E_{i+2} = \delta E_i + (1 - \delta) E_{i+1}$ for some δ, with $0 < \delta < 1$, then $\sigma_{i+2} \leq \delta \sigma_i + (1 - \delta) \sigma_{i+1}$. We can see that the fronties in $(E - V)$ space is also concave. That is, for the same portfolios we show $\sigma_{i+2}^2 \leq \delta \sigma_i^2 + (1 - \delta) \sigma_{i+1}^2$ then

$$\sigma_{i+2}^2 \leq \delta^2 \sigma_i^2 + (1 - \delta)^2 \sigma_{i+1}^2 + 2\delta(1 - \delta) \sigma_i \sigma_{i+1}$$

Hence, $\sigma_{i+2}^2 - [\delta\sigma_i^2 + (1-\delta)\sigma_{i+1}^2] \leq \delta^2\sigma_i^2 + (1-\delta)^2\sigma_{i+1}^2 + 2\delta(1-\delta)\sigma_i\sigma_{i+1} - [\delta\sigma_i^2 + (1-\delta)\sigma_{i+1}^2] = (\delta^2 - \delta)(\sigma_i - \sigma_{i+1})^2 < 0$

Therefore, $\sigma_{i+2}^2 < \delta\sigma_i^2 + (1-\delta)\sigma_{i+1}^2$ from the concavity of efficient frontier in $(E-V)$ space, we can see that if $Y(x(p), z(i))$ is on efficient frontier in $(E-V)$ space, there will be a straight-line tangent to the frontier curve at $Y(x(p), z(i))$. Choosing a as the slope of this line, and maximizing $(E-aV)$ will result in the $(E-V)$ of the portfolio $Y(x(p), z(i))$

$\square$

Proof of Proposition 1

Proof. Let $a_2 > a_1 > 0$ and specify that $Y(x^{a_i} + z^{a_i}) = Y_i \quad for \, i = 1, 2$, then

$$E(Y_1) - a_1 var(Y_1) \geq E[Y_2] - a_1 var(Y_2)$$

$$E(Y_2) - a_2 var(Y_2) \geq E[Y_1] - a_2 var(Y_1)$$

Adding the last two expressions gives

$$(a_2 - a_1) var(Y_1) \geq (a_2 - a_1) var(Y_2)$$

Then $var(Y_1) \geq var(Y_2)$

We can hold that $E[Y_1] - E[Y_2] \geq a_1(var(Y_1) - var < (Y_2)) \geq 0$

$\square$

Proof of Proposition 2

Proof. The Langrangian function for the constrained optimal problem is given by,

$$\partial(x,z) = \iint_{R^2} U(Y|p,\iota) f_{p,\iota}(p,\iota) dp d\iota - \lambda_x \int_{R^2} x(p) g_x(p) dp - \lambda_x \int_{R^2} z(\iota) g_z(\iota) d\iota$$

$$\overrightarrow{grad}(L(x,z)) = \vec{0}$$

With the Lagrange multipliers λ_x, λ_z and the marginal density functions $f_x(p)$ of p and $f_z(\iota)$ of ι under P, by differentiation of $L(x(p))$ with respect to $x(p)$ and $L(z(\iota))$ with respect to $z(\iota)$ results in

$$\frac{\partial L}{\partial x} = E\left[\frac{\partial Y}{\partial x} U'(Y)|p\right] f_x(p) - \lambda_x g_x(p) = 0 \tag{1}$$

$$\frac{\partial L}{\partial z} = E\left[\frac{\partial Y}{\partial z} U'(Y)|\iota\right] f_z(\iota) - \lambda_z g_z(\iota) = 0$$

By the Euler equation from (1) and substituting $\frac{\partial Y}{\partial x} = 1$ and $\frac{\partial Y}{\partial z} = 1$ from (1) yields the first order conditions for the optimal solutions $x^*(p)$ and $z^*(\iota)$ as follows:

$$E\left[U'(Y(p,q,x^*(p),z^*(\iota)))|p\right] = \lambda_x^* \frac{g_x(p)}{f_x(p)}$$

$$E\left[U'(Y(p,q,x^*(p),z^*(\iota)))|\iota\right] = \lambda_z^* \frac{g_z(\iota)}{f_z(\iota)}$$

For an agent who maximizes mean-variance expected utility of profit,

$$U(Y) = Y - \frac{1}{2}a(Y^* - E[Y^*]^2)$$

Then, by substituting $U' = (1 - aY^*)$, the optimal condition is given by:

$$1 - aE\left[Y^*|p\right] = \lambda_x^* \frac{g_x(p)}{f_x(p)}$$

$$1 - aE\left[Y^*|\iota\right] = \lambda_z^* \frac{g_z(\iota)}{f_z(\iota)}$$

Equivalently,

$$f_x(p) - aE\left[Y^*|p\right]f_x(p) = \lambda_x^* g_x(p)$$

$$f_z(\iota) - aE\left[Y^*|\iota\right]f_z(\iota) = \lambda_z^* g_z(\iota)$$

Integrating both sides with respect to p and ι from $-\infty$ to ∞, we obtain $\lambda_x^* = 1 - aE[Y^*]$ and $\lambda_z^* = 1 - aE[Y^*]$ by substituting λ_x^*, λ_z^* and $Y^* = y(p,q) + x^*(p) + z^*(\iota)$ gives,

$$f_x(p) - a\left(E[Y^*|p] + E[x^*|p] + E[z^*|p]\right)f_x(p) = [1 - aE[Y^*]]g_x(p) - a\left(E[x^*] + E[z^*]\right)g_x(p)$$

$$f_z(\iota) - a\left(E[Y^*|\iota] + E[x^*|\iota] + E[z^*|\iota]\right)f_x(\iota) = [1 - aE[Y^*]]g_z(\iota) - a\left(E[x^*] + E[z^*]\right)g_z(\iota)$$

Then,

$$f_x(p) - a\left(E[y(p,q)|p] + x^*(p) + E[z^*|p]\right)f_x(p) = (1 - aE[y(p,q)] - a\left(E[x^*] + E[z^*]\right))g_x(p)$$

$$f_z(\iota) - a\left(E[y(p,q)|\iota] + z^*(\iota) + E[x^*|\iota]\right)f_z(\iota) = ([1 - aE[y(p,q)]] - a(E[x^*] + E[z^*]))g_z(\iota)$$

By rearranging we obtain:

$$x^* = \frac{1}{a} - E\left[y(p,q)|p\right] - E[z^*|p] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_x(p)}{f_x(p)} + (E[x^*] + E[z^*])\frac{g_x(p)}{f_x(p)}$$

$$z^* = \frac{1}{a} - E\left[y(p,q)|\iota\right] - E[x^*|\iota] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)} + (E[x^*] + E[z^*])\frac{g_z(\iota)}{f_z(\iota)}$$

23

If p and ι are uncorrelated then we can establish that:

$$E[z^*|p] = E[z^*]$$
$$E[x^*|\iota] = E[x^*]$$

Finally we have:

$$x^* = \left[\frac{1}{a} - E[y(p,q)|p] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_x(p)}{f_x(p)}\right] + E[x^*]\frac{g_x(p)}{f_x(p)} + E[z^*]\left[\frac{g_x(p)}{f_x(p)} - 1\right]$$

$$z^* = \left[\frac{1}{a} - E[y(p,q)|\iota] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)}\right] + E[z^*]\frac{g_z(\iota)}{f_z(\iota)} + E[x^*]\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]$$

In order to obtain the final formula for the optimal payoff function under mean-variance utility the next system of equations could be utilized:

$$x^* = b_1(p) + a_{11}(p)E[x^*] + a_{12}(p)E[z^*]$$
$$z^* = b_2(\iota) + a_{21}(\iota)E[x^*] + a_{22}(i)E[z^*]$$

We take expectation under Q

$$0 = E^Q[b_1(p)] + E^Q[a_{11}(p)]E[x^*] + E^Q[a_{12}(p)]E[z^*] \tag{2}$$

$$0 = E^Q[b_2(\iota)] + E^Q[a_{21}(\iota)]E[x^*] + E^Q[a_{22}(\iota)]E[z^*] \tag{3}$$

And subtract Eq.(3)*$E^Q[a_{12}(p)]$ from Eq.(2)*$E^Q[a_{22}(\iota)]$

$$0 = E^Q[b_1(p)]E^Q[a_{22}(\iota)] - E^Q[b_2(\iota)]E^Q[a_{12}(p)]$$
$$+ \left[E^Q[a_{11}(p)]E^Q[a_{22}(\iota)] - E^Q[a_{21}(\iota)]E^Q[a_{12}(p)]\right]E[x^*]$$

Where,

$$E[x^*] = \frac{E^Q[b_1(p)]E^Q[a_{22}(\iota)] - E^Q[b_2(\iota)]E^Q[a_{12}(p)]}{E^Q[a_{21}(\iota)]E^Q[a_{12}(p)] - E^Q[a_{11}(p)]E^Q[a_{22}(\iota)]} \tag{4}$$

By substituting $E[x^*]$ in Eq. (3) we obtain,

$$E[z^*] = -\frac{E^Q[b_2(\iota)] + E^Q[a_{21}(\iota)]\frac{E^Q[b_1(p)]E^Q[a_{22}(\iota)] - E^Q[b_2(\iota)]E^Q[a_{12}(p)]}{E^Q[a_{21}(\iota)]E^Q[a_{12}(p)] - E^Q[a_{11}(p)]E^Q[a_{22}(\iota)]}}{E^Q[a_{22}(\iota)]} \tag{5}$$

Moreover, Eq.(4) and Eq.(5) could be expressed as follows:

24

$$E[x^*] = \frac{\left[\frac{1}{a} - E\left[y(p,q)|p\right] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_x(p)}{f_x(p)}\right]\frac{g_z(\iota)}{f_z(\iota)}}{\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right] - \left[\frac{g_x(p)}{f_x(p)}\right]\left[\frac{g_z(\iota)}{f_z(\iota)}\right]}$$

$$- \frac{\left[\frac{1}{a} - E\left[y(p,q)|\iota\right] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)}\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right]}{\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right] - \left[\frac{g_x(p)}{f_x(p)}\right]\left[\frac{g_z(\iota)}{f_z(\iota)}\right]} \tag{6}$$

$$E[z^*] = \frac{\left[\frac{1}{a} - E\left[y(p,q)|\iota\right] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)}\right] + \left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]}{\left[\frac{g_z(\iota)}{f_z(\iota)}\right]\left(\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]\left[\frac{g_x(p)}{f_x(p)} - 1\right] - \left[\frac{g_x(p)}{f_x(p)}\right]\left[\frac{g_z(\iota)}{f_z(\iota)}\right]\right)}$$

$$\left(\left[\frac{1}{a} - E\left[y(p,q)|p\right] + \frac{\left(E[y(p,q)] - \frac{1}{a}\right)g_x(p)}{f_x(p)}\right]E^Q\left[\frac{g_z(\iota)}{f_z(\iota)}\right]\right.$$

$$\left. - \left[\frac{1}{a} - E\left[y(p,q)|\iota\right] + \frac{\left(E[y(p,q)] - \frac{1}{a}\right)g_z(\iota)}{f_z(\iota)}\right]\left[\frac{g_x(\iota)}{f_x(\iota)} - 1\right]\right). \tag{7}$$

For maximizing mean-variance expected utility the optimal solution $x^*(p)$ and $z^*(\iota)$ to problem (4) is given as:

$$x^* = \left[\frac{1}{a} - E\left[y(p,q)|p\right] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_x(p)}{f_x(p)}\right] + E[x^*]\left[\frac{g_x(p)}{f_x(p)}\right] + E[z^*]\left[\frac{g_x(p)}{f_x(p)} - 1\right]$$

$$z^* = \left[\frac{1}{a} - E\left[y(p,q)|p\right] + \left(E[y(p,q)] - \frac{1}{a}\right)\frac{g_z(\iota)}{f_z(\iota)}\right] + E[z^*]\left[\frac{g_z(\iota)}{f_z(\iota)}\right] + E[x^*]\left[\frac{g_z(\iota)}{f_z(\iota)} - 1\right]$$

$\square$

Proof of Proposition 3

Proof. The density function of an n-dimensional normal vector, whose mean is μ and variance-covariance matrix is Σ, is given by:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma(\mathbf{x}-\mu)}$$

We consider a 2-dimensional normal vector (u,v), but, the density function of u knowing v is equal to the joint density of (u,v) divided by the marginal density function of v,

$$f_{u|v}(u|v) = \frac{f_{uv}(u,v)}{f_v(v)}$$

In the case of ι, because (ι, q) are correlated, the variance-covariance matrix is:

$$\Sigma = \begin{bmatrix} \sigma_\iota^2 & \rho_{\iota,q}\sigma_\iota\sigma_q \\ \rho_{\iota,q}\sigma_\iota\sigma_q & \sigma_q^2 \end{bmatrix}$$

The determinant is:

$$\det\Sigma = \sigma_\iota^2\sigma_q^2 - \rho_{\iota,q}^2\sigma_\iota^2\sigma_q^2 = \sigma_\iota^2\sigma_q^2(1-\rho_{\iota,q}^2)$$

Hence, the inverse of variance-covariance matrix is given by:

$$\Sigma^{-1} = \frac{1}{(1-\rho_{\iota,q}^2)\sigma_\iota^2\sigma_q^2}\begin{bmatrix} \sigma_q^2 & -\rho_{\iota,q}\sigma_\iota\sigma_q \\ -\rho_{\iota,q}\sigma_\iota\sigma_q & \sigma_\iota^2 \end{bmatrix}$$

The joint density of $(\ln(p); q)$ is defined by:

$$f(\mathbf{x}) = \frac{1}{2\pi}\frac{1}{\sigma_\iota\sigma_q\sqrt{1-\rho_{\iota,q}^2}}e^{-\frac{1}{2}\frac{1}{\sigma_\iota^2\sigma_q^2(1-\rho_{i,q}^2)}M}$$

Where M is can be formulated such as:

$$M = \begin{bmatrix} \log\iota - \mu_\iota \\ q - \mu_q \end{bmatrix}'\begin{bmatrix} \sigma_q^2 & -\rho_{\iota,q}\sigma_\iota\sigma_q \\ -\rho_{\iota,q}\sigma_\iota\sigma_q & \sigma_\iota^2 \end{bmatrix}\begin{bmatrix} \log\iota - \mu_\iota \\ q - \mu_q \end{bmatrix}$$

We also have the marginal density of $\ln(\iota)$:

$$f_{(\ln\iota)}(\ln\iota) = \frac{1}{2\pi}\frac{1}{\sigma_\iota}e^{-\frac{1}{2}\left(\frac{\log\iota-\mu_\iota}{\sigma_\iota}\right)^2}$$

Then we deduce the density function of q knowing $\ln(\iota)$:

$$f_{q|\ln(\iota)}(q|\ln(\iota)) = \frac{\frac{1}{2\pi}\frac{1}{\sigma_\iota\sigma_q\sqrt{1-\rho_{\iota,q}^2}}e^{-\frac{1}{2}\frac{1}{\sigma_\iota^2\sigma_q^2(1-\rho_{\iota,q}^2)}M}}{\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_\iota}e^{-\frac{1}{2}\left(\frac{\log\iota-\mu_\iota}{\sigma_\iota}\right)^2}}$$

Therefore

$$f_{q|\ln(p)}(q|\ln(p)) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_q\sqrt{1-\rho_{\iota,q}^2}}e^{-\frac{1}{2}N}$$

Where N is defined by:

$$\begin{aligned} N &= \frac{M}{\sigma_\iota^2\sigma_q^2(1-\rho_{\iota,q}^2)} - \left(\frac{\log\iota-\mu_\iota}{\sigma_\iota}\right)^2 \\ &= \frac{1}{\sigma_q^2(1-\rho_{\iota,q}^2)}\left[q-\left(\mu_q+\rho_{\iota,q}\frac{\sigma_q}{\sigma_\iota}(\ln\iota-\mu_\iota)\right)\right]^2 \end{aligned}$$

We finally obtain:

$$f_q^{\{\ln \iota\}}(q) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_q \sqrt{1 - \rho_{\iota,q}^2}} e^{-\frac{1}{2} \frac{\left[q - \left(\mu_q + \rho_{\iota,q} \frac{\sigma_q}{\sigma_\iota}(\ln \iota - \mu_\iota)\right)\right]^2}{\sigma_q^2(1 - \rho_{\iota,q}^2)}}$$

In other words,

$$q_{|\iota} \sim N\left(\mu_q + \rho_{\iota,q} \frac{\sigma_q}{\sigma_\iota}(\ln \iota - \mu_\iota), \sigma_q^2(1 - \rho_{\iota,q}^2)\right)$$

$\square$