RIEMANNIAN WAVEFIELD EXTRAPOLATION

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The city and date(D,M,Y):

To Miguel Angel and Isabella.
Everything has been done for you

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Pspspsps.....To you too Gor!!!

Contents

Chapter 1

Introduction

1.1 Seismic Imaging

Exploration seismology refers to all the research activities that are necessary to obtain a deep knowledge of the structures in the upper crust of the earth. This knowledge is important in several areas, such as shallow depths, ground water and enviromental studies. In the area of oil exploration, the study of hydrocarbon reservoirs is necessary to understand the intrinsic properties of the reservoirs and to properly imaging its location, size and fractures among other features. To this end, seismic imaging concerns to the methodologies used to imaging the structures in the subsurface; roughly speaking, it consists of *listening the earth's interior to look into it*.
The seismic experiment consists of an energy source, which generates waves that propagate into the subsurface, and after reflections due to the heterogeneities of the medium, these "sounds" are collected by geophones making a set of data which contain the information of the heterogeneity of the medium.

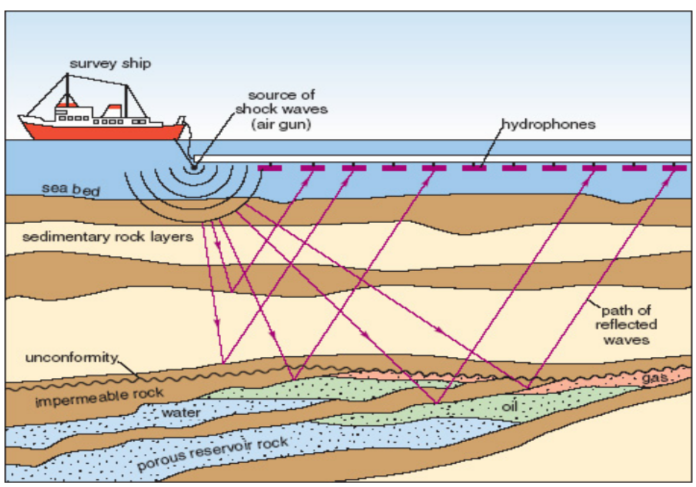


Figure 1.1: The seismic experiment

The reflections are collected on the receivers, which respond to ground motion or pressure, in the form of a seismogram which has to be analized to obtain an image. From figure (1.1), we see that just with the signals recorded the researchers must produce an

accurate image of the earth's interior which accounts for all the features of it. The first step to seismic imaging is wave propagation modeling, but most important than it, is the characterization of the medium on which wave propagation takes place.
The continuum media on which seismic waves propagate is the earth; the earth is at least a viscoelastic medium, in which absorption losses give rise to attenuation and dispersion effects. Moreover, the fractal nature of geologic processes tells that fractures can be expected on all length scales. This posses a constrain on the theory of wave propagation used in seismic imaging since the propagation medium is complex, i.e, the position of layers is not necessarily horizontal, there are fractures and faults, lateral variations of the medium parameters (speed of propagation, density,etc), anisotropy, among others. From this, we get to the approximations of acoustic and elastic wave propagation; the latter more accurate to the physical reality and mathematically complex.
In the particular case of seismic exploration in Colombia, it is being made in geological complex scenarios where the irregular sub-structures and data acquisition surfaces, and the presence of anisotropy and vertical variations of velocity require the construction of general velocity models in depth which can be applied in simulations of wave propagation to get seismic images. This problem justifies the research made in this thesis which is part of the research project titled: *Migración sísmica pre-apilado en profundidad por extrapolación de campos de onda utilizando computación de alto desempeño para datos masivos en zonas complejas*; which is held by Ecopetrol, Colciencias, Instituto Tecnologico Metropolitano (ITM), Universidad de Antioquia (U de A), Universidad Industrial de Santander (UIS) and Universidad de Pamplona, in Colombia.

1.2 Wave propagation and recursive wavefield extrapolation

1.2.1 Elastic wave propagation

Neglecting absorption, the equation which best describes the propagation of seismic waves in the earth is the elastic wave equation, which is framed in terms of tensor operators acting on vector quantities, see [19].
Wave propagation on continuous medium is governed by intrinsic and particular properties of the medium which is being perturbed, these properties include mass density, stiffness, fractures, porosity, anisotropy, permeability, among others. When a medium is perturbed, all of these properties allow the propagation of elastic or acoustic waves through it, reflection, difraction, refraction and scattering phenomena occurs due to all these properties. In particular, the azimuthal anisotropy is associated with fractures and the shear-wave splitting associated with shear-wave propagation in elastic and fractured media, is diagnostic of the fracture orientation and intensity. The elastic wave propagation theory is fundamental in the treatment of anisotropy, even when only P-waves are considered.
For a linearized theory and in presence of small deformations, Hooke's law is valid, which in general and for a Cartesian coordinate system, reads:

$$\sigma_{ij} = \sum_{k,l} C_{ijkl}\varepsilon_{kl},$$

where:

$$\begin{aligned}
\sigma_{ij} &: \text{ is the stress tensor,} \\
C_{ijkl} &: \text{ is the stiffness tensor,} \\
\varepsilon_{kl} &: \text{ is the strain tensor,} \\
\varepsilon_{kl} &= \frac{1}{2}(\partial_k u^l + \partial_l u^k), \\
u^i(x) &: \text{ Components of the displacement vector field.}
\end{aligned}$$

We recognize that the strain tensor is symmetric, and other physical considerations lead to the symmetry of the stress tensor. These symmetries imply that the stiffness tensor is symmetric under interchanges of i with j and k with l. Energy considerations imply symmetry under interchange of (ij) with (kl), then for the stiffness tensor we only have 21 independent elastic coefficients.
From the balance of momentum, we obtain the Cauchy equations of motion

$$\rho(\vec{x})\frac{\partial^2 u_i}{\partial t^2} = \sum_j \frac{\partial}{\partial x_j}\sigma_{ij},$$

which combined with Hook's law leads to

$$\rho(\vec{x})\frac{\partial^2 u_i}{\partial t^2} = \sum_j \frac{\partial}{\partial x_j}\sum_{k,l} C_{ijkl}\varepsilon_{kl},$$

in a homogeneous medium, or one where the spatial variation of the medium parameters is slow when compared with the wavelenght of propagation we have

$$\rho\frac{\partial^2 \vec{u}}{\partial t^2} = C_{ijkl}\sum_j \frac{\partial}{\partial x_j}\frac{1}{2}\left(\partial_k u^l + \partial_l u^k\right).$$

For Isotropic medium we have

$$\sigma_{ij} = \lambda\delta_{ij}\sum_k \varepsilon_{kk} + 2\mu\varepsilon_{ij},$$

and then the equation of motion is

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda+\mu)[\nabla(\nabla\cdot\vec{u})] + \mu\nabla^2\vec{u},$$

since, in general curvilinear coordinates, we have

$$\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u}),$$

and defining

$$\begin{aligned}
\phi &= \nabla\cdot\vec{u}, \\
\psi &= \nabla\times\vec{u}
\end{aligned}$$

we get

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu)\nabla\phi - \mu\nabla \times \psi,$$

where μ and λ are the Lamé parameters. For propagation of P-waves, in the previous equation we take the divergence operator to get

$$\nabla^2\phi - \frac{1}{v_p^2}\frac{\partial^2\phi}{\partial t^2} = 0,$$

where

$$v_p = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}};$$

and for S-waves, take the curl and obtain

$$\nabla^2\psi - \frac{1}{v_s^2}\frac{\partial^2\psi}{\partial t^2} = 0,$$

where

$$v_s = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}.$$

Note that the previous equations are wave equations for acoustic wavefields. The propagation of plane waves in an elastic medium is governed by the Kelvin-Christoffel equation, this equation appears when we consider a plane wave trial solution $\vec{u} = Ue^{i\omega(\vec{s}\cdot\vec{x}-t)}\vec{d}$, where $\vec{d}$ is a unit vector with the direction of displacement and $\vec{s}$ is the slowness vector. This trial solution is tested in the equation

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = C_{ijkl}\sum_j \frac{\partial}{\partial x_j}\frac{1}{2}\left(\partial_k u^l + \partial_l u^k\right),$$

to get

$$\begin{aligned} \left(\Gamma^0_{ik}(\vec{n}) - \nu^2\rho\delta_{ik}\right)d_k &= 0, \\ \left(\Gamma_{ik}(\vec{s}) - \rho\delta_{ik}\right)d_k &= 0, \end{aligned}$$

where $\Gamma^0_{ik}(\vec{n}) = C_{ijkl}\hat{n}_j\hat{n}_l$ and $\Gamma_{ik}(\vec{s}) = C_{ijkl}s_js_l$, are the two forms of the Christoffel matrix and $\hat{n} = \nu\vec{s}$.
Note that this equations are pure eigenvalue problem, for the first equation, and that the eigenvalues are $\nu^2\rho$. Since Γ^0_{ik} is real, positive definite and symmetric for real $\hat{n}$, then all the eigenvalues are real and positive, giving real velocities corresponding to qP , $qS1$ and $qS2$ modes. The eigenvectors are the polarizations $\vec{d}$, which are orthogonal for a given $\hat{n}$. The second form, is more convenient for wavefield extrapolation, since we may fix the horizontal slowness and solve for vertical slowness $s_z = s_3$. The eigenvalues here are the values of s_z, and the characteristic equation $det(\Gamma_{jl} - \rho\delta_{jl}) = 0$ has six roots for s_z. The reader is reffered to [3], [4], [5], [19], [43], [8], among others for more details.

1.2.2 Acoustic wave propagation

The acoustic scalar Helmholtz equation, or the time dependent scalar wave equation have been used in exploration seismology as the modeling equations for seismic imaging due to its relative easy solutions construction, in particular for the constant velocity case, i.e, when the velocity of propagation is constant and the amplitude of the propagating wave only midly varies as function of space.
The acoustic modeling can be constructed as follows, see [18]. The strain tensor can be decomposed into normal and shear components as

$$\sigma_{ij} = (\lambda + \frac{2}{3}\mu)tr(\varepsilon)\delta_{ik} + 2\mu(\varepsilon_{ik} - \frac{1}{3}tr(\varepsilon)\delta_{ik}),$$

where the first term is hydrostatic compression and the second is shear strain. The regime of linear acoustic occurs when we assume that the continuum does not support shear stress, i.e. $\mu = 0$. Then we have $\sigma_{ik} = -p\delta_{ik}$ where p is the pressure field; since $tr(\varepsilon) = \nabla \cdot u$, where u is a small displacement vector field, we have that the system of equations for an acoustic wave propagation is

$$\begin{aligned} \rho\partial_t v &= -\nabla p + f, \\ -\partial_t p &= \kappa \nabla \cdot v, \end{aligned}$$

where κ is the modulus of compression and f represents external energy input and is assume to be irrotational. κ and ρ are material parameters that vary with space and are related through the wave velocity, defined as $c = \sqrt{\frac{\kappa}{\rho}}$. Since $\nabla \times v$ is constant in time and zero, since we assume the medium initially at rest, there exists a potential function $v = -\nabla\phi$, and after eliminating $\nabla \cdot v$, we get to the known equation

$$\left[\frac{1}{c^2(x)}\partial_t^2 - \Delta\right] p(x,t) = f(x,t),$$

which after Fourier transforming in the time variable leads to the scalar Helmholtz wave equation

$$\left[\Delta + \frac{\omega^2}{c^2(x)}\right] \Psi(x,\omega) = 0.$$

1.2.3 Wavefield extrapolation

Many modeling and migration algorithms are based on the concept of extrapolation, in which a wavefield is marched along a specific direction in small steps, which are time steps or depth steps; using a solution of a particular form of the wave equation.
Extrapolation in time is a method which has been used in forward modeling, and is the basis of techniques such as finite difference modeling, pseudospectral modeling, reverse time migration, among others. This extrapolation techniques use the full two-way equation, and can handle multipathing, but this can be a disadvantage when the velocity model is not well known. Extrapolation in depth, is based on the concept of one-way wave equation (OWWE) and uses a decomposition of the wave equation in downgoing

and upgoing waves. The reader is reffered to [6], [7],[8], [3], [18], [19], [43], for a comprehensive treatment of wave modeling and wavefield extrapolation methods.
The one-way wave equation is obtained as follows: Assume that the velocity model is constant for a background medium, then the Helmholtz equation can be factorized as

$$\left(\frac{\partial}{\partial z} + iK_z\right)\left(\frac{\partial}{\partial z} - iK_z\right)\Phi = 0,$$

where the z direction is the direction of propagation and,

$$\begin{aligned} \Phi &= \mathcal{F}(\Psi), \quad \text{Fourier transform} \\ K_z &= \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}. \end{aligned}$$

The one-way wave equations are then:

$$\begin{aligned} \left(\frac{\partial}{\partial z} + iK_z\right)\Phi &= 0, \\ \left(\frac{\partial}{\partial z} - iK_z\right)\Phi &= 0, \end{aligned}$$

one for downgoing waves and the other for upgoing waves depending on the reference system chosen.
This decomposition is achieved since the velocity does not depend on the spatial variables, i.e. a medium with constant velocity of propagation.
In terms of the original wavefield Ψ a strategy that has been successfully used in seismic extrapolation algorithms is to write the one-way wave equations for the wavefield Ψ as

$$\begin{aligned} \left(\frac{\partial}{\partial z} + i\Gamma\right)\Psi &= 0, \\ \left(\frac{\partial}{\partial z} - i\Gamma\right)\Psi &= 0, \end{aligned}$$

where the operator $\Gamma^2 = \frac{\omega^2}{c^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, is the square-root operator and has an explicit meaning since c is constant. The solution of the wave equation is then a linear combination of the solutions of the one-way wave equations.
In the case where $c \equiv c(x)$, many methodologies have been implemented to obtain a one-way wave equation, this means the search for the square-root operator which can factorize the full wave equation. This operator has a simultaneous dependence upon the spatial variables and its Fourier dual variables, say $\Gamma \equiv \Gamma(x, k_x)$, in the case of Fourier transforming the spatial variables. This issues lead us to the rich theory of *pseudodifferential operators*.

1.2.4 Pseudodifferential operators

Pseudodifferential operators are a generalization of differential operators. In this generalization the use of Fourier transform is essential to describe an operator that maps

Schwartz space onto itself. The basic idea arises when we consider a differential operator $P(x,D) = \sum_{|\alpha|} a_\alpha(x) D^\alpha$, to which is associated the principal symbol (polynomial) $p(x,\xi) = \sum_{|\alpha|} a_\alpha(x)(i\xi)^\alpha$, then we can rewritte the differential operator as

$$P(x,D)\phi = F^{-1}[p(x,\xi)F(\phi)],$$

where F and F^{-1} are the Fourier transform and its inverse operator respectively.
Nevertheless, theoricaly the use of symbols that are general than polynomial ones, and the description of differential equations with non constant coefficients, have given rise to pseudodifferential operators that are written in the above way but the symbols belong to a wider class of maps. Among all the properties of such operators are that its inverse is also a pseudodifferential operator, the space of pseudodifferential operators is closed under composition and product, and they are invariant under diffeomorphisms, so they can be defined on a manifold.

Since this is a rich mathematical theory which uses the functional analysis for applications to partial differential equations, see [10], [11], [12], [13], [14], [15], and it is going to be formally presented in Chapter 1, we devote some short lines to this topic in wavefield extrapolation.
As an example, the equation which describes generalized PSPI for the scalar wave equation in two dimensions is

$$\psi(x,z) = \frac{1}{2\pi}\int \alpha(x,k_x,0)\phi(k_x,z)e^{-ixk_x}dk_x,$$

where α is known as the symbol of the operator and is given by

$$\alpha(x,k_x,z = exp\left(iz\sqrt{\left(\frac{\omega^2}{c^2(x)} - k_x^2\right)}\right).$$

The implementation of pseudodifferential operators for the square root operator is not unique. We mention the adjoint form of the integral equation above, which is known as non-stationary phase shift method (NSPS); the anti-standard form known as the Generalized Phase-Shift-Plus-Interpolation method (GPSPI); the use of the Weyl operator which is at an intermediate form between standard and anti-standard forms. These operators in each of its forms have some different properties while being applied to extrapolation and modeling methods, but give good results for large depths steps.
The efficiency of the implementation of these operators requires that they should be approximated using a set of spatially invariant reference operators and an interpolation scheme.
Pseudodifferential operators have been applied to the analysis of phase-screen propagators, see [9].

1.3 Thesis Motivation

We mention some issues that motivate this work:

1. *As seen in figure (1.1), the topography of acquisition is irregular and the layers underneath are curved.*

2. *The wave equation models used so far, are stablished on a Euclidean space.*

3. *Migration by downward continuation imposes strong limitations on the dip of reflectors that can be imaged, see [22].*

4. *The extrapolation operators used in Cartesian coordinates, are of limited accuracy at large propagation angles and can not propagate turning waves, see [25], and do not account for the complexity of the geology, e.g. the irregular geometry of the sub-surface.*

5. *Cartesian coordinates for downward, tilted continuation, or along beams are a mathematical convenience that do not reflect a physical reality, see [22] and [23]; the continuum on which elastic waves are propagated, is not necessarily Cartesian.*

6. *When dealing with non-Cartesian scenarios, the numerical implementations are made in rectangular meshes, and then it is needed a conformal transformation of the physical domain (reality) onto the computational domain.*

7. *The complex geological structures in Colombia for seismic exploration requires a deeper analysis of elastic wave equation, to be applied in migration algorithms to obtain seismic images which are accurate to the real structures and reflect the physical phenomena of elastic waves propagating into the earth.*

We conclude from this aspects, that a formulation of elastic wave propagation in a coordinate system which is general than the Euclidean one, is needed. We call this formulation: *Riemannian wavefield Extrapolation*, which is the formulation of elastic waves propagating in a Riemannian space. This formalism is necessary since the layers in a complex stratified medium should be represented as curves in a plane or surfaces, that is a 2-dimensional or 3-dimensional manifold, and then geometric concepts such as curvature of a curve or surface arise in a fundamental way. More examples and explanations of this theory can be found on [33], [34]. This formulation must accounts for general symmetries (material symmetry), anisotropy parameters and allows to obtain decoupled solutions of the general Riemannian wavefield equation.
The primary objective of the thesis is to obtain an elastic Riemannian wave equation which describes wave propagation in a Riemannian manifold. This approach extends the work done in [22], [23], [25],[26]; on which a formulation of acoustic wave propagation on a Riemannian manifold was proposed as a pure eigenvalue equation without any physical consideration which allows to get to the proposed wave equation. Our approach also extends the work of [44] since we are considering heterogeneous continuum and general symmetries, not just the isotropic one, which is the case considered in that work. We will use the Lagrangian formalism of mechanics on a Riemannian manifold to this end.

In Chapter 2 we present basics aspects of the theory of Fourier Integral Operators and Pseudodifferential operators to be applied in the wave propagation phenomena context. This chapter is devoted to provide an overview of this aspects and contains no results derived from the research.
We consider the phenomena of acoustic wave propagation on a Riemannian manifold in Chapter 3; on which an acoustic Riemannian wavefield is propagatated. We present conformal transformations of Riemannian meshes to Euclidean ones, the Laplace-Beltrami operator, some of its properties and the Riemannian acoustic wave equation as an eigenvalue problem. For the two-way wavefield extrapolation we derive stability and dispersion conditions, based on the Von-Neumann method, and applied it to a finite difference scheme with two different topographic profiles. The results are provided on Section 3.5 showing that for FDTD implementations, the proposed wave equation does not offer any advantage in terms of computational cost and imaging.
Elasticity theory in Riemannian spaces provides a general frame on which one can describe the dynamics of the elastic bodies. In Chapter 4 we present the basic definitions and properties of a deformed elastic body in terms of Riemannian geometry and show and application of this formalism and the theory of pseudodifferential operators, in the isotropic case. In Section 4.4 we present the Lagrangian formalism on a Riemannian manifold which allows to formulate the dynamics of the elastic body in terms of the Lagrangian functional defined for it. The results are presented in Section 4.5, there we propose a Lagrangian density and derive a Noether's condition for an elastic Riemannian body with which we can identify the symmetries of the body, we also obtain the group of symmetries for the particular case of an orthotropic continuum and the constrains equations for the Euclidean case.
The elastic Riemannian wave equation is obtained in Chapter 5, we use the Lagrangian density proposed in the previous chapter, and in Section 5.2 we derive the Euler-Lagrange equations in the manifold, and consider two particular cases, the isotropic Riemannian and the Euclidean, to show the consistency of the equations obtained. In Section 5.3 we use the pseudodifferential operators on a manifold to obtain the desired elastic Riemannian OWWE, we calculate the symbol of the spatial and geometric part of the wave operator, which allows us to get the set of one-way wave equations; we also show that the symbol for the Euclidean isotropic case can be obtained from this general formalism.

Chapter 2

Elastic Wave Equation Imaging Via Tensor Downward Continuation

2.1 Introduction

Decoupling of the wave equation into propagation modes has proved to be an efficient tool for simulating wave propagation into the earth and has been used in global seismology and geophysical imaging since the work by Claerbout [8]. This decoupling which is achieved after factorizing the wave equation is known as "one-way wave equation" which then, allows to separate solutions of the wave equation into down-going and up-going propagating waves describing their propagation in a predetermined direction. Formely, they were not intended to describe wave amplitudes and this restricts the migration process since it is also needed to reconstruct the relative strength of the velocity heterogeneities and to estimate petrophysical parameters then, the one-way equation needed to be refined and one of the problems is the formulation of the square root operator in the case where the velocity depends on the spatial coordinates.
In the latter case, the theory of Fourier Integral Operators and particullary pseudodifferential operators is needed to properly achieve a formulation of the square root operator which decouples the wave equation. This theory has been widely used in the context of differential operators where they are used to formulate solutions of partial differential equations, the reader is remitted to [10], [11], [12], [13], [14], [15], for a deep study of this theory.

In the context of acoustic one-way wave equation, many authors have formulated a diagonalizing pseudodifferential operator which allows to decople the scalar wave equation when the velocity depends on the spatial coordinates. Sheng-Chang and Zai-Tian [16], proposed a high order formula of generalized screen propagator for one-way wave equation using the asymptotic expansion of single-square-root operator. This generalized screen propagator improves not only the calculation precision of conventional generalized screen propagator, but also the suitability of generalized screen propagator to the media with strong lateral-velocity variation, the results are better than those obtained with the generalized screen propagator despite of the increased computational cost. Stolk [17], investigated the damping term of the symbol of the square root operator in a heterogeneous acoustic media finding that incorrectly propagated singularities can be suppresed with this extra term, and the equation can describe singularities propagating along turning rays which are the rays along which the velocity component in the propagating direction changes sign. Op' Root [18] proposed a special form of the square root operator by a symmetrization of the operator using left and right quantizations, and together with a correct normalization of the diagonalizing operator, the one-way wave equations are obtained in the case the velocity has lateral and depth variations, in the high frecuency approximation to be implemented.

For the elastic case, we mention the work of Bale [19], on which the problem of shear-wave splitting is developed in a medium with Horizontal Transverse Isotropy symmetry (HTI) and the solutions come from the Kelvin-Christoffel equations after a high frecuency approximation. In this work, he obtains a one-way extrapolator by diagonalizing the fundamental elasticity matrix in a vertical direction for propagation in a cartesian modeling space. V. Brytik., M. De Hoop., and R. Van Der Hils [20], considered the general elastic wave equation to obtain a diagonal operator which decouples the equation, in this formalism they obtain an oscillatory integral representation for the solution operator of the equation in terms of the flux normalization and the full Green' s tensor; De Hoop and De Hoop [21], developed an operator approach to up/down decomposition for the elastic case in a medium with orthorhombic symmetry; in this work they could also make a decomposition into polarization states for the tensorial one-way equation obtained.
The chapter is presented as follows: First we present some basic definitions and properties of Fourier Integral Operators as given in [12], with an application to wave propagation; then the theory of pseudodifferential operators with the relevant propositions to be used in the case of wave propagation and some examples of this application which can be found in [12] and [13]. The last section brieftly presents the work found in [20], to show the use of the theory in the context of elastic wave propagation.

2.2 Fourier Integral Operators

Let X, Y be open sets in $\mathcal{R}^{\mathcal{N}_{\mathcal{X}}}$ and $\mathcal{R}^{\mathcal{N}_{\mathcal{Y}}}$, and consider the expression

$$Au(x) = \int e^{i\Phi(x,y,\theta)} a(x,y,\theta) u(y) dy d\theta, \tag{2.1}$$

where $u(y) \in C_0^\infty(Y)$, $x \in X$, Φ is a phase function on $X \times Y \times \mathcal{R}^n$ and $a(x,y,\theta) \in S^m_{\rho,\delta}(X \times Y \times \mathcal{R}^n)$ with $\rho > 0, \quad \delta < 1$; under these conditions, the integral

$$\langle Au, v\rangle = \int e^{i\Phi(x,y,\theta)} a(x,y,\theta) v(x) u(y) dx dy d\theta \tag{2.2}$$

is an oscillatory integral, that for fixed u, the expression (2.2) viewed as a functional of v, defines a distribution $Au \in \mathcal{D}'(X)$. Then, a linear operator

$$A : C_0^\infty(Y) \to \mathcal{D}'(X) \tag{2.3}$$

is defined.

Definition 2.2.1. *An operator A of the form (2.1) is called a Fourier Integral Operator (FIO) with phase function Φ.*

Now we will explain some terms involved in the previous definition.

Definition 2.2.2. *Let m, ρ and δ be real numbers such that $0 \leq \delta \leq 1, 0 \leq \rho \leq 1$. the class $S^m_{\rho,\delta}(X \times \mathcal{R}^N)$ consists of functions $a(x,\theta) \in C^\infty(X \times \mathcal{R}^N)$ such that for any multi-indices α, β and any compact set $K \subset X$ exists a constant $C_{\alpha,\beta,K}$ for which*

$$|\partial^\alpha_\theta \partial^\beta_x a(x,\theta)| \leq C_{\alpha,\beta,K} \langle\theta\rangle^{m-\rho|\alpha|-\delta|\beta|}, \tag{2.4}$$

where $x \in K$, $\theta \in \mathcal{R}^N$ and $\langle\theta\rangle = (1+|\theta|^2)^{\frac{1}{2}}$.

Definition 2.2.3. *A funtion $\Phi(x,\theta)$ is called a phase function if $\Phi \in C^\infty(X \times (\mathcal{R}^N \backslash 0))$, is real valued and positively homogeneous of degree 1 in θ and does not have critical points for $\theta \neq 0$ i.e. $\Phi'_{x,\theta}(x,\theta) \neq 0$ for $x \in X$ and $\theta \in \mathcal{R}^N \backslash 0$; ($\Phi'_{x,\theta}(x,\theta)$ is the gradient of $\Phi(x,\theta)$ with respect to x and θ).*

Now, if $a \in S^m_{\rho,\delta}(X \times \mathcal{R}^N)$ and Φ is a phase function, then the integral

$$I_\Phi(au) = \int e^{i\Phi(x,\theta)} a(x,\theta) u(x) dx d\theta,$$

is called an *oscillatory integral* for $u \in C^\infty(X)$.

An important fact about oscillatory integrals is its regularization, which can be achieved provided the following lemma.

Lemma 2.2.1. *There exist on $X \times \mathcal{R}^N$, an operator*

$$L = \sum_{j=1}^N a_j(x,\theta) \frac{\partial}{\partial\theta_j} + \sum_{k=1}^N b_k(x,\theta) \frac{\partial}{\partial x_k} + c(x,\theta),$$

such that $a_j \in S^0(X \times \mathcal{R}^N)$, $b_k, c \in S^{-1}(X \times \mathcal{R}^N)$ and with the formal adjoint given by

$$(L^t)u(x,\theta) = -\sum_{j=1}^N \frac{\partial}{\partial\theta_j}(a_j u) - \sum_{k=1}^N \frac{\partial}{\partial x_k}(b_k u) + cu,$$

we have

$$(L^t)e^{i\Phi} = e^{i\Phi}.$$

Then we have that the oscillatory integral can be written as

$$I_\Phi(au) = \int e^{i\Phi(x,\theta)} L^k(a(x,\theta)u(x)) dx d\theta. \tag{2.5}$$

Consider the sets

$$\begin{aligned} C_\Phi &= \{(x,\theta) : x \in X, \theta \in \mathcal{R}^N \backslash 0, \Phi'_\theta(x,\theta) = 0\}, \\ S_\Phi &= \pi C_\Phi, \\ R_\Phi &= X \backslash S_\Phi, \end{aligned}$$

where $\pi : X \times (\mathcal{R}^N \backslash 0) \to X$ is the natural projection, and the distribution $A \in \mathcal{D}'(X)$ defined as

$$\langle A, u \rangle = I_\Phi(au),$$

then we have that $A \in \mathcal{C}^\infty(R_\Phi)$, or equivalently *sing supp* $A \subset S_\Phi$.

Definition 2.2.4. *The phase function Φ is called non-degenerate if rank $||\Phi''_{\theta\theta}\Phi''_{\theta x}|| = N$ on C_Φ, where*

$$||\Phi''_{\theta\theta}\Phi''_{\theta x}|| = \left\| \begin{matrix} \frac{\partial^2\Phi}{\partial\theta_1\partial\theta_1} & \cdots & \frac{\partial^2\Phi}{\partial\theta_1\partial\theta_N} & \frac{\partial^2\Phi}{\partial\theta_1\partial x_1} & \cdots & \frac{\partial^2\Phi}{\partial\theta_1\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2\Phi}{\partial\theta_N\partial\theta_1} & \cdots & \frac{\partial^2\Phi}{\partial\theta_N\partial\theta_N} & \frac{\partial^2\Phi}{\partial\theta_N\partial x_1} & \cdots & \frac{\partial^2\Phi}{\partial\theta_N\partial x_n} \end{matrix} \right\|$$

Then it is easily proved that if Φ is non-degenerate then, C_Φ is an n-dimensional submanifold in $X \times (\mathcal{R}^N \backslash 0)$

Theorem 2.2.1. *Let Φ be non-degenerate and $a \in S^m_{\rho,\delta}(X \times \mathcal{R}^N)$ and assume we have*

"either $\rho > \delta$ and $\rho + \delta = 1$ or $\rho > \delta$ and Φ is linear in θ"

then

1. *if a has one zero on C_Φ of infinite order then $A(x) \in C^\infty(X)$.*
2. *if $a = 0$ on C_Φ, we can find $b \in S^{m-(\rho-\delta)}_{\rho,\delta}(X \times \mathcal{R}^N)$ such that $I_\Phi(au) = I_\Phi(bu)$ for all $u \in C^\infty_0(X)$*

The latter statement shows that the distribution A can be defined replacing a by b without changing the phase function, provided that $a|_{C_\Phi} = 0$, this results in higher regularity for $A(x)$. The prove of the previous theorem is not straightforward and requires an important lemma regarding the change of variables in $S^m_{\rho,\delta}$.
Assume we are given a diffeomorphism from a conical region $V \subset \mathcal{R}^{n_1} \times \mathcal{R}^{N_1}$ onto a conical region $U \subset \mathcal{R}^n \times \mathcal{R}^N$, commuting with the natural action of the multiplicative group $\mathcal{R}_+$, i.e. the diffeomorphism maps a point $(y,\eta) \in V$ to a point $(x(y,\eta), \theta(y,\eta)) \in U$, where the component functions are positively homogeneous in η of degree 0 and 1 respectively; let

$$b(y,\eta) = a(x(y,\eta), \theta(y,\eta)),$$

we have the following

Lemma 2.2.2. *Let $a \in S^m_{\rho,\delta}(U)$ and assume that one of the following conditions hold:*

1. *$\rho + \delta = 1$;*

2. $\rho + \delta \geq 1$ *and* $x = x(y)$;

3. $x = x(y)$ *and* $\theta = \theta(\eta)$

then $b(y, \eta) \in S^m_{\rho.\delta}(V)$

Proof. From the definition of $b(y, \eta)$ we have

$$\frac{\partial b}{\partial \eta_l} = \frac{\partial a}{\partial x_k}\frac{\partial x_k}{\partial \eta_l} + \frac{\partial a}{\partial \theta_j}\frac{\partial \theta_j}{\partial \eta_l} \tag{2.6}$$

$$\frac{\partial b}{\partial y_r} = \frac{\partial a}{\partial x_k}\frac{\partial x_k}{\partial y_r} + \frac{\partial a}{\partial \theta_j}\frac{\partial \theta_j}{\partial y_r}. \tag{2.7}$$

Since $\frac{\partial \theta_j}{\partial \eta_l}$, $\frac{\partial x_k}{\partial \eta_l}$, $\frac{\partial \theta_j}{\partial y_r}$ and $\frac{\partial x_k}{\partial y_r}$ belong to the classes S^0, S^{-1}, S^1, S^0 respectively in V, we have the estimates

$$\left|\frac{\partial b}{\partial \eta_l}\right| \leq C_K(|\eta|^{m-\rho} + |\eta|^{m+\delta-1}), \qquad \left(y, \frac{\eta}{|\eta|}\right) \in K,$$
$$\left|\frac{\partial b}{\partial y_r}\right| \leq C_K(|\eta|^{m-\rho+1} + |\eta|^{m+\delta}), \qquad \left(y, \frac{\eta}{|\eta|}\right) \in K,$$

where K is a compact set in V.
If $\rho+\delta \leq 1$ we have that $\left|\frac{\partial b}{\partial \eta_l}\right| \leq 2C_K < \eta >^{m-\rho}$. We obtain the same estimate if $x = x(y)$ since $\frac{\partial x_k}{\partial \eta_l} = 0$.
If $\rho+\delta \geq 1$ we have that $\left|\frac{\partial b}{\partial y_l}\right| \leq 2C_K < \eta >^{m+\delta}$. We obtain the same estimate if $\theta = \theta(\eta)$ since $\frac{\partial \theta_j}{\partial y_r} = 0$.
Since $a \in S^m_{\rho,\delta}(U)$ and for arbitrary multi-indices α, β, it can be easily proved that the estimates hold for derivatives of any order k where $|\alpha + \beta| \leq k$. □

Returning to the expression (2.1), denote by $K_A \in \mathcal{D}'(X \times Y)$ the distribution

$$\langle K_A, w\rangle = \int e^{i\Phi(x,y,\theta)} a(x, y, \theta) w(x, y) dx dy d\theta, \tag{2.8}$$

called, the kernel of the FIO A.

Proposition 2.2.1. *1.* $K_A \in R_\Phi$, *where* $R_\Phi = \{(x, y) : \Phi'_\theta(x, y, \theta) \neq 0, \quad \theta \in \mathcal{R}^N \backslash 0\}$,

2. *if* $a = 0$ *in a conical neighbourhood of the set*

$$C_\Phi = \{(x, y, \theta) : \Phi'_\theta(x, y, \theta) = 0\},$$

then $K_A \in C^\infty(X \times Y)$

It is easy to verify that the kernel K_A is uniquely determined by A and conversely; also that it is the usual kernel of A in the sense of Schwartz since for $u \in C_0^\infty(Y)$ and $v \in C_0^\infty(X)$, we have

$$\langle Au, v\rangle = \langle K_A, u(y)v(x)\rangle .$$

Definition 2.2.5. *A phase function $\Phi(x, y, \theta)$ is called an operator phase function, if the following holds:*

$$\Phi'_{y,\theta} \neq 0 \quad for \quad \theta \neq 0, \quad x \in X, \ y \in Y \tag{2.9}$$

$$\Phi'_{x,\theta} \neq 0 \quad for \quad \theta \neq 0, \quad x \in X, \ y \in Y \tag{2.10}$$

Proposition 2.2.2. *If (2.9) holds then, the operator (2.1) continuously maps $C_0^\infty(Y)$ into $C_0^\infty(X)$*

Proposition 2.2.3. *Let $\mathcal{E}'(Y)$ denote the dual space of $C^\infty(Y)$ and assume condition (2.10) holds then, the operator (2.1) extends by continuity to a continuous map:*

$$A : \mathcal{E}'(Y) \to \mathcal{D}'(X). \tag{2.11}$$

Proof. The operator

$$(A)^t v(y) = \int e^{i\Phi(x,y,\theta)} a(x, y, \theta) v(x) dx d\theta \tag{2.12}$$

defines a map $(A)^t : C_0^\infty(X) \to C^\infty(Y)$, by Proposition 2.2.2. Then, defining A by

$$\langle Au, v \rangle = \left\langle u, (A)^t v \right\rangle,$$

with $u \in \mathcal{E}'(Y), \quad v \in C_0^\infty(X)$ the proposition is proved. □

Now we will show an application of this concept.

Example 2.2.1. *Consider the Cauchy problem*

$$\frac{\partial^2 f}{\partial t^2} = \Delta f \tag{2.13}$$

$$f|_{t=0} = 0 \qquad f'_t|_{t=0} = u(x) \tag{2.14}$$

where $x \in \mathcal{R}^n$ and $u \in C_0^\infty(\mathcal{R}^n)$. To solve the system (2.13)-(2.14) we use the Fourier transform with respect to the variable x,

$$\tilde{f} = \int e^{-iy\cdot\xi} f(t, y) dy,$$

to transform into the system

$$\frac{\partial^2 \tilde{f}}{\partial t^2} = |\xi|^2 \tilde{f} \tag{2.15}$$

$$\tilde{f}|_{t=0} = 0 \qquad \tilde{f}'_t|_{t=0} = \tilde{u}(\xi), \tag{2.16}$$

which is easily solve by

$$\tilde{f}(t, \xi) = \tilde{u}(\xi) \frac{\sin(t|\xi|)}{|\xi|}.$$

By the Fourier inversion formula we get

$$\begin{aligned} f(t,x) &= \int e^{i(x-y)\cdot\xi}|\xi|^{-1}\sin(t|\xi|)u(y)dyd\xi \\ &= \int e^{i(x-y)\cdot\xi}(2i|\xi|)^{-1}(e^{it|\xi|}-e^{-it|\xi|})u(y)dyd\xi. \end{aligned}$$

If we split this last integral into two parts separating the exponents, this would lead us to a singularity at $\xi = 0$*, then we use a cut-off function* $\chi(\xi) \in C_0^\infty(\mathcal{R}^n)$ *such that* $\chi(\xi) = 1$ *near* 0*, and split the integral into three parts*

$$\begin{aligned} f(t,x) &= f_+(t,x) - f_-(t,x) + r(t,x) \\ f_+(t,x) &= \int e^{i[(x-y)\cdot\xi+t|\xi|]}(1-\chi(\xi))(2i|\xi|)^{-1}dyd\xi, \\ f_-(t,x) &= \int e^{i[(x-y)\cdot\xi-t|\xi|]}(1-\chi(\xi))(2i|\xi|)^{-1}dyd\xi, \\ r(t,x) &= \int e^{i(x-y)\cdot\xi}\chi(\xi)|\xi|^{-1}\sin(t|\xi|)dyd\xi. \end{aligned}$$

It is clear that $f_+ = Au$ *where* A *is a FIO with phase function*

$$\Phi(t,x,y,\xi) = (x-y)\cdot\xi + t|\xi|,$$

and since $\Phi'_\xi = (x-y) + t\frac{\xi}{|\xi|}$*, we get that this is an operator phase function and*

$$\begin{aligned} C_\Phi &= \left\{(t,x,y,\xi) : y - x = t\frac{\xi}{|\xi|}\right\}, \\ S_\Phi &= \{(t,x,y) : |x-y|^2 = t^2\}. \end{aligned}$$

For the second term we have $f_- = \tilde{A}u$ *with phase function*

$$\tilde{\Phi}(t,x,y,\xi) = (x-y)\cdot\xi - t|\xi|,$$

and $S_{\tilde{\Phi}} = S_\Phi$*.*
For the third term $r = Ru$ *where* R *has a* C^∞ *Schwartz kernel* $K_R(t,x,y)$*. Any such operator can be written as a FIO with an arbitrary choice of the phase function and an amplitude* $a \in S^{-\infty}$*.*
So, each integral term can be presented as a result of the application of a FIO to the inicial condition u*. In particular, we can define* $f(t,x)$ *for every* $u \in \mathcal{E}'(\mathcal{R}^n)$ *by Proposition 2.2.3; and the singularities of* $f(t,x)$ *belong to*

$$\{(t,x) : \exists u \in \textit{sing supp } u,\ |x-y|^2 = t^2\}.$$

Example 2.2.2. *Let*

$$A = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha. \tag{2.17}$$

where $a_\alpha \in C^\infty(X)$*,* X *is open in* $\mathcal{R}^n$ *and* $D = i^{-1}\frac{\partial}{\partial x}$*.*

We know that

$$D^\alpha u(x) = \int \xi^\alpha e^{i(x-y)\cdot\xi} u(y) dy d\xi,$$

hence

$$Au(x) = \int e^{i(x-y)\cdot\xi} \sigma_A(x,\xi) u(y) dy d\xi,$$

where $\sigma_A(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ *is called the principal symbol of the operator* A*. Since* $\sigma_A \in S^m(X \times \mathcal{R}^n)$ *it is clear that every linear differential operator is a FIO with phase function* $\Phi(x,y,\xi) = (x-y)\cdot\xi$.

2.3 Pseudodifferential Operators

The last example of the previous section is a motivation for defining the pseudodifferential operators, which are a particular class of FIO and have been widely used in the context of acoustic and elastic wave propagation.

Definition 2.3.1. *Let* $X = Y$ *be a subset of* $\mathcal{R}^N$*. A FIO with phase function* $\Phi(x, y.\xi) = (x-y)\cdot\xi$ *is called a pseudodifferential operator (ΨDO) with* $a(x,y,\xi) \in S^m_{\rho,\delta}(X\times X\times\mathcal{R}^N)$.

In this section we will focus on the explanation of the so called clasical ΨDO, for which $\rho = 1, \quad \delta = 0$ and the symbol $a(x,\xi) \in C^\infty(X \times \mathcal{R}^N)$.

Definition 2.3.2. *Let* $\sigma \in \bigcup_{m\in\mathcal{R}} S^m$ *be a symbol. The pseudodifferential operator* T *associated to* σ *is defined by*

$$(T_\sigma\varphi)(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)d\xi \tag{2.18}$$

We list some important properties of this operators and its symbols, the proofs are straightforward.

Proposition 2.3.1.

1. *If* $P(x,D)$ *is a linear partial differential operator whose coefficients* $a_\alpha(x) \in C^\infty$ *and have bounded derivatives of all orders, then the symbol* $P(x,\xi) \in S^m$.
2. *If* $\sigma \in S^{m_1}$ *and* $\tau \in S^{m_2}$ *the* $\sigma + \tau \in S^{m_1+m_2}$.
3. *If* $\sigma \in S^m$ *then* $D^\beta_x D^\alpha_\xi \sigma \in S^{m-|\alpha|}$.
4. *Let* σ *be a symbol,* $\varphi \in \mathcal{S}(\mathcal{R}^N)$ *and define* $\tau(x,\xi) = \sigma(x,\xi)\varphi(\xi)$ *for all* $x,\xi \in \mathcal{R}^N$ *then,* $\tau(x,\xi) \in \bigcap_{m\in\mathcal{R}} S^m$.
5. *Let* σ *be a symbol then,* $T_\sigma\varphi \in \mathcal{S}(\mathcal{R}^N)$

Now we define the asymptotic expansion of a symbol which is a concept which allows to aproximate the values of a symbol and analize its behavior at infinity.

Definition 2.3.3. *Let* $\sigma \in S^m(\mathcal{R}^N)$. *Suppose we can find* $\sigma_j \in S^{m_j}$ *where* $m = m_0 > m_1 > \cdots > m_j \to -\infty \quad as \quad j \to \infty$ *such that*

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in S^{m_N}$$

for every positive integer N. *We call* $\sum_{j=0}^{\infty} \sigma_j$ *an asymptotic expansion of* σ *and we write*

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_j.$$

The following is an important proposition whose proof shows the way to construct a symbol which is asympotically expanded by a sequence of symbols.

Proposition 2.3.2. *Let* $m_0 > m_1 > \cdots > m_j \to -\infty$ *as* $j \to \infty$. *Suppose* $\sigma_j \in S^{m_j}$. *Then there exist a symbol* $\sigma \in S^{m_0}$ *such that* $\sigma \sim \sum_{j=0}^{\infty} \sigma_j$.

Definition 2.3.4. *Let* A *be a pseudodifferential operator and* K_A *its kernel.* A *is called properly supported if the projections* $\Pi_1, \Pi_2 : supp\ K_A \to X$ *are proper maps*

Note that linear differential operators are properly supported ΨDO. In this case $suppK_A = \triangle$, the diagonal in $X \times X$.

Proposition 2.3.3. *Let* A *be a properly supported* ΨDO, *Then* A *defines a map*

$$A : C_0^\infty(X) \to C_0^\infty(X)$$

which extends to continuous maps

$$A : \mathcal{E}'(X) \to \mathcal{E}'(X).$$

In definition 2.3.1 we have that in general a ΨDO is written as

$$Au(x) = \int e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y) dy d\xi \tag{2.19}$$

The following proposition gives an expression for the symbol of a properly supported pseudodifferential operator in terms of $a(x, y, \xi)$

Proposition 2.3.4. *Let* A *be a properly supported* ΨDO *in the form (2.19), and* $\sigma_A(x, \xi)$ *its symbol. Then*

$$\sigma_A(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x, y, \xi)|_{y=x}. \tag{2.20}$$

Consider an operator in the form (2.19), and define its transposed operator A^t as

$$\langle Au, v\rangle = \langle u, A^t v\rangle, \tag{2.21}$$

where $u, v \in C_0^\infty(X)$ and

$$< u, v >= \int u(x)v(x)dx.$$

It is clear that for A we have that

$$A^t v(y) = \int e^{i(x-y)\cdot\xi} a(x, y, \xi)v(x)dxd\xi,$$

and if $\eta = -\xi$ we have

$$A^t v(y) = \int e^{i(y-x)\cdot\eta} a(x, y, -\eta)v(x)dxd\eta,$$

from which A^t is a ΨDO.

Proposition 2.3.5. *Let A be a properly supported ΨDO with symbol σ_A and $\sigma_A^t(x,\xi)$ be the symbol of A^t then,*

$$\sigma_A^t(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \sigma_A(x, -\xi)$$

Another important concept is the composition formula or product of two pseudodifferential operators.

Proposition 2.3.6. *Let $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$. Then the product $T_\sigma T_\tau$ is a ΨDO whose symbol $\lambda \in S^{m_1+m_2}$ and has the following asymptotic expansion*

$$\lambda \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha \sigma(x,\xi))(D_x^\alpha \tau(x,\xi)).$$

2.3.1 ΨDO in Wavefield Extrapolation

Pseudodifferential operators naturally appear in the context of wave propagation. In this subsection we will show some examples in which these operators are used to obtain extrapolators to simulate wave propagation in the acoustic case.

Example 2.3.1. *Let $\sigma(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$. It is a straightforward exercise to prove that $\sigma \in S^m(\mathcal{R}^n)$.*
If $m = 2$ we have a pseudodifferential operator given by:

$$\begin{aligned}(T_\sigma\varphi)(x) &= (2\pi)^{-\frac{n}{2}}\int e^{i(x\cdot\xi)}(1+|\xi|^2)\hat{\varphi}(\xi)d\xi \\ &= \left[(1+|\xi|^2)\hat{\varphi}(\xi)\right]^{\check{}}(x) \\ &= \left[\hat{\varphi}(\xi) + \widehat{\left(\sum_{i=1}^n \frac{\partial^2\varphi}{\partial x^2}\right)}\right]^{\check{}}(\xi) \\ &= (id_n + \Delta)\varphi(x).\end{aligned}$$

We can prove that the asymptotic expansion for the symbol σ *is given by*

$$\sigma(\xi) = (1 + |\xi|^2)^{\frac{m}{2}} \sim \sum_{j=0}^{\infty} \begin{pmatrix} m \\ j \end{pmatrix} |\xi|^{2j},$$

where

$$\begin{pmatrix} m \\ j \end{pmatrix} = \frac{m(m-1)(m-2)\cdots(m-j+1)}{j!}.$$

Now, consider the wave equation

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{v^2(x,z)} \right) \hat{P}(x,z,\omega) = 0, \tag{2.22}$$

where $\hat{P}(x,z,\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int e^{-it\omega} P(x,z,t) dt$. *We must consider the following cases:*

1. *When the velocity of propagation of the medium is constant* $v(x,z) = c$, *we can decouple the wave equation into propagation modes, known as upgoing and downgoing waves as*

$$\begin{aligned} \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2} \right) \hat{P}(x,z,\omega) &= 0 \qquad (2.23) \\ \left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - k_x^2 \right) \hat{P}(k_x,z,\omega) &= 0 \qquad (2.24) \\ \left(\frac{\partial}{\partial z} - ik_{z_0} \right) \left(\frac{\partial}{\partial z} + ik_{z_0} \right) \hat{P}(k_x,z,\omega) &= 0 \qquad (2.25) \end{aligned}$$

   where $k_{z_0}^2 = \frac{\omega^2}{c^2} - k_x^2$.

   Each factor of the last equation represents a one way wave equation, and for the downgoing wave, the solution is

$$\hat{P}(k_x,z,\omega) = \hat{P}(k_x,z_0,\omega) e^{ik_{z_0}(z-z_0)}.$$

2. *Suppose* $v(x,z)$ *is not constant, then take the wave operator as* $A(x,z,D_t,D_x) = -\frac{1}{v^2(x,z)} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$, *whose symbol is* $a(x,z,\omega,k_x) = \frac{\omega^2}{v^2(x,z)} - k_x^2$.

   To achieve the decoupling of the equation we need to obtain the square root operator $\sqrt{A}$ *which is achieved by its symbol. Nevertheless we can consider an approximation by supposing we can decouple the equations as*

$$\frac{\partial}{\partial z} \hat{P}(k_x,z,\omega) = i\sqrt{a} \hat{P}(k_x,z,\omega),$$

   in a subinterval $[z_i, z_i + \triangle z]$ *for which* $v^2(x,z) \equiv v^2(x)$. *Note that*

$$\begin{aligned} k_z &= \omega s(x,z_i) \sqrt{1 - \frac{k_x^2}{\omega^2 s^2}} \\ k_{z_0} &= \omega s_0 \sqrt{1 - \frac{k_x^2}{\omega^2 s_0^2}}, \end{aligned}$$

where the last equation stands for the constant velocity case. With this equations we have

$$k_z = k_{z_0}\sqrt{1 + \frac{\omega^2}{k_{z_0}^2}(s^2 - s_0^2)},$$

and taking $\xi^2 = 1 + \frac{\omega^2}{k_{z_0}^2}(s_0^2 - s^2)$ *and if* $\xi^2 < 1$ *we have that, according to example 2.3.1, the asymptotic expansion for this symbol is*

$$k_z = k_{z_0} + k_{z_0}\sum_{j=1}^{\infty}(-1)^j \left(\begin{array}{c} \frac{1}{2} \\ n \end{array}\right)\left[\frac{\omega^2}{k_{z_0}^2}(s_0^2 - s^2)\right]^j,$$

this expression is incerted into the extrapolator

$$\hat{P}(k_x, z_0 + \triangle z, \omega) = e^{ik_z \triangle z}\hat{P}(k_x, z_0, \omega)$$

to obtain a better approximation of the propagator as $n \to \infty$*..*

3. *Let* A *be the wave operator as above, and consider the following vectorial operators:*

$$\begin{aligned} [A] &= \begin{bmatrix} 0 & 1 \\ -A & 0 \end{bmatrix}, \\ \vec{\psi} &= \begin{bmatrix} \psi \\ \partial_z \psi \end{bmatrix}, \\ \vec{f} &= \begin{bmatrix} 0 \\ f(x,z,t) \end{bmatrix} \end{aligned}$$

with these operators the wave equation is reduced to the first order partial differentuial equations system:

$$\frac{\partial}{\partial z}\vec{\psi} = [A]\vec{\psi} - \vec{f}.$$

The operator $[A]$ *is diagonalizable when the operators* $\sqrt{A}$ *and* $A^{-\frac{1}{2}}$ *exist, and then*

$$[A] = VB\Lambda,$$

where $\Lambda = V^{-1}$, *and*

$$B = \begin{bmatrix} i\sqrt{A} & 0 \\ 0 & -i\sqrt{A} \end{bmatrix}$$

and

$$\Lambda = \frac{1}{2}\begin{bmatrix} 1 & -iA^{-\frac{1}{2}} \\ 1 & iA^{-\frac{1}{2}} \end{bmatrix}.$$

After performing the transformation $\vec{\psi} = V\vec{\mu}$, *and if the medium is invariant with depth we obtain the system*

$$\frac{\partial}{\partial z}\vec{\mu} = B\vec{\mu} - \Lambda\vec{f},$$

and for μ_2, the vertical component of $\vec{\mu}$, we have the decoupled OWWE equation

$$\frac{\partial}{\partial z}\mu_2 = -i\sqrt{A}\mu_2 - \frac{i}{2}A^{-\frac{1}{2}}f.$$

The main goal is to find the symbol of the square root of the operator A. Notice that the symbol $a(x,z,\omega,k_x) = \frac{\omega^2}{v^2(x,z)} - k_x^2$ has order 1 and is a first candidate for the symbol of the square root operator; nevertheless the subprincipal symbol in the asymptotic expansion is important since it is needed to describe wave amplitudes. Take $a \equiv a_0$ for the principal symbol, which determines the rays, and a_1 for the subprincipal symbol in the asymptotic expansion.
Let B be the operator asociated to the symbol $a = a_0 + a_1$ then, for the operator $A - B^2$ to have a 0^{th} order symbol, we find from the composition formula of the symbol $(a_0+a_1)(a_0+a_1)$ that:

$$a_1 = i\frac{k_x\partial_x v}{2\omega}\left(1 - \frac{v^2(x,z)k_x^2}{\omega^2}\right)^{-\frac{3}{2}}. \tag{2.26}$$

The operator asociated to this symbols describes the rays and the amplitudes, ensuring that the operator $B^2 - A$ is a pseudodifferential operator of order 0.
Aditional procedures are made to this work. On one hand a symmetric quantization is made to the square root operator in order to preserve true amplitudes and use the principal symbol only. Then a normalization is made to achieve decoupling when the velocity is $z-$depending and to perform some numerical implementations.

This examples show the need of the pseudodifferential operators and FIO theory to accurately describe the wave propagation phenomena on the acoustic case.

2.4 Elastic Wave up/down Decomposition

Now we are going to show the application of the previous concepts to the case of elastic wave propagation as it is done in [20].
The propagation and scattering of seismic waves is governed by the elastic wave equation, which is writen in the form

$$P_{il}u_l = 0, \tag{2.27}$$
$$u_l|_{t=0} = 0, \quad \partial u_l|_{t=0} = h_l, \tag{2.28}$$

with $u_l = \sqrt{\rho}(U_l)$ where U_l is the displacement field of the particles of the medium and

$$P_{il} = \delta_{il}\frac{\partial^2}{\partial t^2} + A_{il}, \tag{2.29}$$
$$A_{il} = -\frac{\partial}{\partial x_j}\frac{C_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k} + l.o.t \tag{2.30}$$

where l.o.t means lower order terms. Here $x \in \mathcal{R}^n$, $i,j,k,l \in \{1,....,n\}$; $C_{ijkl}(x)$ is the stiffness tensor and $\rho(x)$ is density of mass. Decoupling of the propagation modes is made by diagonalizing the system; it is accomplished by transforming the system using some appropiate matrix-valued pseudodifferential operators, $Q(x, D_x)_{iM}$ where M stands for some polarization mode. It is clear that we only need to diagonalize the operator A_{il}, then we must find Q_{iM} and A_M such that

$$Q(x, D_x)^{-1}_{Mi} A_{il}(x, D_x) Q(x, D_x)_{lN} = diag(A_M(x, D_x); M = 1, ...n)_{MN}, \tag{2.31}$$

where M, N represent the mode of propagation then, uncoupled system of equations

$$P_M(x, Dx, D_t)u_M = 0 \tag{2.32}$$
$$u_M|_{t=0} = 0, \quad \partial u_M|_{t=0} = h_M, \tag{2.33}$$

is satisfied by

$$u_M = Q(x, D_x)^{-1}_{Mi} u_i, \tag{2.34}$$
$$h_M = Q(x, D_x)^{-1}_{Mi} h_i. \tag{2.35}$$

The properties of stiffness guarantee that the principal symbol of $A_{il}(x, D_x)$ is a positive and symmetric matrix, which then can be diagonalized by an orthogonal matrix. For the symbols we have that

$$Q^{prin}_{Mi}(x, \xi)^{-1} A^{prin}_{il}(x, \xi) Q^{prin}_{lN}(x, \xi) = diag(A^{prin}_M(x, \xi))_{MN}, \tag{2.36}$$

where $A^{prin}_M(x, \xi)$ are the eigenvalues of $A^{prin}_{il}(x, \xi)$ and Q^{prin}_{Mi} is the matrix whose columns are the orthonormalized polarizations vectors associated with the modes of propagation. Let $x = (x', x_n)$ where $x' = (x_1, ..., x_{n-1})$ and $x_n = z$ be the normal coordinates where $\zeta = \xi_n$, and for $x_n = 0$ the boundary normal coordinates defined in a bounded open subset of the boundary where the receivers are placed, Σ; and introduce $B_M(x, D_x) = \sqrt{A_M(x, D_x)}$, with principal symbol $B^{prin}_M(x, \xi) = \sqrt{A^{prin}_M(x, \xi)}$.
Let the polarized Green' s function G_N, be the solution of

$$P_N(x, D_x, D_t) G_N(x, \tilde{x}, t) = \delta_{\tilde{x}}(x)\delta_0(t) \tag{2.37}$$

and is identified with the incident field, and d_{MN} the $N - M$ converted data for a given source at $\tilde{x}$, that are observed on $\Sigma \times (0, T)$. The reverse-time continued field is the anticausal solution of the equation

$$\left[\partial_t^2 + A_M(x, D_x)\right] v_r(x, t) = \delta(x_n) \mathcal{N}_M(x', D_{x'}, D_t) \Psi_{\mu,\Sigma}(x', t, D_{x'}, D_t) d_{MN}(x', t; \tilde{x}), \tag{2.38}$$

where

$$\mathcal{N}_M(x', D_{x'}, D_t) = -2iD_t \frac{\partial B^{prin}_M}{\partial \xi_n}, \tag{2.39}$$

and $\Psi_{\mu,\Sigma}$ is a pseudodifferential cutoff. Define the operators

$$\Xi_0(x, \xi, \tau) = \tau, \qquad \Xi_j(x, \xi, t) = \xi_j \tag{2.40}$$

$$\Theta_0(x, \xi, \tau) = \tau, \qquad \Theta_j(x, \xi, \tau) = \tau \frac{\partial B^{prin}_M}{\partial \xi_j}(x, \xi). \tag{2.41}$$

For imaging the contrast in stiffness tensor we have an operator H_{MN} given by:

$$(H_{MN}d_{MN})_{ijkl}(x) = -\frac{1}{2\pi}\int \frac{2\Omega(\tau)}{i\tau|\hat{G}_N(x,\tilde{x},\tau)|^2}\sum_{p=0}^{n}\left(\frac{\partial}{\partial x_k}Q(x,D_x)_{lN}\Xi_p(x,D_x,\tau)\overline{\hat{G}_N(x,\tilde{x},\tau)}\right)\times \left(\frac{\partial}{\partial x_j}Q(x,D_x)_{iM}\Theta_p(x,D_x,\tau)\hat{v}_r(x,\tau)\right)d\tau.$$

A similar expression holds for imaging the density contrast and in the case of mode conversions ($M \neq N$), in which the use of the operator $\mathcal{N}_N$ is used. Note that this operator extends to the elastic case the operator associated to the symbol (2.26), constructed in the acoustic case.

Now, consider the equivalent system

$$\frac{\partial}{\partial t}\begin{pmatrix} u_M \\ \frac{\partial u_M}{\partial t}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -A_M(x,D_x) & 0\end{pmatrix}\begin{pmatrix} u_M \\ \frac{\partial u_M}{\partial t}\end{pmatrix} \tag{2.42}$$

which can be decoupled with the operators

$$V_M(x,D_x) = \begin{pmatrix} 1 & 1 \\ -iB_M(x,D_x) & iB_M(x,D_x)\end{pmatrix} \tag{2.43}$$

$$\Lambda_M(x,D_x) = \begin{pmatrix} 1 & iB_M(x,D_x)^{-1} \\ 1 & -iB_M(x,D_x)^{-1}\end{pmatrix}; \tag{2.44}$$

then we have that for

$$P_{M,\pm}(x,D_x,D_t) = \partial_t \pm iB_M(x,D_x) \tag{2.45}$$

with

$$P_{M,+}P_{M,-} = P_M,$$

the distribution

$$u_{M,\pm} = \frac{1}{2}u_M \pm \frac{1}{2}iB_M(x,D_x)^{-1}\frac{\partial u_M}{\partial t} \tag{2.46}$$

satisfies the first-order (half wave) system of equations

$$P_{M,\pm}(x,D_x,D_t)u_{M,\pm} = 0 \tag{2.47}$$

$$u_{M,m}|_{t=0} = h_{M,\pm}, \qquad h_{M,\pm} = \pm\frac{1}{2}iB_M(x,D_x)^{-1}h_M. \tag{2.48}$$

The solution to this IVP admits a representation $u_{M,\pm}(y,t) = (S_{M,\pm}(t)h_{M,\pm})(y)$ where

$$u_M(y,t) = ([S_{M,+}(t) - S_{M,-}(t)]\frac{1}{2}iB_M^{-1}h_M)(y) \tag{2.49}$$

and the operators $S_{M,\pm}$ are Fourier integral operators which can be written as

$$S_M(t) = V_M\begin{pmatrix} S_{M,+}(t) & 0 \\ 0 & S_{M,-}(t)\end{pmatrix}\Lambda_M \tag{2.50}$$

and have the oscillatory integral representation

$$(S_{M,\pm}(t)h_{M,\pm})(y) = (2\pi)^{-n}\int a_{M,\pm}(y,t,\xi)e^{i\phi_{M,\pm}(y,t,x,\xi)}h_{M,\pm}(x)dxd\xi \tag{2.51}$$

where

$$\phi_{M,\pm}(y,t,x,\xi) = \alpha_{M,\pm}(y,t,\xi) - < \xi, x >,$$

and the amplitudes $a_{M,\pm}$ and phase operator $\alpha_{M,\pm}$ are found by solving the Hamilton's equations generated by the principal symbol $\pm B_M^{prin}(x,\xi)$, (which also are the propagation of singularities):

$$\frac{\partial y^t}{\partial t} = \pm \frac{\partial B_M^{prin}(y^t, \eta^y)}{\partial \eta}, \qquad \frac{\partial \eta^t}{\partial t} = \mp \frac{\partial B_M^{prin}(y^t, \eta^t)}{\partial y.} \tag{2.52}$$

Chapter 3

Riemannian Acoustic Wave Equation

3.1 Introduction

One of the main challenges of wave-equation migration methods is to imaging complex geologic structures which appear to be a result of anisotropy, spatial dependence of the velocity of propagation and/or irregular sub-structures of the earth; this complexity posses some problems in the migration methods such as the innaccuracy of extrapolation operators which can not propagate turning waves, the limitations on the dip of reflectors, large propagation angles, irregular 3D domain boundary surfaces and mesh interiors that are best described by non-cartesian geometry. An approach to partially solving this issue is to consider a theory of elastic wave propagation in general coordinate systems, i.e, elastic wave equation in a Riemannian manifold. This formalism is necessary since, for example, the layers in a complex stratified medium should be represented as curves in a plane or surfaces, that is a 2-dimensional or 3-dimensional manifold, and then geometric concepts such as curvature of a curve or surface arise in a fundamental way.

Sava and Fomel [22],[23], recognized that Cartesian coordinates for downward, tilted continuation, or along beams of limited spatial extent do not reflect a physical reality. Their main idea was to use Riemannian coordinates that conform with the general direction of wave propagation, so the coordinate system can follow the waves, which may overturn, see also [24]. This is based on an extrapolator operator that is not necessarily a solution for the acoustic wave equation proposed, since that equation is not the result of physical considerations. In this approaches it is used known mappings between 3D generalized and Cartesian coordinate systems to specify the Laplacian differential operator which governs the 3D acoustic wave equation. Some particular examples are provided in the works of Sava and Fomel for 3D semiorthogonal Riemannian spaces, in which the extrapolation direction is orthogonal to the other directions. Shragge [25], provided an extension to non-orthogonal coordinate systems such as 2D shared Cartesian coordinates systems, polar ellipsoidal coordinates systems, 2D non-orthogonal coordinate systems and 2D orthogonal coordinates systems; in this work, the extrapolator is also implemented in a propagator of phase-shift type and the inaccuracy of the propagation of amplitudes in laterally varying media are corrected by considering the kinematic terms on the extrapolator.

Shragge [26], uses a Finite Difference Tiem-Domain (FDTD) methodology to solve the full wave equation in a 3D Riemannian coordinate system on which he could handle scenarios exhibiting irregular geometry. The FDTD scheme implemented in the work consist on a solution to the two-way wave equation of order $O(\Delta t^2, \Delta x^8)$. A stability condition is derived in a heuristic way taking as a starting point the Courant stability condition and translating it to the transformed scenario by means of the chain rule of calculus, but this is not formally a stability condition and indeed the resulting rule it gives to calculate the time sampling needed to ensure stability of the FD solution is quite far of being true.
Arias, Coimbra, Quiceno and Tygel [27], applied the Von-Neumann method to obtain a stability criteria for a second order and fourth order FDTD scheme of the 2D Riemannian acoustic wave equation and compare it with the heuristic one used by Shragge [26]. For numerical comparison, we also perform a wave propagation using two different topography profiles: a Gaussian 2D profile and the Canadian Foothills profile, a synthetic velocity model representing a zone in the British Columbia (Canada), that shows several geological complexities common in that region. This velocity model allows us to show the dependence of the stability criteria on the smoothness of the profile. Finally we analyze the numerical dispersion for the generalized wave equation and compare it with the Cartesian acoustic wave equation.

The chapter is presented as follows: In the first section we show some examples of meshes transforms given in [22], [23], [24], with the respective metric tensors which are conformal to the Euclidean one. Section 3.3 develops the Laplace.-Beltrami operator defined in a Riemannian manifold, some basic properties of this operator, which are found in F.Wong [28], and the resulting form with each of the metrics constructed in the previous section. In Section 3.4 we show the Riemannian acoustic wave equation proposed in [22], [23], and derive the extrapolator for every coordinate transformation; we end the section by showing the FDTD scheme used in [26]. The last section presents the analysis of stability given in [27], with the application of the Von-Neumann method and the numerical examples.

3.2 Meshes Transforms

The space on which wave propagation occurs is in general a curved three dimensional space, which is called the physical domain; in this space is where a wave equation is proposed and solved. For numerical implementations it is customary to use a regular (rectangular) mesh on which the computations are made; this mesh represents a Cartesian space usually called the computational domain. Then, to properly simulate a physical situation on a computational setting we need to properly transform these domains, i.e., find a correct homeomorphism which maps the physical domain into the computational one and then the numerical approximations should be accurate to the physical phenomena. Let $\mathbf{x} = [x_1, x_2, x_3]^T$ be the coordinates of a curved physical domain on which we want to solve the wave equation and $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]^T$ a regular (rectangular) computational domain on which one actually computes the acoustic wave field. Thus, we use a function $\mathbf{x} = \boldsymbol{\phi}(\boldsymbol{\xi})$ that maps the computational domain onto physical domain.

Example 3.2.1. 3D-Semiorthogonal coordinate system
In this coordinate system we have that one coordinate, say ξ_3 is orthogonal to the other coordinates (ξ_1, ξ_2); we can take this orthogonal coordinate as the propagation direction. The local representation of the metric tensor is $[g_{ij}] = \sum_k \frac{\partial \phi_k}{\partial \xi_i}\frac{\partial \phi_k}{\partial \xi_j}$ we have that for this system

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{11} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \tag{3.1}$$

and the inverse can be found by the simple relation

$$[g^{ij}] = \frac{1}{|\mathbf{g}|}\begin{bmatrix} g_{22}g_{33} - g_{23}^2 & g_{13}g_{23} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ g_{13}g_{23} - g_{12}g_{33} & g_{11}g_{33} - g_{13}^2 & g_{12}g_{13} - g_{11}g_{23} \\ g_{12}g_{23} - g_{13}g_{22} & g_{12}g_{13} - g_{11}g_{23} & g_{11}g_{22} - g_1^2 \end{bmatrix} \tag{3.2}$$

where $|\mathbf{g}| = g_{11}g_{22}g_{33} - g_{12}^2 g_{33} - g_{23}^2 g_{11} - g_{13}^2 g_{22} + 2g_{12}g_{13}g_{23}$, it follows that, with the notations of [22],

$$[g^{ij}] = \begin{bmatrix} \frac{G}{J^2} & -\frac{F}{J^2} & 0 \\ -\frac{F}{J^2} & \frac{E}{J^2} & 0 \\ 0 & 0 & \frac{1}{\alpha^2} \end{bmatrix} \tag{3.3}$$

where

$$\begin{aligned} E &= g_{11} \\ F &= g_{12} \\ G &= g_{22} \\ \alpha^2 &= g_{33} \\ J^2 &= EG - F^2 \\ |\mathbf{g}| &= \alpha^2 J^2 \end{aligned}$$

Example 3.2.2. Cylindrical and Spherical coordinate systems
For the Cylindrical, which means propagation along radial direction, the transformation map is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \xi_3 \cos(\xi_1) \\ \xi_3 \sin(\xi_1) \\ \xi_2 \end{bmatrix} \tag{3.4}$$

which gives the metric tensors:

$$[g_{ij}] = \begin{bmatrix} \xi_3^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

$$[g^{ij}] = \begin{bmatrix} \frac{1}{\xi_3^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.6}$$

where clearly $|\mathbf{g}| = \xi_3^2$.
For the spherical coordinate system where the radius is the propagation direction, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \xi_3 \sin(\xi_1)\cos(\xi_2) \\ \xi_3 \sin(\xi_1)\sin(\xi_2) \\ \xi_3 \cos(\xi_1) \end{bmatrix} \tag{3.7}$$

and

$$[g_{ij}] = \begin{bmatrix} \xi_3^2 & 0 & 0 \\ 0 & \xi_3^2 \sin^2(\xi_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

with $J = \xi_3^2 \sin(\xi_1)$.

Example 3.2.3. 2D sheared Cartesian coordinate system
For this system we let the coordinate transformation to be:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & \cos(\theta) \\ 0 & \sin(\theta) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{3.9}$$

where ξ_2 *is the propagation direction and* θ *is the shear angle. The metric tensor and its inverse are easily calculated as*

$$[g_{ij}] = \begin{bmatrix} 1 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix} \tag{3.10}$$

$$[g^{ij}] = \frac{1}{\sin^2(\theta)} \begin{bmatrix} 1 & -\cos(\theta) \\ -\cos(\theta) & 1 \end{bmatrix}. \tag{3.11}$$

Note that in this system the metric tensor is coordinate invariant.

Example 3.2.4. Polar ellipsoidal coordinates
This system is specified by the transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a(\xi_2)\xi_1 \cos(\xi_2) \\ a(\xi_2)\xi_1 \sin(\xi_2) \end{bmatrix} \tag{3.12}$$

where ξ_1 *is the radius from the center focus and* ξ_2 *is polar angle; the function* $a(\xi_2)$ *is a smooth function controlling coordinate system ellipticity with curvature parameters* $b = \frac{\partial a}{\partial \xi_2}$ *and* $c = \frac{\partial^2 a}{\partial \xi_2^2}$. *The metric tensor is given as:*

$$[g_{ij}] = \begin{bmatrix} a^2 & \xi_1 ab \\ \xi_1 ab & \xi_1^2(a^2 + b^2) \end{bmatrix} \tag{3.13}$$

and

$$[g^{ij}] = \begin{bmatrix} \frac{a^2+b^2}{a^4} & -\frac{b}{a^3\xi_1} \\ -\frac{b}{a^3\xi_1} & \frac{1}{a^2\xi_1^2} \end{bmatrix}. \tag{3.14}$$

In this case the metric tensor is nonstationary.

Example 3.2.5. Topography transformation system
We consider a topography transformation which represents the acquisition surface and it is worked on [27]. The transformation that maps the rectangular domain with coordinates onto the physical domain is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \phi_1(\xi_1, \xi_2) \\ \phi_2(\xi_1, \xi_2), \end{bmatrix} \tag{3.15}$$

where

$$\phi_1(\xi_1, \xi_2) = \xi_1\,; \tag{3.16}$$
$$\phi_2(\xi_1, \xi_2) = \xi_2 + \psi(\xi_1) \tag{3.17}$$

and ψ is a smooth function that represents the curved upper boundary of the physical domain. This function must be, at least, twice differentiable. Thus, under that condition the function $\boldsymbol{\phi} = (\phi_1, \phi_2)$ is a coordinate chart for the physical domain which is being modeled on a regular Euclidean space. With this transformation we get to

$$[g_{ij}] = \begin{bmatrix} (1 + (\psi')^2) & \psi' \\ \psi' & 1 \end{bmatrix} \tag{3.18}$$

and

$$[g^{ij}] = \begin{bmatrix} 1 & -\psi' \\ -\psi' & (1 + (\psi')^2) \end{bmatrix} \tag{3.19}$$

and $|\mathbf{g}| = 1$.

3.3 The Laplace-Beltrami Operator

We present the definition and some important properties of the Laplace-Beltrami operator which allows to formulate and approximation of the wave equation in a Riemannian manifold. The theory, which started with the work of Riemann, is wide and can be found in fundamental books as [29],[30],[31],[32], among others. We do not pretend to formulate this theory and then we just give some basic definitions.

A smooth manifold M is said to be *Riemannian* if it is endowed with a metric, g, a family of smoothly varying, positive-definite inner products g_x defined on every tangent space T_xM for all $x \in M$. The metric g_x is a bilinear form on the tangent space and then is an element of $T^*_xM \otimes T^*_xM$ and since it is smooth, g is a section of the tensor bundle $T^*M \otimes T^*M$ and a symmetric $(0,2)-$tensor.

Definition 3.3.1. *Let (M, g) be a Riemannian manifold and f a smooth function on M. The gradient of f is defined as the vector field $\nabla f = grad(f)$, such that $g(\nabla f, X) = df(X)$ for all $X \in TM$. In other words, ∇f raises an index on the one-form df.*

It is clear that for every $X \in TM$ and $f \in \mathcal{F}(M)$ we have in local coordinates:

$$\begin{aligned} g_{ij}(\nabla f)^j X^i &= \frac{\partial f}{\partial x_j} X^i \\ (\nabla f)^j &= g^{ij} \frac{\partial f}{\partial x_i} \\ (\nabla f) &= g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \end{aligned}$$

Definition 3.3.2. *Let $X \in \mathcal{X}(M)$ be a vector field. The divergence of X at the point $p \in M$ is defined as*

$$div(X)_p = \sum_{i=1}^{n} g_p(\nabla_{E_i} X, E_i),$$

where $E_1, ..., E_n$ is an orthonormal basis for T_pM and ∇ is the Levi-Civita connection with respect to g.

It is clear from the definition that

$$div(X) = Trace(Y \to g(\nabla_Y X, Y)),$$

and using the Christoffel symbols is easy to get the expression in local coordinates as

$$div(X) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} (\sqrt{|g|} X^j.)$$

Consider the space of 1-forms $\Lambda^1(M)$ and the operator, exterior derivative,

$$d : C^\infty(M) \to \Lambda^1(M),$$

mapping a function to its differential

$$\begin{aligned} df &\in \Lambda^1(M) \\ df &= \frac{\partial f}{\partial x_i} dx_i, \end{aligned}$$

and the operator $\delta : \Lambda^1(M) \to C^\infty(M)$ as its dual $\delta := d^*$.

Definition 3.3.3. *The Laplace-Beltrami operator on functions on a Riemannian manifold M is defined as*

$$\Delta \ : \ C^\infty(M) \to C^\infty(M) \tag{3.20}$$
$$\Delta \ = \ -\delta \cdot d. \tag{3.21}$$

It is clear that $\Delta f = -div(\nabla f)$.
Some properties of the operator (3.21) are listed in the next proposition.

Proposition 3.3.1. *Let Δ be the Laplace-Beltrami operator on a Riemannian manifold M then,* *1.* $\Delta^* = \Delta$;

2. Δ *conmutes with isometries, i.e. If* $T : M \to M$ *preserves the metric on tangent spaces and* $\hat{T} : C^\infty(M) \to C^\infty(M)$ *is given by* $\hat{T}f = f \circ T$, *then*

$$\Delta\hat{T} = \hat{T}\Delta$$

.

3. In local coordinates

$$\delta\left(a_i dx^i\right) = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_j}(\sqrt{|g|}g^{ij}a_i),$$

and then

$$\Delta f = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_j}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x_i}\right).$$

If we take the Laplace-Beltrami operator on the space $L^2(M)$ we have

Proposition 3.3.2. *If* $f, g \in C_0^\infty(M)$ *and* $\partial M = \emptyset$, *then*

$$\int_M f\Delta g\mu = -\int_M \langle \nabla f, \nabla g\rangle \mu = \int_M g\Delta f\mu,$$

where $\mu = \sqrt{|g|}dx^1 ... dx^n$ *is the volume element defined on* M.

This proposition (Green' s second identity) shows that in $L^2(M)$ the operator is self-adjoint with respect to the inner product $\langle \Delta f, g\rangle_{L^2} = \langle f, \Delta g\rangle_{L^2}$. Moreover $\int_M \Delta f\mu = 0$ for $f \in C_0^\infty(M)$ and

$$\langle \Delta f, f\rangle = -||\nabla f||^2 \tag{3.22}$$

$$||f||^2 \leq c||\nabla f||^2, \quad \text{for} \quad c > 0. \tag{3.23}$$

Theorem 3.3.1. *Let* (M, g) *be a compact Riemannian manifold. Then the eigenvalue problem*

$$\Delta f + \lambda f = 0, \qquad f \in L^2(M)$$

has countably many eigenvalue-eigenfunction pair of solutions $(\lambda, f) = (\lambda_n, f_n)$ *for which* $\langle f_n, f_m\rangle = \delta_n^m$ *and* $\lambda_n\delta_n^m = \langle \Delta f_n, f_m\rangle = -\langle \nabla f_n, \nabla f_m\rangle = \lambda_m\delta_n^m$. *If* $\partial M \neq \emptyset$, *all eigenvalues are positive, and* $\lim_{n\to\infty} \lambda_n = \infty$. *We also have for* $g \in L^2(M)$ *and* $h \in C_0^\infty(M)$

$$g = \sum_{i=0}^{\infty} \langle g, f_i\rangle f_i, \quad \text{and} \quad \langle \Delta h, h\rangle = \sum_{i=1}^{\infty} \lambda_i \langle h, f_i\rangle^2.$$

We will show some basic examples to illustrate the use of the Laplace-Beltrami operator.

Example 3.3.1. On the submanifold S^1.
Consider $\Delta := \frac{d^2}{d\theta^2}$. *Since* $-\Delta e^{in\theta} = n^2 e^{in\theta}$, *it follows that* $n^2 \in \mathcal{Z}$ *are the eigenvalues of* $-\Delta$. *An orthonormal basis for* $L^2(S^1)$ *is the set* $\{e^{in\theta}\}$. *The eigenvalue 0 occurs with multiplicity 1 and the others with multiplicity 2. For an arbitrary* $f \in L^2(S^1)$ *we have*

$$f = \sum_n \left\langle f, e^{in\theta}\right\rangle e^{in\theta},$$

which is the usual Fourier decomposition for f.

Example 3.3.2. On the disc $\mathcal{D}$.
Consider the unit disc $\mathcal{D} = \{x \in \mathcal{R}^2 : |x| \leq 1\}$, on which is clear that $-\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$. Assume that we have $f(r,\theta) = g(r)\phi(\theta)$ and then we solve the eigenvalue problem by separating the variables

$$\begin{aligned} -\Delta f &= \lambda f \\ g''(r)\phi(\theta) + \frac{1}{r}g'(r)\phi(\theta) + \frac{1}{r^2}g(r)\phi''(\theta) &= \lambda g(r)\phi(\theta) \\ r^2\left[\lambda + \frac{g''(r)}{g(r)}\right] + r\left[\frac{g'(r)}{g(r)}\right] &= -\frac{\phi''(\theta)}{\phi(\theta)}. \end{aligned}$$

It is a straightforward exercise to show that the equation for g can be written in the form of the Bessel's equation and that the eigenfunctions are given in terms of Bessel's functions as $f_k^\lambda(r,\theta) = \phi_k(\theta)J_k(\sqrt{\lambda}r)$ where $\phi_k(\theta) = a_k\cos(k\theta) + b_k\sin(k\theta)$, where λ is the eigenvalue corresponding to f_k^λ.
With Dirichlet boundary conditions, $f|_{\partial M} = 0$, $\sqrt{\lambda}$ must be a zero of J_k; and with Neumann boundary conditions, $\partial_{\vec{n}}f = 0$ with $\vec{n}$ normal to ∂M, the value $\sqrt{\lambda}$ must be a zero of J_k'. The generalization, say the Laplace operator on the sphere S^n can be found in [12].

3.4 Riemannian Acoustic Wave Equation

In this section we show the works of [22], [23] and [26] on which is used the Laplace-Beltrami operator to model acoustic wave propagation.
Consider the monochromatic wave equation for an acoustic wavefield

$$\Delta_\xi U = -\omega^2 s_\xi^2 U \tag{3.24}$$

$$\Delta_\xi U = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial \xi_i}\left(\sqrt{|g|}g^{ij}\frac{\partial U}{\partial \xi_j}\right) \tag{3.25}$$

and g_{ij} is the metric tensor associated with a chart or coordinate transformation which is described by a homeomorphism $x_i = f_i(\xi_j)$. The equation can be written as

$$n^j\frac{\partial U}{\partial \xi_j} + m^{ij}\frac{\partial^2 U}{\partial \xi_j \partial \xi_i} = -\sqrt{|g|}\omega^2 s_\xi^2 U, \tag{3.26}$$

where s_ξ is the slowness field in the Riemannian coordinates, n^j and m^{ij} are geometrical factors depending on the metric and are calculated as

$$\begin{aligned} m^{ij} &= \sqrt{|g|}g^{ij} \\ n^j &= \frac{\partial m^{ij}}{\partial \xi_i}. \end{aligned}$$

Following Claerbout, [8], we can obtain a dispersion relation by replacing the differential operators action on U with their Fourier domain wavenumber duals ($\xi_\nu \leftrightarrow k_{\xi_\nu}$), to obtain the equation

$$\left(m^{ij}k_{\xi_i} - in^j\right)k_{\xi_j} = \sqrt{|g|}\omega^2 s_\xi^2. \tag{3.27}$$

This equation represents the dispersion relation requiered to propagate a wave field in a generalized 3D coordinate system. The operator m^{ij} is a measure of the inner product of wavenumber vectors and n^j represents a scaling of wavenumber caused by local expansion of the coordinate system in the jth direction. An important observation is that in this work it is assumed that the operators m^{ij} and n^j are stationary in the Fourier domain, which is an approximation to obtain a dispersion relation since they explicitly depend on the coordinates.

Equation (3.26), can be used to describe two-way propagation of acoustic waves in a semiorthogonal Riemann space. To get a one-way wavefield extrapolation the use of equation (3.27) is needed after isolating a specified propagation direction. This extrapolator, with mixed domain fields, is inserted in the propagator

$$U(\xi_3 + \triangle\, \xi_3, k_{\xi_1}, k_{\xi_2}; \omega) = U(\xi_3, k_{\xi_1}, k_{\xi_2}; \omega) e^{ik_{\xi_3} \triangle \xi_3}, \tag{3.28}$$

which is an $(\omega - \mathbf{k}_\xi)$ phase-shift extrapolator in Fourier domain with ξ_3 being the propagation direction. Now we need to isolate the k_{ξ_3} component of the wave vector in the equation (3.27).

Expanding equation (3.27), and solving for k_{ξ_3} we get

$$\begin{aligned} m^{33} k_{\xi_3}^2 + (-in^3 + 2m^{13} k_{\xi_1} + 2m^{23} k_{\xi_2}) k_{\xi_3} &= \sqrt{|g|}\omega^2 s_\xi^2 + i(n^1 k_{\xi_1} + n^2 k_{\xi_2}) \\ &\quad -m^{11} k_{\xi_1}^2 - m^{22} k_{\xi_2} - 2m^{12} k_{\xi_1} k_{\xi_2}, \end{aligned}$$

then

$$\begin{aligned} k_{\xi_3} &= -\frac{B}{2A} \pm \left[\left(\frac{B}{2A} \right)^2 - \frac{C}{A} \right]^{\frac{1}{2}} \\ A &= m^{33} \\ B &= -in^3 + 2m^{13} k_{\xi_1} + 2m^{23} k_{\xi_2} \\ C &= -\sqrt{|g|}\omega^2 s_\xi^2 - i(n^1 k_{\xi_1} + n^2 k_{\xi_2}) + m^{11} k_{\xi_1}^2 + m^{22} k_{\xi_2} + 2m^{12} k_{\xi_1} k_{\xi_2}, \end{aligned}$$

after the calculations it follows

$$k_{\xi_3} = -a_1 k_{\xi_1} - a_2 k_{\xi_2} + ia_3 \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} + ia_8 k_{\xi_1} + ia_9 k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}, \tag{3.29}$$

where the a_ν coefficients depend on the operators n^j and m^{ij}.

It is clear that taking $g^{ij} = \delta^{ij}$ we get $m^{ij} = \delta^{ij}$,$n^j = 0$ and equations (3.26) and (3.29) reduce to the Cartesian acoustic case

$$\begin{aligned} \nabla^2 U &= -\omega^2 s^2 U \\ k_{\xi_3} &= [s^2 \omega^2 - k_{\xi_1}^2 - k_{\xi_2}^2]^{\frac{1}{2}}. \end{aligned}$$

3.4.1 Kernels

We refer to the works of [23] and [25]. By Kernels we mean the extrapolator field used for different geometries or meshes transform.

Example 3.4.1. 3D Coordinate system
Equation (3.29) represents the extrapolator for a 3D nonorthogonal coordinate system, the kinematic part is an approximation which reduces computational cost, see [25], and is defined as the real part of the extrapolator, then the 3D kinematic extrapolator is

$$\hat{k_{\xi_3}} = -a_1 k_{\xi_1} - a_2 k_{\xi_2} \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}.$$

Example 3.4.2. 3D Semiorthogonal
In this case assume that coordinate k_{ξ_3} is orthogonal to the other two coordinates, then the extrapolator reduces to

$$k_{\xi_3} = ia_3 \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} + ia_8 k_{\xi_1} + ia_9 k_{\xi_2} - a_{10}^2]^{\frac{1}{2}},$$

and the kinematic extrapolator is

$$\hat{k_{\xi_3}} = \pm[a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}$$

.

Example 3.4.3. 2D Nonorthogonal
This situation is dealt by defining $\xi_2 = 0$ and then all derivatives with respect to this coordinate are zero in (3.29), resulting in the extrapolator

$$k_{\xi_3} = -a_1 k_{\xi_1} + ia_3 \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 + ia_8 k_{\xi_1} - a_{10}^2]^{\frac{1}{2}},$$

with kinematic part

$$\hat{k_{\xi_3}} = -a_1 k_{\xi_1} \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 - a_{10}^2]^{\frac{1}{2}}.$$

Example 3.4.4. 2D Sheared Cartesian *For the 2D sheared system, with metric given by (3.10), we get the kinematic extrapolator*

$$\begin{aligned} k_{\xi_3} &= -\frac{g^{13}}{g^{33}} k_{\xi_1} \pm \left[\frac{s^2\omega^2}{g^{33}} - \left(\frac{g^{11}}{g^{33}} - \left(\frac{g^{13}}{g^{33}}\right)^2\right) k_{\xi_1}^2\right]^{\frac{1}{2}} \\ k_{\xi_3} &= \cos(\theta) k_{\xi_1} \pm \sin(\theta)\sqrt{s^2\omega^2 - k_{\xi_1}^2} \end{aligned}$$

Example 3.4.5. Polar Ellipsoidal Coordinates *For this system, using the transformation (3.12) and the metric tensor (3.14), we get*

$$\begin{aligned} k_{\xi_3} &= \frac{\xi_1 b}{a} k_{\xi_1} \pm \left[a^2\xi_1^2 s^2\omega^2 - \xi_1^2 k_{\xi_1}^2 - i\xi_1 k_{\xi_1}\left(\frac{a^2+2b^2-ac}{a^2}\right)\right]^{\frac{1}{2}} \\ \hat{k_{\xi_3}} &= \xi_1\left[\frac{b}{a} k_{\xi_1} \pm \sqrt{a^2 s^2\omega^2 - k_{\xi_1}^2}\right]. \end{aligned}$$

3.5 Finite Difference Approach. Analysis of Stability and Dispersion

In this section we present the results of the work in [27], regarding a finite difference scheme to the 2D Riemannian acoustic wave equation and the analysis of stability and dispersion. This later analysis is compared with the stability condition proposed by [26] and is obtained by the application of the Von-Neumann stability criteria for the 2D acoustic Riemannian wave equation associated with a topographic transform.
Using equations (3.15), (3.18), (3.19) into equation (3.26) we get the 2D acoustic Riemannian wave equation:

$$\left[\frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial \xi_1^2} - 2\psi' \frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial \xi_1 \partial \xi_2} + \left[1 + (\psi')^2\right] \frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial \xi_2^2} - \psi'' \frac{\partial U_{\boldsymbol{\xi}}}{\partial \xi_2}\right] - \frac{1}{v_{\boldsymbol{\xi}}^2} \frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial t^2} = F_{\boldsymbol{\xi}}. \tag{3.30}$$

The stability condition for the acoustic wave equation is widely known as the Courant-Friderichs-Lewy condition (CFL) and is given in the case of two spatial dimensions by

$$\Delta t \leq \frac{\Delta r}{v(\mathbf{x})}, \tag{3.31}$$

where

$$\Delta r = \left[\Delta x_1^{-2} + \Delta x_2^{-2}\right]^{-\frac{1}{2}} \tag{3.32}$$

is the root-mean-square (RMS) of the spatial sampling and $v(\mathbf{x})$ is the value maximum of the velocity model in physical grid.

In the Riemannian case the only approach to it has been made by [26] who uses the chain rule to obtain from Equation (3.31) the following expression:

$$\Delta t \leq \frac{1}{v_{\boldsymbol{\xi}}} \times \underset{\boldsymbol{\xi}}{\operatorname{argmin}} \left\{ \left[\left(\frac{\partial \phi_1}{\partial \boldsymbol{\xi}^T} \Delta \boldsymbol{\xi} \right)^{-2} + \left(\frac{\partial \phi_2}{\partial \boldsymbol{\xi}^T} \Delta \boldsymbol{\xi} \right)^{-2} \right]^{-\frac{1}{2}} \right\}. \tag{3.33}$$

To derive the appropriated stability condition we make use of the Von-Neumann method as follows:

The differential operators in Equation (3.26), are expanded in a second order finite difference scheme as:

$$
\begin{aligned}
\frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial t^2} &= \frac{U^{n+1}_{\nu,k} - 2U^n_{\nu,k} + U^{n-1}_{\nu,k}}{(\Delta t)^2}\,, \\
\frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial \xi_1 \partial \xi_2} &= \frac{U^n_{\nu+1,k+1} - U^n_{\nu-1,k+1} - U^n_{\nu+1,k-1} + U^n_{\nu-1,k-1}}{4\Delta\xi_1\Delta\xi_2}\,, \\
\frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial \xi_1^2} &= \frac{U^n_{\nu+1,k} - 2U^n_{\nu,k} + U^n_{\nu-1,k}}{(\Delta\xi_1)^2}\,, \\
\frac{\partial^2 U_{\boldsymbol{\xi}}}{\partial \xi_2^2} &= \frac{U^n_{\nu,k+1} - 2U^n_{\nu,k} + U^n_{\nu,k-1}}{(\Delta\xi_2)^2}\,, \\
\frac{\partial U_{\boldsymbol{\xi}}}{\partial \xi_1} &= \frac{U^n_{\nu+1,k} - U^n_{\nu-1,k}}{2\Delta\xi_1}\,, \\
\frac{\partial U_{\boldsymbol{\xi}}}{\partial \xi_2} &= \frac{U^n_{\nu,k+1} - U^n_{\nu,k-1}}{2\Delta\xi_2},
\end{aligned}
\tag{3.34}
$$

where n, ν, and k are the discretization variables for t, ξ_1, and ξ_2, respectively. The system of equations (3.34) replaced in the Riemannian acoustic wave equation (3.30) with $F_{\boldsymbol{\xi}} = 0$ produces the recursive scheme

$$
\begin{aligned}
U^{n+1}_{\nu,k} = -U^{n-1}_{\nu,k} - \frac{\Delta t^2 v^2_{\boldsymbol{\xi}}}{2\Delta\xi_1^2\Delta\xi_2^2}\big[&-\Delta\xi_1\Delta\xi_2 U^n_{\nu-1,k-1}g_{12} + \Delta\xi_1\Delta\xi_2 U^n_{\nu-1,k+1}g_{12} \\
&+ \Delta\xi_1\Delta\xi_2 U^n_{\nu+1,k-1}g_{12} - \Delta\xi_1\Delta\xi_2 U^n_{\nu+1,k+1}g_{12} \\
&+ U^n_{\nu,k}(-4\Delta\xi_1^2\Delta\xi_2^2 + 4\Delta\xi_2^2 g_{11} + 4\Delta\xi_2^2 g_{22}) \\
&+ (U^n_{\nu+1,k} + U^n_{\nu-1,k})(-2\Delta\xi_2^2 g_{11} - \Delta\xi_1\Delta\xi_2^2\zeta_1) \\
&+ (U^n_{\nu,k+1} + U^n_{\nu,k-1})(-2\Delta\xi_1^2 g_{22} - \Delta\xi_1^2\Delta\xi_2\zeta_2)\big]\,,
\end{aligned}
\tag{3.35}
$$

valid up to order two in space and time. Now consider a trial solution in the form

$$
U^n_{\nu,k} = u^n_{\nu,k} + \epsilon^n_{\nu,k}\,, \tag{3.36}
$$

where $U_{\boldsymbol{\xi}}$ is the exact solution (if it exists), u is the approximation to the solution and ϵ is the error introduced up to the desired order when approximating the solution. Assuming that equation (3.35) admits harmonic solutions in the form:

$$
u^n_{\nu,k} = u^n e^{i(\kappa_1\nu\Delta\xi_1 + \kappa_2 k\Delta\xi_2)}\,, \tag{3.37}
$$

where $i = \sqrt{-1}$ is the imaginary number and κ_1, and κ_2 are wave numbers, we can expect that the error behaves in the same way:

$$
\epsilon^n_{\nu,k} = \epsilon^n e^{i(\kappa_1\nu\Delta\xi_1 + \kappa_2 k\Delta\xi_2)}\,. \tag{3.38}
$$

Inserting Equation (3.36) in the FD scheme Equation (3.35) and taking into account that u satisfies the wave equation within the specified order of accuracy, the resulting equation only involves ϵ:

$$\begin{aligned}\epsilon_{\nu,k}^{n+1} &= 2\epsilon_{\nu,k}^{n} - \epsilon_{\nu,k}^{n-1} \\ &+\epsilon_{\nu,k}^{n}(c\Delta t)^2\left(\frac{\zeta^1}{\Delta\xi_1} i\sin(\kappa_1\Delta\xi_1) + \frac{\zeta^2}{\Delta\xi_2} i\sin(\kappa_2\Delta\xi_2) + \frac{g^{11}}{(\Delta\xi_1)^2}[2\cos(\kappa_1\Delta\xi_1)-2]\right. \\ &+\left.\frac{g^{22}}{(\Delta\xi_2)^2}[2\cos(\kappa_2\Delta\xi_2)-2] + \frac{g^{12}}{2\Delta\xi_1\Delta\xi_2}2i\sin(\kappa_1\Delta\xi_1)2i\sin(\kappa_2\Delta\xi_2)\right), \end{aligned} \quad (3.39)$$

that can be written as

$$\epsilon^{n+1} = B\epsilon^{n} - \epsilon^{n-1}\,. \quad (3.40)$$

Denote

$$R = \frac{\epsilon^{n+1}}{\epsilon^{n}} = \frac{\epsilon^{n}}{\epsilon^{n-1}}\,, \quad (3.41)$$

then the previous expression is written as $R^2 - BR + 1 = 0$, from which

$$R = \frac{B \pm \sqrt{B^2-4}}{2}\,. \quad (3.42)$$

We want ϵ^n, to be decreasing, so $|R| \leq 1$ which implies that the stability condition for the 2D acoustic Riemannian wave equation is

$$\left|B \pm \sqrt{B^2-4}\right| \leq 2. \quad (3.43)$$

If B is real, this expression could be replaced by two inequalities that would allow to solve for Δt. Nevertheless we should evaluate this inequality numerically since it is not possible to solve for Δt because it contains nonstationary terms associated with the metric tensor, explicitly depends on the space variables and, contains complex terms arising from mixed derivatives.

For each fixed x_1-value, we calculate B from Equation 3.39, for a range of Δt from 0 to a value above the CFL-limit. In this interval we take about a hundred values in an increasing way and, select for each value of x_1 the maximum Δt that satisfies the inequality 3.43; the results of this procedure are shown in Figure 4 and Figure 8 for two different topographic profiles.

We also calculate the stability condition for the generalized acoustic wave equation using 4^{th} order finite differences equations for the spatial derivatives. Again we obtain a relation between ϵ^{n+1}, ϵ^n and ϵ^{n-1}, because they come from the second derivative in time which is still approximated in a second order finite-difference equation. Then, this produces a 2^{nd} order polynomial in R, therefore a stability condition similar to (3.43) given by the expression:

$$
\begin{aligned}
B = 2 \quad &+ \quad v^2(\Delta t)^2 \left\{ \left(\frac{2\cos(\kappa_1 \Delta\xi_1)\sin^2\left(\frac{1}{2}\kappa_1\Delta\xi_1\right)}{3\Delta\xi_1^2} - \frac{14\sin^2\left(\frac{1}{2}\kappa_1\Delta\xi_1\right)}{3\Delta\xi_1^2} \right) g_{11} \right. \\
&+ \quad \left(\frac{4\sin(2\kappa_1\Delta\xi_1)\sin(\kappa_2\Delta\xi_1)}{9\Delta\xi_1\Delta\xi_2} - \frac{32\sin(\kappa_1\Delta\xi_1)\sin(\kappa_2\Delta\xi_2)}{9\Delta\xi_1\Delta\xi_2} \right. \\
&+ \quad \left. \frac{\sin(\kappa_1\Delta\xi_1)\sin(2\kappa_2\Delta\xi_2)}{9\Delta\xi_1\Delta\xi_2} - \frac{\sin(2\kappa_1\Delta\xi_1)\sin(2\kappa_2\Delta\xi_2)}{18\Delta\xi_1\Delta\xi_2} \right) g_{12} \\
&+ \quad \left(\frac{\cos(\kappa_2\Delta\xi_2)\sin^2\left(\frac{1}{2}\kappa_2\Delta\xi_1\right)}{3\Delta\xi_2^2} - \frac{14\sin^2\left(\frac{1}{2}\kappa_1\Delta\xi_1\right)}{3\Delta\xi_2^2} \right) g_{11} \\
&+ \quad i\left(\frac{4\sin(\kappa_1\Delta\xi_1)}{3\Delta\xi_2} - \frac{\sin(2\kappa_1\Delta\xi_1)}{6\Delta\xi_1} \right) \\
&+ \quad \left. i\left(\frac{4\sin(\kappa_2\Delta\xi_2)}{3\Delta\xi_2} - \frac{\sin(2\kappa_2\Delta\xi_2)}{6\Delta\xi_2} \right) \right\}.
\end{aligned} \tag{3.44}
$$

3.5.1 Numerical dispersion analysis

The numerical dispersion analysis is performed by the relation between the phase velocity and the frequency or equivalently, between phase velocity and the number of points per wavelength (ppw). In the absence of dispersion, the phase velocity is constant. To find this relation, the plane wave solution (3.37) is substituted in Equation (3.35) which, up to order 2, gives the following equation:

$$
e^n_{\nu,k} = e^{i(\kappa_1\nu\Delta\xi_1 + \kappa_2 k\Delta\xi_2 - \omega n\Delta t)}\,. \tag{3.45}
$$

To derive the dispersion relation, the harmonic plane wave in Equation (3.45) is used in equations (3.34), and obtain

$$
\begin{aligned}
2\cos(\omega\Delta t) = 2 \quad &+ \quad v^2(\Delta t)^2 \left[\frac{-2g_{11}}{\Delta\xi_1^2} + \frac{2\cos(d\xi_1 p\cos(\theta))g_{11}}{d\xi_1^2} \right. \\
&- \quad \frac{\sin(d\xi_1 p\cos(\theta))\sin(d\xi_2 p\sin(\theta))g_{12}}{d\xi_1 d\xi_2} - \frac{2g_{22}}{d\xi_2} + \frac{2\cos(d\xi_2 p\sin(\theta))g_{22}}{d\xi_2^2} \\
&+ \quad \left. i\frac{\sin(d\xi_1 p\cos(\theta))\zeta_1}{d\xi_1} + i\frac{\sin(d\xi_2 p\sin(\theta))\zeta_2}{d\xi_2} \right] \qquad (3.46) \\
&= \quad f(\xi_1, \theta, \Delta\xi_1, \Delta\xi_2, p), \qquad (3.47)
\end{aligned}
$$

where p is the modulus of the wavenumber vector and θ is its argument. Equation (3.47) allows to find the relation between the grid points per wavelength $G_1 = \lambda/\Delta\xi_1$, $G_2 = \lambda/\Delta\xi_2$ and the normalized phase velocity $C_p/v = (\omega/k)/v$ since

$$
\frac{C_p}{v} = \frac{\omega}{k}\frac{1}{v} = \frac{\omega}{2\pi}\frac{\lambda}{\Delta\xi}\frac{\Delta\xi}{v} = \frac{\omega}{2\pi}G\Delta\xi, \tag{3.48}
$$

$$
p\Delta\xi = \frac{2\pi}{G}, \tag{3.49}
$$

and

$$\omega\Delta t = \frac{C_p}{v}\frac{2\pi}{G}\frac{v\Delta t}{\Delta\xi} = \frac{C_p}{v}\frac{2\pi c}{G} \tag{3.50}$$

where $c = v\Delta t/\Delta\xi$ is the Courant-Friedrich-Levy number.
Replacing the expression (3.50) in Equation (3.47) we get

$$\frac{C_p}{v} = \frac{G}{4\pi c}\arccos(f(\xi_1, \theta, \Delta\xi, p)) \tag{3.51}$$

where we take $\Delta\xi_1 = \Delta\xi_2 = \Delta\xi$ for simplicity and f is defined by Equation (3.47).

3.5.2 Numerical Results

In this section we investigate, the 2D acoustic wave propagation for two topographic coordinates, in the following aspects: (i) stability criteria and, (ii) numerical dispersion. These computational meshes provide informative tests of the generalized 2D acoustic wave equation theory and of the implementation of the 2D FDTD numerical scheme described above.

Numerical stability

As a first example we solved the acoustic Riemannian wave equation 3.30 for a domain with constant velocity and an upper boundary given by $\psi(x_1) = he^{-a^2x_1^2}$ with $h = 3$ and $a = 0.5$, shown in Figure 3.1. Several snapshots for the propagation in that domain are shown in Figure 3.2. The stability condition was evaluated numerically using values for Δt from 0.6 s to 0 s and taking the biggest value of Δt that satisfies (3.43) at each point ξ_1. Since in transformation (9), $\xi_1 = x_1$, we can evaluate the stability condition in terms of x_1 directly. The result are shown in Figure 3.3. In Figure 3.3 the numeric limit is found by solving numerically the expression (3.43) and the heuristic limit is the one given by the equation (3.33).

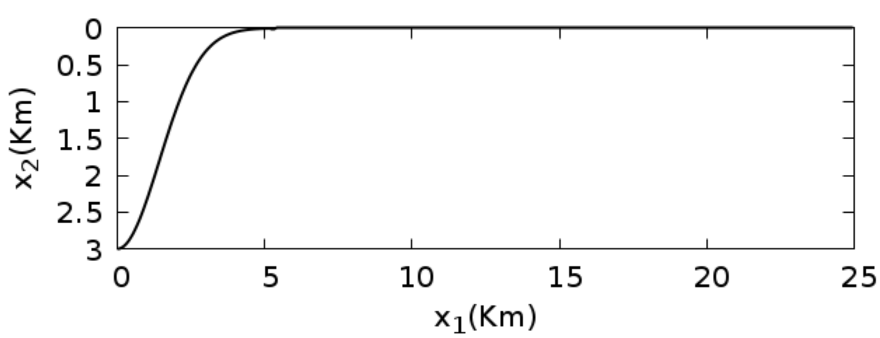
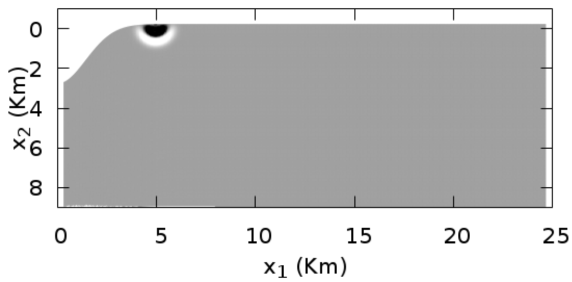
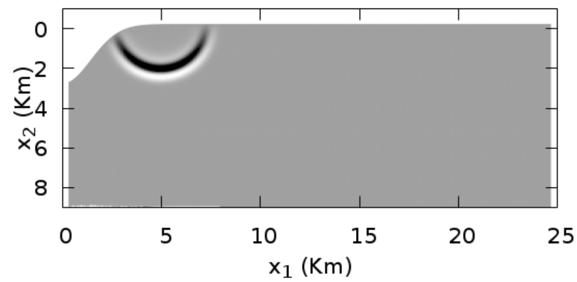


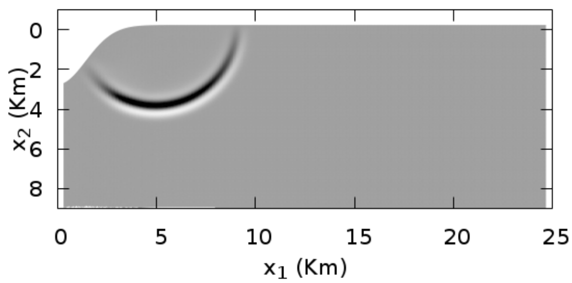
Figure 3.1: Mountain profile given by $\psi(x_1) = he^{-a^2x_1^2}$.



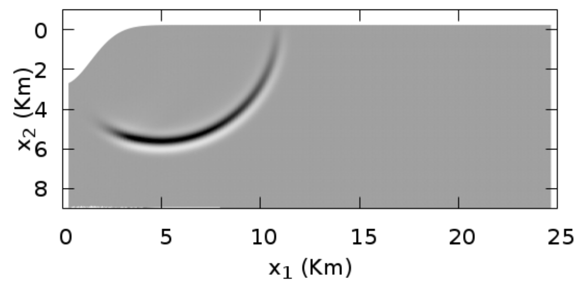
(a) Propagation of the Ricker pulse at 30 ms



(b) Propagation of the Ricker pulse at 60 ms



(c) Propagation of the Ricker pulse at 90 ms



(d) Propagation of the Ricker pulse at 120 ms

Figure 3.2: Snapshot for the propagation of a Ricker pulse in a medium with constant velocity and a upper boundary given by $\psi(x_1) = he^{-a^2 x_1^2}$. The value of Δt used for the propagation is $4 \times 10^{-3} s$, which is in agreement with the numeric limit shown in Figure 3.3 but not with the heuristic limit.

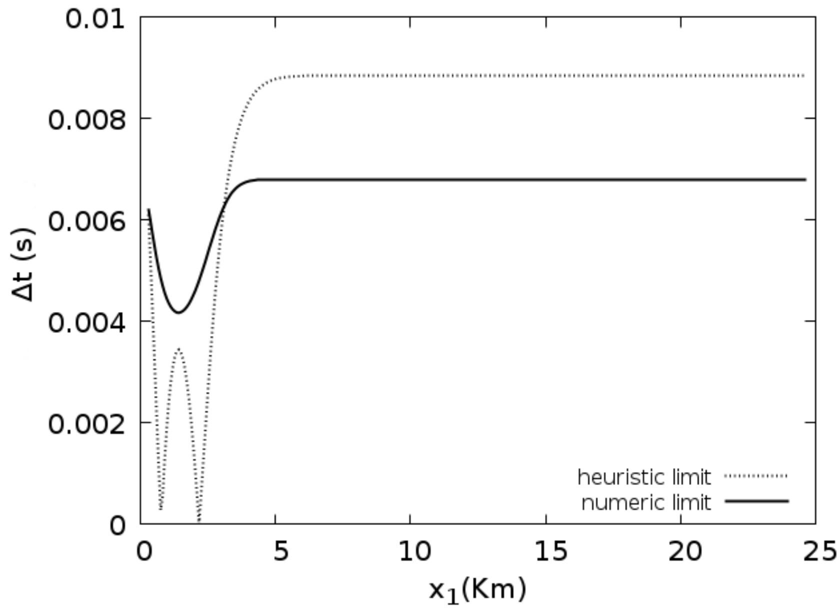


Figure 3.3: Stability condition for the Riemannian acoustic wave equation with an upper boundary given by a Gaussian function.The solid line corresponds to the numerical solution of the expression 3.43. The minimum Δt for the numeric limit is $4.1 \times 10^{-3}s$ and for the heuristic limit is 1.8×10^{-6}.

As a second example we solved Equation 3.30 for a domain with a constant velocity (4 Km/s) and an upper boundary corresponding to the Canadian Foothills velocity model. The size of the model is 1668x1000, nevertheless for our analysis we used a sub-sampled version, taking one sample for each 5 points of the model in both directions. The size of the sub-sampled version used in this work is 334x200. Several snapshots of the propagation of a Ricker pulse in this model are shown in Figure 3.6.
Since the calculations of the coefficients ζ_i requires the use of second derivatives, the mountain boundary should be smooth in order to avoid divergences, so we smoothed the original boundary of the sub-sampled model that is shown if Figure 3.4 using a simple moving average: the height of each point of the mountain was recalculated as the simple mean of the two nearest neighbors. Figure 3.5 shows the result of the application of this moving average after smoothing 5 times. The profile on Figure 3.5 is the one used for the propagation in Figure 3.6. As shown in Figure 3.7, the numeric limit for Δt is bigger than the heuristic limit. Let us define here the *degree of smoothness* of the profile as the number of times it was smoothed with the simple moving average. We evaluated the stability condition (3.43) for different degrees of smoothness for the profile of Figure 3.4 and found that the minimum Δt depends on this degree. The profiles for the several degrees of smoothness we used are shown in Figure 3.8 and the respective limits for Δt are shown in Figure 3.9.

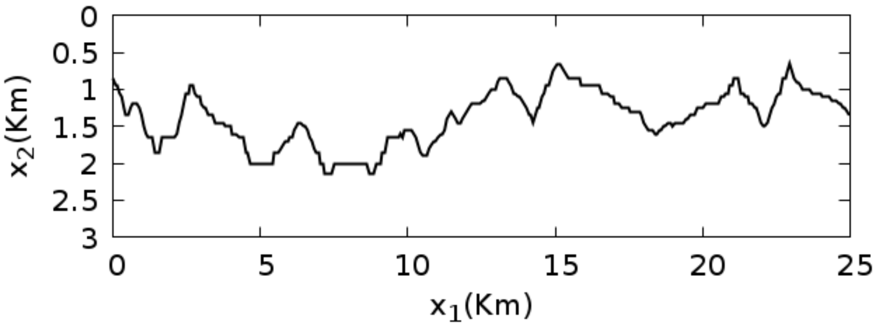


Figure 3.4: Boundary of a sub-sampled model of the Canadian Foothills with size 334x200.

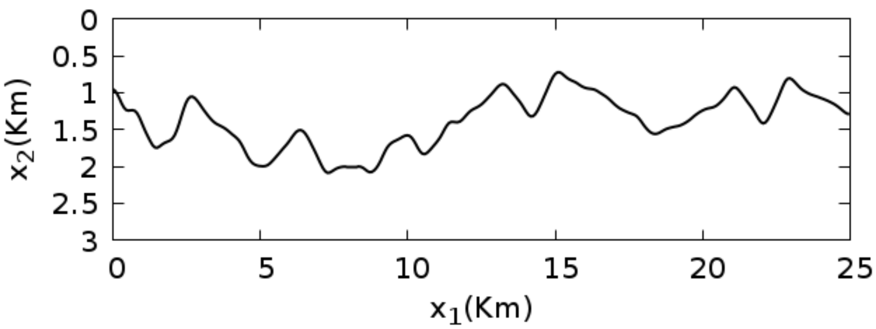
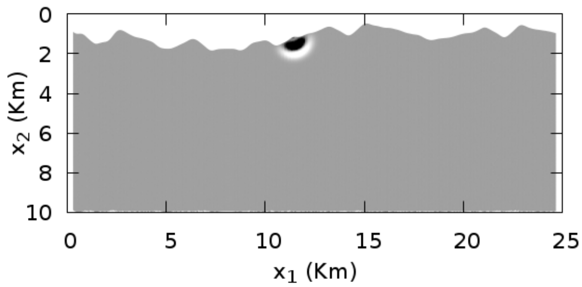
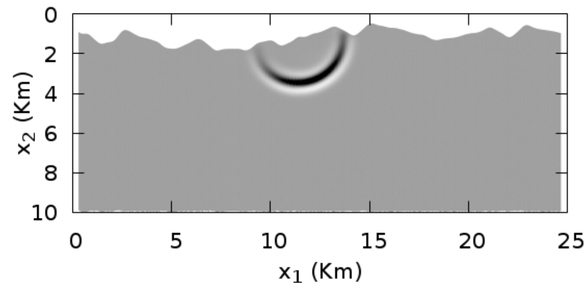


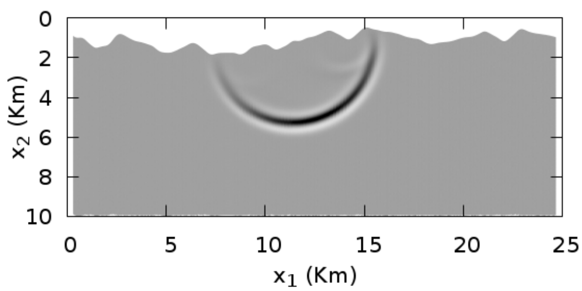
Figure 3.5: Smoothed boundary using a moving average, for the sub-sampled model of the Canadian Foothills model.



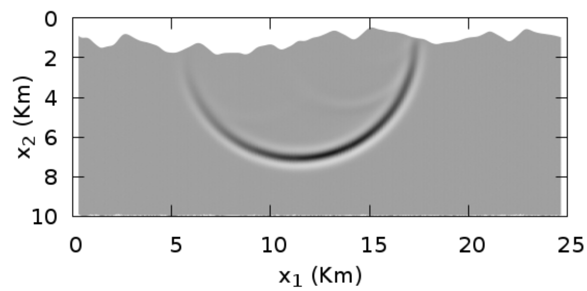
(a) Propagation of the Ricker pulse at 30 ms



(b) Propagation of the Ricker pulse at 60 ms



(c) Propagation of the Ricker pulse at 90 ms



(d) Propagation of the Ricker pulse at 120 ms

Figure 3.6: Snapshots of the propagation of a Ricker pulse in a constant velocity model with a upper boundary corresponding to a sub-sampled version of the Canadian Foothills with size 334x200. The value of Δt used for the propagation was $10^{-3}s$, which is in agreement with the numeric limit shown in Figure 3.7 but not with the heuristic limit. The CFL limit for a Cartesian version of this model would be $5.9 \times 10^{-3}s$ which implies that the computational cost is 143% bigger using the Riemannian acoustic wave equation. The mountain boundary was smoothed with a moving average. A secondary wavefront is due to the reflection of the wave in the rugged surface.

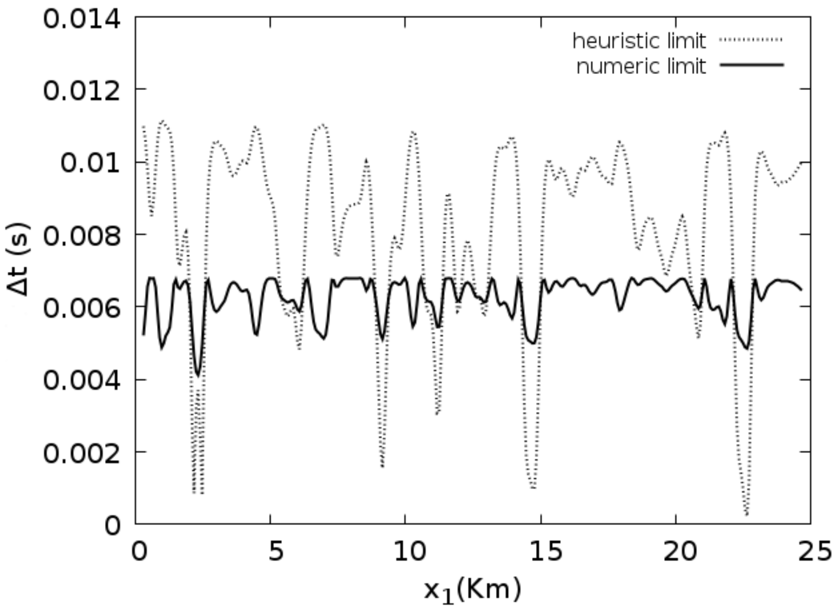


Figure 3.7: Stability condition for the Riemannian acoustic wave equation for the sub-sampled Canadian Foothill velocity model. The solid line is the numerical solution of the expression 3.43. The minimum Δt for the numeric limit is $4 \times 10^{-3}s$ and for the heuristic limit is $2 \times 10^{-4}s$.

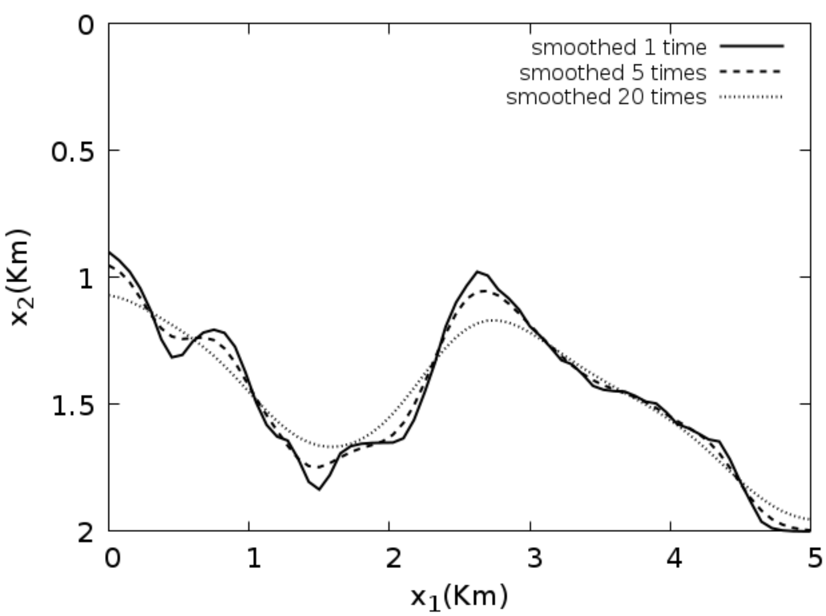


Figure 3.8: Smoothed sections for the sub-sampled Canadian Foothills model with different degree of smootheness.

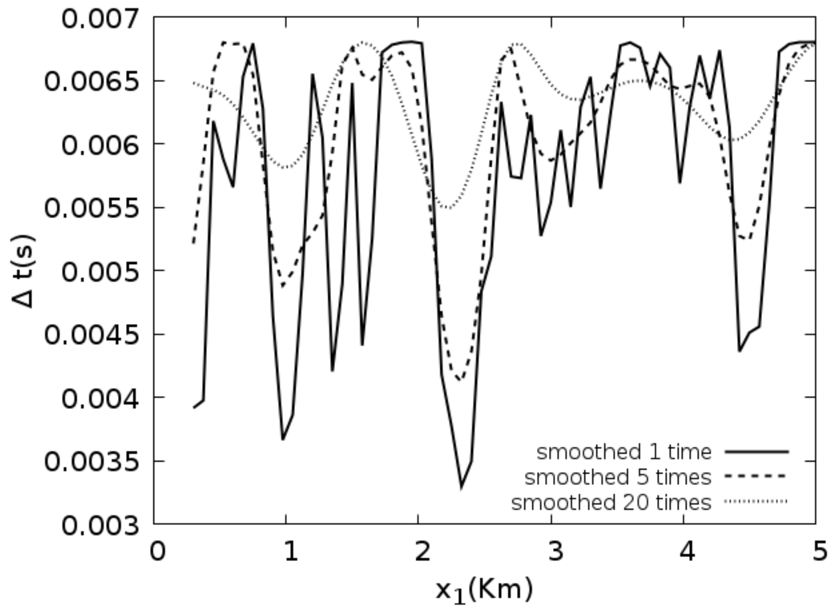


Figure 3.9: Maximum Δt allowed for different degrees of smoothness for a section the sub-sampled Canadian Foothills model. The graphs correspond to the boundary shown in Figure 3.4, smoothed 1, 5 and 20 times and, the respective limits for Δt are 0.0032 s, 0.0041 s and 0.0054 s.

Additionally we studied the dependence of the condition (3.43) on the frequency and found that the maximum Δt allowed, strongly depends on the particular values of the frequency used. For the propagation we used a Ricker pulse of central frequency $6Hz$ but it is obviously composed of a wide range of frequencies. Viewing the power spectrum of this pulse we identified a range from $1Hz$ to $80Hz$ approximately, so we evaluated Equation (3.43) for different values in this range. The results are shown in Figure 3.10.

The results for the stability condition for a section of the Canadian Foothills model at second and fourth order approximations are shown in Figure 3.11.

Dispersion analysis

Finally, we performed a dispersion analysis using Equation (3.51), which allows to observe the variation of the normalized velocity C_p/v as the number of points per wavelength varies. Figure 13 shows the result when we take the an angle $\theta = 0$ for the wavenumber and Figure 14 shows the result when the angle is $\pi/2$. For comparisons, we calculated the dispersion for the acoustic wave equation in the Euclidean case and, the result shown in Figure 15, shows a behavior that is expected: that for a number of points per wavelength large enough the normalized velocity tends to 1. In Figure 15, after a value of the grid point per wavelength around 14, the normalized velocity remains stabilized. Contrary to the expected result, Figure 13 shows that no matter the number of point per grid taken,

C_p/v is not stabilized, which means that different frequencies have different values of the phase velocity C_p producing wavefronts that are deformed as time runs. This effect is more dramatic for an angle of the wavenumber vector of $\theta = \pi/2$ as shown in Figure 14.

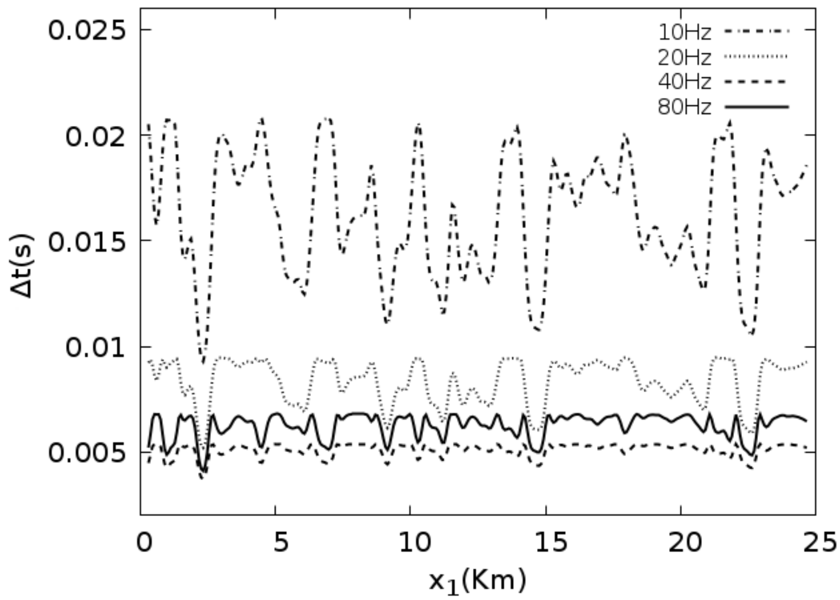


Figure 3.10: Maximum Δt allowed for different frequencies. For frequencies bigger than $80Hz$ the curves lie between those of top and the bottom shown.

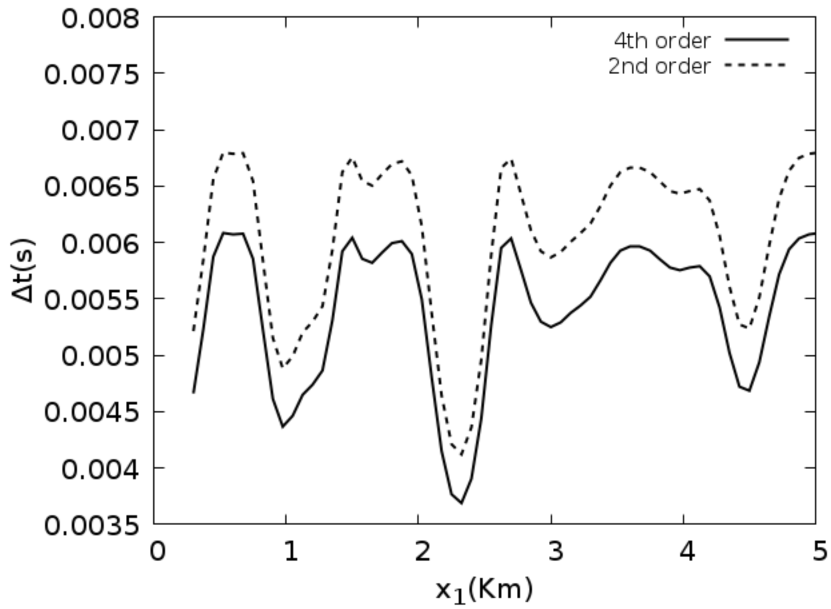


Figure 3.11: Comparison of the stability conditions of order 2 and 4 in the spatial derivatives for the Riemannian acoustic wave equation. The maximum Δt allowed for the 2nd order scheme is 0.0041 s and for the 4th order is 0.0037 s. The frecuency used is 40 Hz.

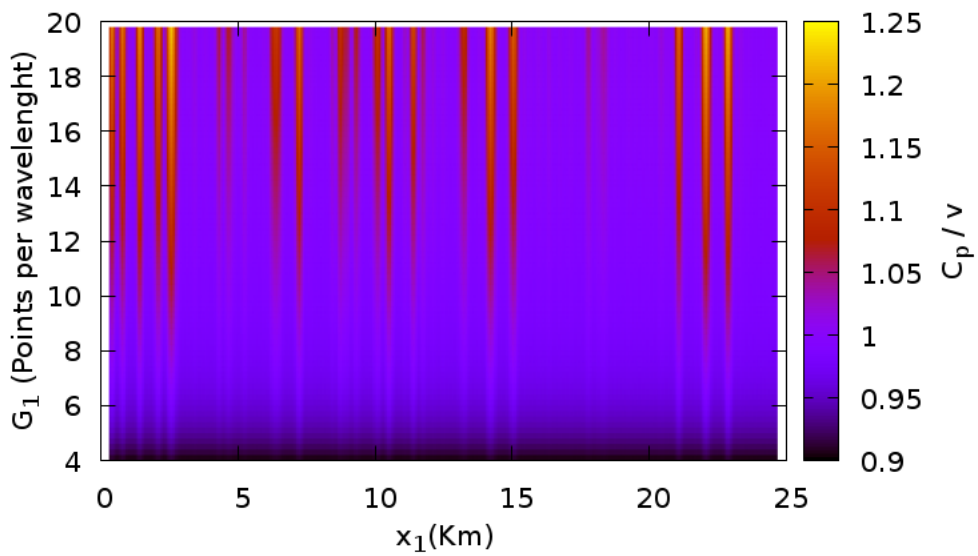


Figure 3.12: Dispersion analysis for the sub-sampled Canadian Foothills model with size 334x200. The graph shows that the normalized velocity C_p/v does not get the expected value of 1 for a large number of grid points per wavelength. This plot is for $\theta = 0$.

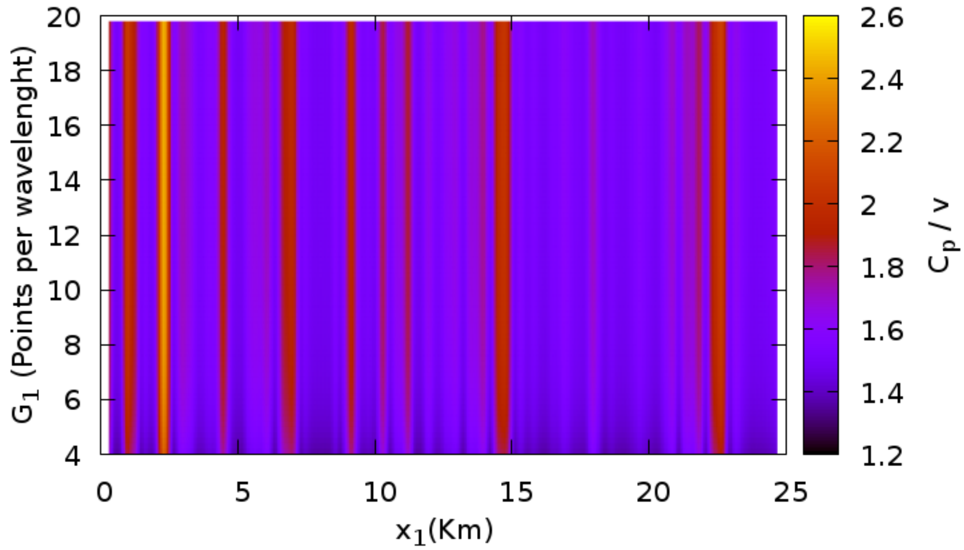


Figure 3.13: Dispersion analysis for the sub-sampled Canadian Foothills model of size 334x200. The graph shows that the normalized velocity C_p/v does not get the expected value of 1 for a large number of grid points per wavelength. This plot is for $\theta = \pi/2$.

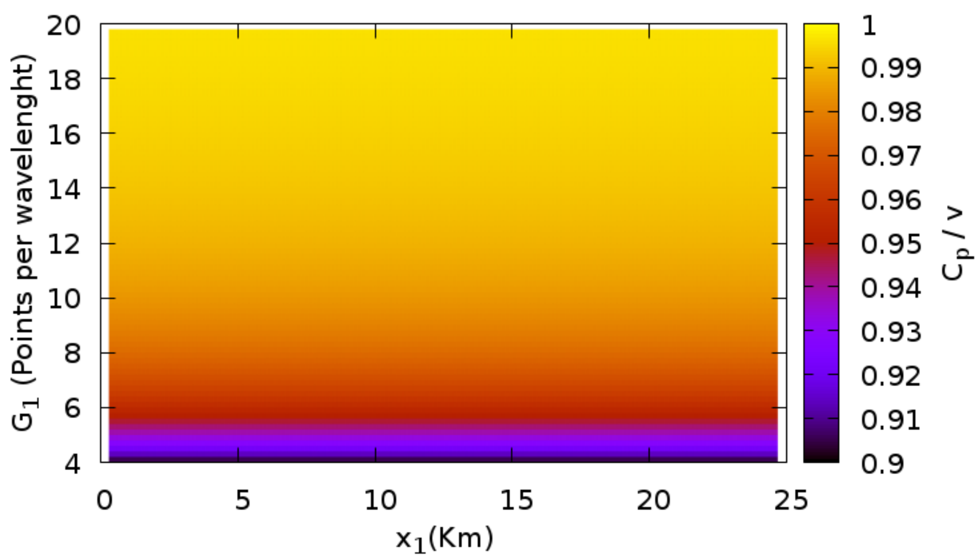


Figure 3.14: Dispersion analysis for a constant velocity model with Cartesian metric and flat surface with size 334x200. The graph shows that the normalized velocity C_p/v get the expected value of 1 for a large number of grid points per wavelength.

3.6 Conclusions

The numerical experiments show that the time step implied by the stability condition strongly depends on the degree of smoothness of the topographic profile so, to obtain

time steps suitable for calculations we must represent the topography with curves that may not perfectly match each point of the true profile.
There is a clear difference between the time step given by the heuristic stability condition and the time step given by the limit obtained rigorously.
The limits obtained for the time sampling show that the computational cost of the propagation using a FDTD scheme for the Riemannian 2D acoustic wave equationis is bigger than the same simulation with the usual acoustic wave equation (around 143%) bigger for a particular case shown.
Different transformations from the physical domain onto the computational domain, imply different metric tensors and then, different limits for the time step required for stability.
This kind of transformations imply very strong numerical dispersion which suggests that the Riemannian approach to the solution of the acoustic wave equation does not seem to be convenient for RTM or FWI.
The stability and numerical dispersion analysis for other kind of transformation of domains can be achieved using the same general expressions given in this work, just by replacing the corresponding metric tensor.

Chapter 4

The Elastic Riemannian Manifold

4.1 Introduction

The use of differential geometry allow us to study some spaces in general coordinate systems, not necessarily orthogonal curvilinear ones, which are a special case of coordinate maps, and then we can generalize concepts as directional derivative, integral theory, energy functionals, least action curves (geodesics), etc. So, we can study spaces which as a hole, looks complex in geometry, but locally are like the model space which is well known as an Euclidean space, Banach space, Orlicz space, among others. In the theory of elasticity the role of Riemannian geometry is fundamental, in the sense that many of the concepts involved in the theory are modeled or described by fundamental concepts of Riemannian geometry. For example, the work done in bending a flexible rod is proportional to the integral of the square of the curvature along the rod, and the rod itself should be modeled as a curve in $\mathcal{R}^2$. The concept of Lie derivative occurs in elasticity theory in computations of stress rates; this is related to the concept of the rate of deformation tensor, which is merely the Lie derivative of the Riemannian metric. The Cauchy-Green tensor must be studied as the pull-back of the Riemannian metric on space by the deformation, and this important tensor generalizes the stress tensor defined in the Cartesian picture. A last, by definitely not least, motivation for this formalism is to recognize that real elastic waves propagate through many layers which do not necessarily are located in a good order, say horizontally or vertically, therefore the need to extend the elastic wave equations on Euclidean spaces to equations on Riemannian spaces is fundamental. For a deep study on the formulation of elasticity theory in the context of differential geometry, the reader is referred to [33], [34], [35], [36], among others.

The formulation of the theory of elasticity can be done in the formalism of differential geometry leading to balance laws, constitutive equations, nonlinear elasticity and its linearization; nevertheless the Lagrangian approach and the calculus of variations, lead us to the equations of motions associated to a particular phenomena, after an adequate formulation of the Lagrangian functional. This formulation allows to write the equations of motion in a general way on which the form of the equations is invariant since they depend on the way we define a functional, called the action, and after the application of the principle of least action we obtain the Euler-Lagrange equations which are the equations of motion for a particular system. This formulation leads to non-dynamical symmetries arising from the way in which we formulate the action, and this symmetries give rise to mathematical identities. For a deep study of Lagrangian formulation of mechanics and variational principles in mechanics, see [38], [39], [40]. Symmetry transformations are changes in the coordinates or in the variables that leave the action invariant and the connection of the symmetries with the conserved quantities, through Noether's theorem, can lead to an algebraic solution of the equations of motion, see [41], [42]. It shows the need of a Lagrangian formulation of the elastic wave equation in a Riemannian manifold.

The chapter starts with the basics concepts of elasticity in the context of Riemannian manifolds. In section 4.3 we show the work of Yasutomi on which is presented the elastic wave equation on a Riemannian manifold for the isotropic case. In section 4.4 we give the necessary concepts for a Lagrangian formulation on a Riemannian manifold, see [37], and we end the chapter with the proposal of a Lagrangian functional, that describes propagation of elastic waves in a Riemannian manifold, and we study its symmetries by the application of a Noether's condition on a Riemannian manifold.

4.2 The Body and Configuration Manifolds

Let a solid continuum body to be an open subset of three-dimensional Euclidean space. We follow the general presentation of Marsden, [33], and analyze the mathematical description of the deformation and motions of an elastic material. Every deformation of a simple body leads to a new configuration space and a motion is a time-dependent family of configurations. We describe this concepts and the ones derived from them, in terms of Riemannian manifolds.

Definition 4.2.1. *A simple body is an open set $\mathfrak{B} \subset \mathcal{R}^3$, on which a configuration is a map $\phi : \mathfrak{B} \to \phi(\mathfrak{B})$.*

A configuration represents a deformed state of the body, then the set of all configurations is a subset of $\mathcal{R}^3$. As the body moves, we obtain a family of configurations depending on time.

Definition 4.2.2. *A motion is a curve on the space of configurations, i.e, a map $t \to \phi_t$, for $t \in \mathcal{R}$. The configurations and motions of a simple body are glued together in a map $\phi(X, t) : \mathfrak{B} \times \mathcal{R} \to \phi(\mathfrak{B})$.*

To get to the geometric picture we consider $\mathfrak{B}$ as an n-manifold with coordinate chart $(\psi, U.)$ This means that for every $X \in U \subset \mathfrak{B}$ there exists a homeomorphism $\psi : U \to \Omega \subset \mathcal{R}^n$ such that $\psi(X) = Z$, which in components looks like $Z^i = \psi^i(X^j)$; and the change of coordinates is of some regularity. For $\mathfrak{B} \subset \mathcal{R}^n$, the tangent space to $\mathfrak{B}$ at a point X is simply the vector space $\mathcal{R}^n$ as vectors coming from the point X; it is denoted as $T_X\mathfrak{B}$. The coordinate charts induces a basis on the tangent space as follow:

Let $Z \in \Omega$, then

$$\begin{aligned} Z^A &= \psi^A(X^j) \\ dZ &= dZ^A \hat{I}_A, \quad \text{canonical} \\ dZ &= \frac{\partial \psi^A}{\partial X^j} dX^j \hat{I}_A, \end{aligned}$$

then the induced basis on the tangent space $T_X\mathfrak{B}$ is given by

$$\hat{E}_j = \frac{\partial \psi^A}{\partial X^j} \hat{I}_A. \tag{4.1}$$

It is clear that if $W^j_{\mathfrak{B}}$ are the components of a tangent vector on $T_X\mathfrak{B}$ and $W^A_{\mathcal{R}^n}$ are the corresponding components on the modeling space, we have that

$$W^A_{\mathcal{R}^n} = W^j_{\mathfrak{B}} \frac{\partial \psi^A}{\partial X^j} \tag{4.2}$$

Definition 4.2.3. *Let $\phi(X,t)$ be a regular C^2 motion and define the material velocity and acceleration as*

$$\begin{aligned} V_t &: \mathfrak{B} \to \mathcal{R}^n & (4.3) \\ V_t(X) &= \frac{\partial \phi}{\partial t} & (4.4) \\ A_t &: \mathfrak{B} \to \mathcal{R}^n & (4.5) \\ A_t(X) &= \frac{\partial V(X,t)}{\partial t}, & (4.6) \end{aligned}$$

if the derivatives exist.

Note that $V(X,t)$ and $A(X,t)$ are evaluated at X and then lie on a manifold $\mathcal{S}$ such that $\phi(\mathfrak{B}) \subset \mathcal{S}$. Points on $\mathcal{S}$ are denoted by lowercase letters x and the coordinate chart by $\theta(x) = z \in \mathcal{R}^n$, then the induced basis on the tangent space $T_x\mathcal{S}$ is given by

$$\hat{e}_j = \frac{\partial \theta^a}{\partial x^j} \hat{i}_a.$$

Working out the components of $\mathbf{A}$ on $\mathcal{S}$ we have that:

$$\mathbf{A} = \left[\frac{\partial V^a_{\mathcal{S}}}{\partial t} + \frac{\partial^2 z^i}{\partial x^b \partial x^c} \frac{\partial x^a}{\partial z^i} V^b_{\mathcal{S}} V^c_{\mathcal{S}} \right] \hat{e}_a,$$

where $x = \theta^{-1}(z)$ and the second term are clearly the Christoffel symbols on $\mathcal{S}$ given on coordinates by

$$\gamma^a_{bc} = \frac{\partial^2 z^i}{\partial x^b \partial x^c} \frac{\partial x^a}{\partial z^i}.$$

Definition 4.2.4. *Let $\phi(X,t)$ be a regular C^2 motion and define the spatial velocity and acceleration on $\mathcal{S}$ by*

$$\begin{aligned} \mathbf{v}_t &= \mathbf{V}_t \circ \phi^{-1} & (4.7) \\ \mathbf{a}_t &= \mathbf{A}_t \circ \phi^{-1}. & (4.8) \end{aligned}$$

Note that for the acceleration $\mathbf{A} = \left[\frac{\partial V^a_{\mathcal{S}}}{\partial t} + \gamma^a_{bc} V^b_{\mathcal{S}} V^c_{\mathcal{S}}\right] \hat{e}_a$, we have that

$$\begin{aligned} \mathbf{a} &= \frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{v}} \mathbf{v} \\ a^a &= \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma^a_{bc} v^b v^c. \end{aligned}$$

This vector fields will be relevant latter.

Definition 4.2.5. *Let $\mathfrak{B}$ be open and $\mathcal{S} = \mathcal{R}^n$. If $\phi : \mathfrak{B} \to \mathcal{S}$ is C^1, the tangent map of ϕ is*

$$T\phi : T\mathfrak{B} \to T\mathcal{S}, \quad \text{where} \quad T\phi(X, W) = (\phi(X), D\phi(X) \cdot W). \tag{4.9}$$

Note that the following diagram commutes

$$\begin{array}{ccc} T\mathfrak{B} & \xrightarrow{T\phi} & T\mathcal{S} \\ \downarrow{\scriptstyle \pi_{\mathfrak{B}}} & & \downarrow{\scriptstyle \pi_{\mathcal{S}}} \\ \mathfrak{B} & \xrightarrow{\phi} & \mathcal{S} \end{array}$$

where $\pi_{\mathcal{A}}$ are the projections $T\mathcal{A} \to \mathcal{A}$.

Proposition 4.2.1. *Let $\{\psi^A\}$ and $\{\theta^a\}$ be coordinate systems for $\mathfrak{B}$ and $\mathcal{S}$ resp. For $W \in T_X\mathfrak{B}$ the components of the tangent map are*

$$(T\phi \cdot W_X)^a_{\mathcal{S}} = \frac{\partial \phi^a_{\mathfrak{B}}}{\partial X^j} W^j_{\mathfrak{B}}.$$

The proof is straightforward, noting that the coordinate representation of the motion is $\phi^a_{\mathcal{R}^n} = \theta^a \phi^b_{\mathfrak{B}} (\psi^{-1})^j$.

Proposition 4.2.2. *If $\lambda(t)$ is a curve in $\mathfrak{B}$ and $W_X = \lambda(0)$ then*

$$T\phi \cdot W_X = \frac{d}{dt} \phi(\lambda(t))|_{t=0}.$$

The tangent map is an important object for an elastic body in particular when a Riemannian structure is on the hand.

Definition 4.2.6. *The metric tensor on $\mathcal{S}$, denoted g is given by the expression*

$$g_{ij} = \frac{\partial \theta^k}{\partial x^i} \frac{\partial \theta^k}{\partial x^j}. \tag{4.10}$$

For $\mathfrak{B}$ we have

$$G_{ij} = \frac{\partial \psi^k}{\partial X^i} \frac{\partial \psi^k}{\partial X^j}. \tag{4.11}$$

and the inverses are given by the equations $g_{ij} g^{jk} = \delta^k_i$ and $G_{ij} G^{j\nu} = \delta^\nu_i$.

Proposition 4.2.3. *Denote $F = T\phi$ and the restriction of F to $T_X\mathfrak{B}$ as $F(X) : T_X\mathfrak{B} \to T_{\phi(X)}\mathcal{S}$. We already know that the representation with respect to the coordinate basis $\{\hat{e}_a\}$, $\{\hat{E}_A\}$ of $T_{\phi(X)}\mathcal{S}$ and $T_X\mathfrak{B}$, resp., are*

$$F^a_A(X) = \frac{\partial \phi^a_{\mathfrak{B}}}{\partial X^A}.$$

If ϕ is a C^1 configuration of $\mathfrak{B}$, then the matrix of the adjoint F^T is given by

$$(F^T(x))^A_a = g_{ab}(x) F^b_B(X) G^{AB}(X), \tag{4.12}$$

where $x = \phi(X)$.

Definition 4.2.7. *The deformation tensor (right Cauchy-Green tensor)* $\mathbf{C}$ *is defined as:*

$$\mathbf{C}(X) : T_X\mathfrak{B} \to T_X\mathfrak{B}, \qquad \mathbf{C}(X) = F^T(X)F(X). \tag{4.13}$$

Proposition 4.2.4. *Let ϕ be a C^1 configuration*

1. *In the coordinate systems $\{\psi\}$ and $\{\theta\}$ we have*

$$C^A_B = (F^T)^A_a F^a_B = g_{ab} G^{AC} \frac{\partial \phi^b_{\mathfrak{B}}}{\partial X^C} \frac{\partial \phi^a_{\mathfrak{B}}}{\partial X^B}.$$

2. $\mathbf{C}$ *is symmetric and positive-semidefinite.*

Definition 4.2.8. *Let ϕ be a regular C^1 configuration. The finger deformation tensor (left Cauchy-Green tensor) is defined as*

$$\mathbf{b}(x) : T_x\phi(\mathfrak{B}) \to T_x\phi(\mathfrak{B}), \qquad \mathbf{b}(x) = F(X)F^T(X), \tag{4.14}$$

where $X = \phi^{-1}(x)$.

For this tensor we have.

Proposition 4.2.5. *If ϕ is regular and C^1 then:*

1. $b^a_b = g_{bc} G^{AB} \frac{\partial \phi^a_{\mathfrak{B}}}{\partial X^A} \frac{\partial \phi^a_{\mathfrak{B}}}{\partial X^B}$.

2. $\mathbf{b}$ *is symmetric and positive-definite.*

An important fact regarding the right and left Cauchy-Green tensors is the singular values decomposition of the deformation tensor and it is associated with the square root tensors $\mathbf{U}$, $\mathbf{V}$ of $\mathbf{C}$ and $\mathbf{b}$ res. Note that $\mathbf{U}(X) : T_X\mathfrak{B} \to T_X\mathfrak{B}$, and $\mathbf{V}(x) : T_x\mathcal{S} \to T_x\mathcal{S}$.

Proposition 4.2.6. *Let ϕ be regular. For each $X \in \mathfrak{B}$ there exists an orthogonal transformation $\mathbf{R}(X) : T_X\mathfrak{B} \to T_x\mathcal{S}$, i.e, $\mathbf{R}^T(X)\mathbf{R}(X) = id_{T_X\mathfrak{B}}$ and $\mathbf{R}(X)\mathbf{R}^T(X) = id_{T_x\mathcal{S}}$, such that:*

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \qquad \mathbf{F}(X) = \mathbf{R}(X)\mathbf{U}(X) \tag{4.15}$$

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \qquad \mathbf{F}(X) = \mathbf{V}(x)\mathbf{R}(X) \tag{4.16}$$

and each decomposition is unique.

This proposition shows that $\mathbf{U}$ and $\mathbf{V}$ are similar transformations, and that the corresponding eigenvalues can be defined in terms of $\sqrt{\mathbf{F}^T\mathbf{F}}$. A geometric property is provided by the following proposition which shows that the stretch tensor $\mathbf{U}$, respectively the right Cauchy-Green tensor $\mathbf{C}$ contains the information about deformed curves, changes in lengths and angles due to a deformation. This is the same information contained in the stress tensor for the Euclidean case, see [43].

Proposition 4.2.7. *Let σ be a C^1 curve in $\mathfrak{B}$ and ϕ a C^1 configuration of $\mathfrak{B}$ in $\mathcal{S}$. Let $\tilde{\sigma} = \phi \circ \sigma$.*

1. *The length of $\tilde{\sigma}$ depends only on σ and the stretch tensor $\mathbf{U}$, say*

$$||\tilde{\sigma}'(t)|| = ||\mathbf{U}_{\sigma(t)}\sigma'(t)||.$$

2. *If σ_1 and σ_2 are C^1 curves such that $\sigma_1(t_1) = \sigma_2(t_2) = X$. Then the angle θ between the deformed curves $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ is given by*

$$\cos\theta = \frac{\langle \mathbf{U}_X\sigma_1'(t_1), \mathbf{U}_X\sigma_2'(t_2)\rangle_X}{||\mathbf{U}_X\sigma_1'(t_1)|| \cdot ||\mathbf{U}_X\sigma_2'(t_2)||}.$$

Assume the existence of a mass density function ρ.

Definition 4.2.9. 1. *Let $\mathfrak{B}$ be a simple body and $\phi(X,t)$ a motion. A function $\rho(x,t)$ is said to obey conservation of mass if for all open sets $\mathfrak{U} \subset \mathfrak{B}$ with C^1 piecewise boundary,*

$$\frac{d}{dt}\int_{\phi(\mathfrak{U})} \rho(x,t)dv = 0,$$

where dv is the Euclidean volume element.

2. *The determinant of the linear transformation $\mathbf{F}(X,t)$ is called the Jacobian and denoted $\mathbf{J}(X,t)$.*

In this Definition we are assuming that $\mathfrak{B} \subset \mathcal{R}^3$ and $\mathcal{S} = \mathcal{R}^3$; then the motions are embeddings.

Proposition 4.2.8. *Assume ϕ_t is a C^1 regular motion. Then the following are equivalent:*

i. *ϕ_t is volume preserving;*

ii. *$\mathbf{J}(X,t) = 1$;*

iii. *div $\mathbf{v} = 0$, where $\mathbf{v}$ is the spatial velocity.*

Proposition 4.2.9. *Assume ϕ_t is a C^1 regular motion and $\rho(x,t)$ is a C^1 function, then the following are equivalent*

i. *Conservation of mass;*

ii. *$\rho(x,t)\mathbf{J}(X,t) = \rho(X,0)$ where $x = \phi(X,t)$;*

iii. the equation of continuity,

$$\frac{\partial \rho}{\partial t} + div\ (\rho \mathbf{v}) = 0.$$

Assume that $\mathfrak{B}$ and $\mathcal{S}$ are of the same dimension, and that $f(x,t) : \phi_t(\mathfrak{U}) \times \mathcal{R}_+ \to \mathcal{R}$ is C^1, then

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} f dv = \int_{\phi_t(\mathfrak{U})} \left(\dot{f} + f\ div\ \mathbf{v}\right) dv = \int_{\phi_t(\mathfrak{U})} \left(\frac{\partial f}{\partial t} + div\ (f\mathbf{v})\right) dv. \tag{4.17}$$

This result (transport theorem) is easily proved by the change of variables formula and recalling that the material time derivative $\dot{f} = \frac{\partial f}{\partial t} + df \cdot \mathbf{v}$.
An important concept which allows to prove Cauchy's theorem is the master balance law.

Definition 4.2.10. Master Balance Law in $\mathcal{S}$
Let $a(x,t)$, $b(x,t)$ be scalar functions defined on $\phi_t(\mathfrak{U}) \times \mathcal{R}_+$ and $\mathbf{c}(x,t)$ be a vector field. We say that a, b and $\mathbf{c}$ satisfy the master balance law if and only if

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} a dv = \int_{\phi_t(\mathfrak{U})} b dv + \int_{\partial\phi_t(\mathfrak{U})} < \mathbf{c}, \mathbf{n} > ds,$$

where $\mathbf{n}$ is the unit normal outward to $\partial\phi_t(\mathfrak{U})$ and ds is the area element on this surface.

It is easily proved, by the transport theorem, that the master balance law is equivalent to

$$\frac{\partial a}{\partial t} + div\ (a\mathbf{v}) = b + div\ \mathbf{c}.$$

If we incorporate mass density ρ and assuming conservation of mass, the transport theorem takes the form:

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} f\rho dv = \int_{\phi_t(\mathfrak{U})} \dot{f}\rho dv,$$

and defining $a = \rho\tilde{a}$ and $b = \rho\tilde{b}$ in the master balance law, then

$$\rho\dot{\tilde{a}} = \rho\tilde{b} + div\ \mathbf{c}.$$

Proposition 4.2.10. *Cauchy's Theorem*
Let $a(x,t)$ be C^1, and $b(x,t)$, $c(x,t,\mathbf{n})$ be continuous functions for all $t \in \mathcal{R}$, $x \in \phi(\mathfrak{B})$ and unit vectors $\mathbf{n}$ at x. Assume that a, b and c satisfy the master balance law as

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} a dv = \int_{\phi_t(\mathfrak{U})} b dv + \int_{\partial\phi_t(\mathfrak{U})} c(x,t,\mathbf{n}) ds,$$

then there exist a unique vector field $\mathbf{c}(x,t)$ on $\phi_t(\mathfrak{U})$ such that

$$c(x,t,\mathbf{n}) = < \mathbf{c}(x,t), \mathbf{n} > .$$

Assume the existence of a vector field $\tau(x,t,\mathbf{n})$ which implicitly depends on the motion ϕ. We say that balance of momentum is satisfied if for $\phi(X,t)$, $\rho(x,t)$, $\tau(x,t,\mathbf{n})$ and $\mathbf{b}(x,t)$ given we have

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} \rho\mathbf{v} dv = \int_{\phi_t(\mathfrak{U})} \rho\mathbf{b} + \int_{\partial\phi_t(\mathfrak{U})} \tau ds,$$

note that in this definition, the vector field $\mathbf{b}$ represents an external force field.

Proposition 4.2.11. *Assume that balance of momentum holds, ϕ is C^1 and τ is a continuous function of its arguments. There is a unique $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor field $\sigma(x,t)$ such that*

$$\tau(x,t,\mathbf{n}) =< \sigma(x,t), \mathbf{n} > . \tag{4.18}$$

In coordinates $\{\theta^a\}$ on $\mathcal{S}$, we have

$$\tau^a(x,t,\mathbf{n}) = g_{bc}\sigma^{ac}(x,t)n^b = \sigma^a_b n^b. \tag{4.19}$$

The tensor σ^{ab} is called the *Cauchy stress tensor.*

Proposition 4.2.12. *Assume that balance of momentum and conservation of mass hold. Then*

$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} + div\ \sigma. \tag{4.20}$$

The Cauchy stress tensor is symmetric in the sense that $\sigma^{ab} = \sigma^{ba}$. This important property is equivalent to the balance of moment of momentum which is stablished as an independent hypothesis, let $\mathbf{x}$ denote the position vector to the point x, then balance of moment of momentum is satisfied if

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} \rho(\mathbf{x}\times\mathbf{v})dv = \int_{\phi_t(\mathfrak{U})} \rho(\mathbf{x}\times\mathbf{b})dv + \int_{\partial\phi_t(\mathfrak{U})} \mathbf{x}\times < \sigma, \mathbf{n} > ds.$$

We define balance of energy, which allows us to obtain an important equation relating the cauchy stress tensor with the metric tensor defined on $\mathcal{S}$.
Assume the existence of functions $e(x,t)$ representing mechanical energy stored internally in the body, $r(x,t)$ representing incoming heat energy and $h(x,t,\mathbf{n})$ representing the rate of heat conduction across a surface with unit normal $\mathbf{n}$. We say that balance of energy holds iff

$$\frac{d}{dt}\int_{\phi_t(\mathfrak{U})} \rho(e + \frac{1}{2} < \mathbf{v}, \mathbf{v} >)dv = \int_{\phi_t(\mathfrak{U})} \rho(< \mathbf{b}, \mathbf{v} > + r)dv + \int_{\partial\phi_t(\mathfrak{U})} (< \tau, \mathbf{v} > + h)ds.$$

This states, see [33], that the rate of increase of the total energy of any portion of the deformed body equals the rate of work done on that portion plus the rate of increase of heat energy. This priciple makes sense on manifolds and can be used as a covariant basis for elasticity theory.
Let $\mathcal{O}_g$ denote the orbit of g defined as

$$\mathcal{O}_g = \{\eta^* g | \eta : \mathcal{S} \to \mathcal{S} \quad \text{is a diffeomorphism}\},$$

since the changes of metrics on $\mathcal{S}$ affect the accelerations of the particles, the internal energy E must depend parametrically on the metric, say $e(x,t,g)$. This function is a differentiable function of $\tilde{g} \in \mathcal{O}_g$, then it depends only on the point values of $\tilde{g}$.

Proposition 4.2.13. *Let $(\mathcal{S}, g)$ be a Riemannian manifold and $\phi_t : \mathfrak{B} \to \mathcal{S}$ a motion. Let $\rho, e, ..$ such that satisfy balance of energy and that for every Newtonian slicing and associated maps $\xi_t; \mathcal{S} \to \mathcal{S}$, the energy functional is $e'(x,t,\xi_t^* g)$ balance of energy holds in every slicing. Then there exist $\sigma, \mathbf{q}$ such that $\tau =< \sigma, \mathbf{n} >$ and $h = - < \mathbf{q}, \mathbf{n} >$, and* *i.* *conservation of mass,*

ii. *balance of momentum,*

iii. *balance of moment momentum,*

iv. *balance of energy, and*

v. $\sigma = 2\rho\left(\frac{\partial e}{\partial g}\right)$.

This proposition leads to an invariant formulation of elasticity.

4.3 Elastic Wave Equation for an Isotropic Riemannian Manifold

In this section we show the work of Yasutomi [44], on which an equation for wave propagation in Riemannian manifolds is proposed and the solution is decomposed into polarizations modes using the theory of pseudodifferential operators on the manifold. We restrict this presentation to the Riemannian case, the reader interested in the Complex and Kähler cases must see the reference.
Let $(\mathcal{M}, g)$ be an n-dimensional Riemannian manifold. Consider an elastic body G in $\mathcal{M}$, and $\widetilde{\mathcal{M}} := \mathcal{M}\times\mathcal{R}_+$. A motion of G is identified with an open subset $\tilde{G}\subset\tilde{\mathcal{M}}$ with a one parameter family of diffeomorphism ϕ_t.
Consider the motion given by

$$\phi_t^i(x) = x^i + u^i(x,t)$$

in a local coordinate system $(x^1, ...x^n)$, where $u = \sum u^i(x,t)\hat{e}_i$ is a small displacement vector field. Clearly:

$$\begin{aligned} d\phi : T_x\mathcal{M} &\rightarrow T_{\phi(x)}\mathcal{M} \\ d\phi(\xi^i) = \xi^j\left(\frac{\partial\phi^i}{\partial x^j}\right) &= \xi^i + \frac{\partial u^i}{\partial x^j}\xi^j = \eta^i \end{aligned}$$

and its dual

$$\begin{aligned} d\phi^* : T^*_{\phi(x)}\mathcal{M} &\rightarrow T^*_x\mathcal{M} \\ d\phi^*(\eta_i) = \eta_i\left(\frac{\partial\phi^i}{\partial x^j}\right) &= \eta_j + \frac{\partial u^i}{\partial x^j}\eta_i = \xi_j. \end{aligned}$$

The strain tensor is calculated as the difference of the line elements after the motion and before. The deformed line element is the pull-back of the line element by the motion, then

$$\varepsilon_{ij}dx^i\otimes dx^j := \frac{1}{2}\left\{\phi^* ds(x)^2 - ds(x^2)\right\},$$

from which

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}(g_{km}\partial_l u^m + g_{ml}\partial_k u^m + u^m\partial_m g_{kl}) \\ &= \frac{1}{2}(g_{km}\nabla_l u^m + g_{ml}\nabla_k u^m). \end{aligned} \tag{4.21}$$

Note that

$$\varepsilon_{ij} = \mathfrak{L}_u g, \tag{4.22}$$

That is the Lie derivative of the metric in the direction of displacement vector field.
In the limit of small deformations we have Hooke's law, relating the stress and strain tensors as

$$\frac{\sigma^{ij}}{\sqrt{|g|}} = C^{ijkl}\varepsilon_{kl},$$

where C^{ijkl} is called the stiffness tensor. For an isotropic Riemannian elastic body it is known that:

$$C^{ijkl} = \lambda g^{ij}g^{kl} + \mu g^{ik}g^{jl} + \mu g^{il}g^{kj},$$

where λ and μ are the Láme parameters. Then, using (4.21), we have:

$$\frac{\sigma^{ij}}{\sqrt{|g|}} = \lambda g^{ij}\nabla_l u^l + \mu g^{jl}\nabla_l u^i + \mu g^{il}\nabla_l u^j.$$

Consider a neighborhood V of $x \in \mathcal{M}$, $S = \partial V$ and let df^i be the external force vector for the surface element dS_j. By Newton's law we have:

$$\int_V \rho\frac{\partial^2 u^i}{\partial t^2}dv = -\int_S df^i,$$

where dv is the volume element of the manifold. Since $df^i = \frac{\sigma^{ij}}{\sqrt{|g|}}$, and by the divergence theorem, we have *the Riemannian wave equation for a homogeneous and isotropic elastic boby* as:

$$\begin{aligned}
\rho\frac{\partial^2 u^i}{\partial t^2} &= \nabla_j\left(\frac{\sigma^{ij}}{\sqrt{|g|}}\right) = \lambda g^{ij}\nabla_j\nabla_k u^k + \mu g^{jk}\nabla_j\nabla_k u^i + \mu g^{ik}\nabla_j\nabla_k u^j, && (4.23)\\
P_{\mathfrak{R}}\, u &= 0. && (4.24)
\end{aligned}$$

To obtain the decomposition of any solution distribution we need to fix some notation.
Let $\Lambda^{(p)}(T^*\mathcal{M})$ be the vector bundle of p-differential forms on $\mathcal{M}$, $\mathcal{E}^{(p)}_{\mathcal{M}}$ be a sheaf of p-forms on $\mathcal{M}$ and $\mathcal{D}b^{(p)}_{\mathcal{M}}$ be the space of p-forms with distribution coefficients. This spaces are extended to $\widetilde{\mathcal{M}}$ in a natural way.
The operator $P_{\mathfrak{R}}$ in equation (4.24), is defined as:

$$\begin{aligned}
P_{\mathfrak{R}} : \widetilde{\mathcal{D}b^{(p)}_{\mathcal{M}}} &\to \widetilde{\mathcal{D}b^{(p)}_{\mathcal{M}}} && (4.25)\\
P_{\mathfrak{R}}\, u &= \rho\frac{\partial^2 u}{\partial t^2} + (\lambda + 2\mu)d\delta u + \mu\delta du, && (4.26)
\end{aligned}$$

where d and δ are defined like as in Definition 3.3.2, and extended to p-forms. For $u \in \widetilde{\mathcal{D}b^{(p)}_{\mathcal{M}}}$ define the equations

$$\mathfrak{M}^{\mathfrak{R}} \ :\ P_{\mathfrak{R}}\, u = 0, \tag{4.27}$$

$$\mathfrak{M}^{\mathfrak{R}}_1 \ :\ \begin{cases} P_{\mathfrak{R}}\, u = 0 \\ du = 0 \end{cases} \tag{4.28}$$

$$\mathfrak{M}^{\mathfrak{R}}_2 \ :\ \begin{cases} P_{\mathfrak{R}}\, u = 0 \\ \delta u = 0 \end{cases} \tag{4.29}$$

Lemma 4.3.1. *For any solution $u \in Sol(\mathfrak{M}^{\mathfrak{R}})$, and the set*

$$V_{\mathfrak{R}} = \bigcup_{k=1,2} \left\{(t,x;\tau,\xi)|\tau^2 - c_k g^{ij}\xi_i\xi_j = 0\right\},$$

we have $WF(u) \subset V_{\mathfrak{R}}$ where $c_1 = \alpha = \frac{(\lambda+2\mu)}{\rho}$ and $c_2 = \beta = \frac{\mu}{\rho}$.

Proof. The symbol of the second-order operator $\rho^{-1}P_{\mathfrak{R}}(t,x,\partial_t,\partial_x)$ at $(x',t') \in T^*\mathcal{M}$ is defined as the linear operator

$$\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t') : \Lambda^{(p)}T^*_{x'}\mathcal{M} \to \Lambda^{(p)}T^*_{x'}\mathcal{M}$$

as

$$\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')U = \lim_{\lambda\to\infty}\frac{1}{\lambda^2}e^{-\lambda i(<x,\xi'>+t\tau')}\rho^{-1}P_{\mathfrak{R}}(t',x',\partial_t,\partial_x)\left(e^{\lambda i(<x,\xi'>+t\tau')}U\right),$$

where the coefficients of U are assumed to be constant in a local coordinate system. Suppose $(\tau',\xi') \neq (0,0)$, then we get

$$\begin{aligned}
\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')U &= \sigma_2(\partial_t^2 + \alpha d\delta + \beta\delta d)U\\
\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')U &= -\tau'^2U + (-1)^{n(p-1)}\alpha(\tilde{\xi}\wedge(*(\tilde{\xi}\wedge *U)))\\
&+ (-1)^{np}\beta(*(\tilde{\xi}\wedge *(\tilde{\xi}\wedge U))),
\end{aligned}$$

where $\tilde{\xi} = \sum \xi'_j dx^j$. Assume a decomposition of U as $U = U_1 + U_2$ in an orthonormal basis $\{\omega\}$ of $T^*_{x'}\mathfrak{M}$, from which we get

$$\begin{aligned}
U_1 &= \sum_{1\in I} U_{1I}\omega^I\\
U_2 &= \sum_{1\notin I} U_{2I}\omega^I\\
\tilde{\xi}\wedge U_1 &= 0\\
\tilde{\xi}\wedge *U_2 &= 0\\
*(\tilde{\xi}\wedge *(\tilde{\xi}\wedge U)) &= (-1)^{np}|\tilde{xi}|^2U_2\\
\tilde{\xi}\wedge(*(\tilde{\xi}\wedge *U)) &= (-1)^{n(p-1)}|\tilde{\xi}|^2U_2\\
\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')U &= (-\tau'^2 + \alpha|\tilde{\xi}|^2)U_1 + (-\tau'^2 + \beta|\tilde{\xi}|^2)U_2,
\end{aligned}$$

then the eigenvalues of σ_2 are the coefficients in the last equation and

$$det[\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')] = (-\tau'^2 + \alpha|\tilde{\xi}|^2)^{{}_{n-1}C_{p-1}}(-\tau'^2 + \beta|\tilde{\xi}|^2)^{{}_{n-1}C_p}.$$

If $\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')$ is an isomorphism, then (x',t') does not belong to the characteristic variety of $P_{\mathfrak{R}}\,u = 0$. Then for a characteristic point (x',t') we have $det[\sigma_2(\rho^{-1}P_{\mathfrak{R}})(x',t')] = 0$, hence, $WF(u) \subset V_{\mathfrak{R}}$. □

Lemma 4.3.2. *For every $u \in \widetilde{\mathcal{D}b^{(p)}_{\mathcal{M}}}$, there exist $\omega \in \widetilde{\mathcal{D}b^{(p)}_{\mathcal{M}}}$ such that $\Delta\omega = u$.*

Proof. For $u = \sum u_i(x,t)dx^i$, write $\Delta u = \sum[P_{ij}(x,\partial_x)u_j(x,t)]dx^i$.
Since Δ is an elliptic operator on $\mathcal{D}b^{(p)}_{\mathcal{M}}$, then in a neighborhood of (x',t') there exist integral kernels $[K_{j\mu}(x,y)]_{j\mu}$ such that

$$\begin{aligned} \sum P_{ij}(x,\partial_x)K_{j\mu}(x,y) &= \delta^i_\mu \cdot \delta(x-y) \\ WF(K_{j\mu}) &\subset \{(x,y;\xi,\eta)|x=y, \quad \xi=-\eta\}, \end{aligned}$$

where δ^i_μ is a Kronecker's delta. Now take

$$\omega = \sum \left(\int K_{j\mu}(x,y)u_\mu(x,t)\psi(y)dy \right) dx^i,$$

where $\psi \in C^\infty_0(\mathfrak{M})$ is a cut-off function with $\psi(y) = 1$ near x'; then it follows that $\Delta\omega = u$. □

This lemmas allow us to prove the next theorem, on which a decomposition of $u \in Sol(\mathfrak{M}^{\mathfrak{R}})$ is achieved.

Theorem 4.3.1. *For any $u \in Sol(\mathfrak{M}^{\mathfrak{R}})|_{(x',t')}$ there exist some $u_j \in Sol(\mathfrak{M}^{\mathfrak{R}}_j)|_{(x',t')}$ with $j = 1,2$ such that $u = u_1 + u_2$.*

Proof. Let $u \in \widetilde{\mathcal{D}b^{(p)}_{\mathcal{M}}}$, $u_2 = \delta v$ and $u_1 = u - \delta v$, where $v \in \widetilde{\mathcal{D}b^{(p+1)}_{\mathcal{M}}}$.
Since $P_{\mathfrak{R}}(\delta v) = (\partial_t^2 + \beta\delta d)\delta v = \delta(\partial_t^2 + \beta\Delta)v$ holds, we impose the following conditions

$$\begin{cases} \partial_t^2 v = -\beta du, \\ \Delta v = du, \\ dv = 0. \end{cases}$$

From the previous lemmas, we have ω such that $\Delta\omega = u$ and $(x',t';\pm dt) \notin WF(\omega)$. It is sufficient to have :

$$\begin{cases} \partial_t^2(v - d\omega) = -d(\beta u + \partial_t^2\omega), \\ \Delta(v - d\omega) = 0, \\ d(v - d\omega) = 0. \end{cases}$$

Therefore we take v as

$$v - d\omega = -d\int_{t'}^{t} ds \int_{t'}^{s} (\beta u(x,\tau) + \partial_t^2\omega(x,\tau))d\tau.$$

Then u_2 satisfies the equation

$$\rho\frac{\partial^2}{\partial t^2}u_2 + \beta\Delta u_2 = 0,$$

and the theorem is proved. □

We note that for the case of 1-forms ($p = 1$), this decomposition means $u^i = u^i_1 + u^i_2 \in \widetilde{\mathcal{D}b^{(1)}_{\mathcal{M}}}$ satisfying the conditions

$$\nabla_i \ u^i_1 = 0, \quad \nabla^i \ u^j_2 - \nabla^j \ u^i_2 = 0.$$

4.4 Lagrangian Formalism on Riemannian Manifolds

The steps we usually use for a Lagrangian formulation for any mechanical system are:

- Find a suitable Lagrangian, which in the simplest case is the difference between the kinetic and the potential energy of the system.
- Write down the Euler -Lagrange equations, the Hamilton equations and the Hamilton-Jacobi equation.
- Choose one of the above equations which can be studied from the point of view of existence, uniqueness, and regularity of solutions.

Consider a functional defined on the tangent bundle of a configuration space

$$\mathfrak{L}(x, \dot{x}, t) : \mathcal{C} \to \mathcal{R},$$

and the action integral:

$$S = \int_{t_1}^{t_2} \mathfrak{L} dt.$$

We need to solve the problem

$$\delta S = 0.$$

The latter equation means that the action is stationary when the trajectory is the actual trajectory followed by the system. After solving the last equation we get the Euler-Lagrange equations for a single particle

$$\frac{\partial \mathfrak{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right).$$

For the Riemannian formulation, consider $\psi : \mathcal{M} \to \mathcal{N}$ be a map between Riemannian manifolds $(\mathcal{M}, G)$ and $(\mathcal{N}, g)$.

Definition 4.4.1. *Let $D \subset \mathcal{M}$ be bounded and closed. A variation of ψ on D is a one-parameter family of functions $\psi(s, x)$ where $s \in (-\epsilon, \epsilon)$, $x \in \mathcal{M}$ such that*

$$\begin{aligned} \psi(0, x) &= \psi(x), \\ \psi(s, x) &= \psi(x), \qquad \forall x \in \mathcal{M} \backslash D. \end{aligned}$$

Denote

$$\delta \psi^i(x) = \frac{\partial \psi^i(s, x)}{\partial s}|_{s=0}.$$

Definition 4.4.2. *A lagrangian is a function $\mathfrak{L} : T\mathcal{N} \to \mathcal{R}$. The Lagrangian associated to a map $\psi : \mathcal{M} \to \mathcal{N}$ is a scalar function of ψ and the components of the push-forward of ψ evaluated on the basis tangents of $T\mathcal{M}$.*

Definition 4.4.3. *The integral*

$$I = \int_D \mathfrak{L} dv_g, \tag{4.30}$$

is called stationary under any variation of ψ *if*

$$\frac{dI}{ds}|_{s=0} = 0.$$

Where dv_g *is the volume element on* $\mathcal{N}$.

Proposition 4.4.1. *The integral (4.30) is stationary under any variation of* ψ *if and only if the following equations are satisfied*

$$\sum_{k=1}^{m} \left(\frac{\partial \mathfrak{L}}{\partial(\psi^i_{;k})} \right)_{;k} = \frac{\partial \mathfrak{L}}{\partial \psi^i}, \tag{4.31}$$

where $m = dim(\mathcal{M})$ *and* $\psi^i_{;k}$ *are the components of the vector field* $\psi_* \hat{E}$.

Proof. By the chain rule

$$\frac{dI}{du}|_{u=0} = \sum_i \int_D \left[\frac{\partial \mathfrak{L}}{\partial \psi^i} \delta\psi^i + \frac{\partial \mathfrak{L}}{\partial(\psi^i_{;e})} \delta(\psi^i_{;e}) \right] dv_g.$$

As $\delta(\psi^i_{;e}) = (\delta\psi^i)_{;e}$ and by Leibnitz's rule we have that the second term is:

$$\sum_i \int_D \left(\left[\frac{\partial \mathfrak{L}}{\partial(\psi^i_{;e})} \delta\psi^i \right]_{;e} - \left[\frac{\partial \mathfrak{L}}{\partial(\psi^i_{;e})} \right]_{;e} \delta\psi^i \right) dv_g.$$

By the divergence theorem applied to the components of the vector field

$$X = \left(\sum_i \frac{\partial \mathfrak{L}}{\partial(\psi^i_{;e})} \delta\psi^i \right) \hat{E}_e,$$

we have

$$\int_D \left(\sum_i \frac{\partial \mathfrak{L}}{\partial(\psi^i_{;e})} \delta\psi^i \right)_{;e} dv_g = 0,$$

and then

$$\frac{dI}{du}|_{u=0} = \int_D \left[\frac{\partial \mathfrak{L}}{\partial \psi^i} - \left(\frac{\partial \mathfrak{L}}{\partial(\psi^i_{;e}} \right)_{;e} \right] \delta\psi^i dv_g,$$

for all variations of ψ. □

The system of equations (4.31), are called the Euler-Lagrange equations on the Riemannian manifold $(\mathcal{N}, g)$.
Two well known examples are the Laplace's and Poisson equation on $(\mathcal{M}, g)$. Let $f, \rho \in \mathfrak{F}(\mathcal{M})$ then, for the Lagrangian

$$\mathfrak{L} = \frac{1}{2} |\nabla f|^2,$$

the Euler-Lagrange equations are

$$\Delta f = 0;$$

and for the Lagrangian

$$\mathfrak{L} = \frac{1}{2}|\nabla f|^2 - \rho f,$$

the Euler-Lagrange equations are

$$\Delta f = \rho,$$

on the manifold.
An interesting description of the Euler-Lagrange equations on a Riemannian manifold is provided by geodesics. Let $I \subset \mathcal{R}$ be an interval, $(\mathcal{M}, g)$ a Riemannian manifold. Consider a curve $\phi : I \to \mathcal{M}$ and the Lagrangian

$$\mathfrak{L}(\phi, \dot{\phi}) = \frac{1}{2}|\dot{\phi}|_g^2 = \frac{1}{2}g_{ij}\dot{\phi}^i\dot{\phi}^j.$$

The following propositions are straightforward to prove.

Proposition 4.4.2. *The extremizers of the integral*

$$J(\phi) = \int_I \frac{1}{2}|\dot{\phi}|_g^2 dt,$$

are solutions of the equation:

$$\ddot{\phi}^l + \Gamma^l_{is}\dot{\phi}^i\dot{\phi}^s = 0. \tag{4.32}$$

Proposition 4.4.3. *Let the tangent field along the curve $\phi(t)$ be*

$$\dot{\phi}(t) = \phi_* \left(\frac{d}{dt}\right)$$

then ,

$$\nabla_{\dot{\phi}}\dot{\phi} = \left(\ddot{\phi}^l + \Gamma^l_{is}\dot{\phi}^i\dot{\phi}^s\right)\hat{E}_s.$$

The vector field $\nabla_{\dot{\phi}}\dot{\phi}$ is interpreted as acceleration along the curve, then the solution to the Euler-Lagrange equation are the curves that satisfy

$$\nabla_{\dot{\phi}}\dot{\phi} = 0,$$

and such curves are called geodesics on the Riemannian manifold $(\mathcal{M}, g)$.
Consider the Lagrangian $\mathfrak{L}(\phi, \dot{\phi}) = \frac{1}{2}g(\dot{\phi}, \dot{\phi}) - U(\phi)$, where $U : \mathcal{M} \to R$ is called the potential. The curve ϕ is an extremizer of the action

$$\int_{t_1}^{t_2} \mathfrak{L} dt,$$

if it satisfy the Newton's equation

$$\nabla_{\dot{\phi}}\dot{\phi} = -\nabla U.$$

Le us define the one-forms $\omega_\phi, w_\phi \in T^*\mathcal{M}$ as

$$\begin{aligned} \omega_\phi(V) &= g(\dot{\phi}, V), \quad \text{momentum in the } V-\text{direction}, && (4.33)\\ w_\phi(V) &= g(\nabla_{\dot{\phi}}\dot{\phi}, V), \quad \text{work in the } V\text{-direction}. && (4.34) \end{aligned}$$

Using the fact that ∇ is metric connection

$$\dot{\phi} g(\dot{\phi}, V) = g(\nabla_{\dot{\phi}}\dot{\phi}, V) + g(\dot{\phi}, \nabla_{\dot{\phi}} V),$$

we write the work in terms of momentum

$$w_\phi(V) = \dot{\phi}\omega_\phi(V) - \omega_\phi(\nabla_{\dot{\phi}} V).$$

Proposition 4.4.4. *Let $\phi(t)$ be a geodesic. Then,*

1. $w_\phi(V) = 0$,
2. $\omega_\phi(\dot{\phi})$ *is preserved along the geodesic.*

Proposition 4.4.5. *If $\phi(t)$ satisfies Newton's equation then,*

1. $w_\phi(V) = g(-\nabla U, V)$,
2. $\omega_\phi(V)$ *is constant along ϕ, where V is a killing vector field s.t* $w_\phi(V) = 0$,
3. $w_\phi(\nabla_{\dot{\phi}} V) = 0$,
4. $|\dot{\phi}|$ *is constant along ϕ if and only if U is constant along ϕ.*

The last item of the previous proposition follows directly from the equation:

$$\nabla_{\dot{\phi}} g(\dot{\phi}, \dot{\phi}) = 2g(\nabla_{\dot{\phi}}\dot{\phi}, \dot{\phi}) = 2w_\phi(\dot{\phi}).$$

4.5 Symmetries and Noether's Condition for an Elastic Body

Now we are going to propose a Lagrangian function for the elastic body to study the symmetries of the action. Symmetry (material symmetry) is a linear isomorphism $\lambda : T\mathcal{S} \to T\mathcal{S}$ which preserves the metric and can be obtained by the group of transformations which leave the action integral invariant. The connection between symmetry and conserved quantities is provided by Noether's theorem and can be studied in books like [38], [39], [40] and papers like [42], [41], among others. The aim of this section is to characterize the symmetries and conserved quantities for an elastic Riemannian body.

Let $(\mathfrak{B}, G)$, $(\mathcal{S}, g)$ be the body and configuration Riemannian manifolds as in section 4.2, with metrics given in definition 4.2.6 and ϕ a motion given by definitions 4.2.1 and 4.2.2. Assume that Proposition 4.2.13 holds.

Definition 4.5.1. *Let* $\mathbf{v}$ *be the velocity vector field given in definition 4.2.4. The Lagrangian function is defined as* $\mathfrak{L}(x, \mathbf{v}, t) : T\mathcal{S} \to \mathcal{R}$ *given by*

$$\mathfrak{L}(x, \mathbf{v}, t) = \frac{1}{2} < \mathbf{v}, \mathbf{v} >_g - e(x, t, g). \tag{4.35}$$

From the previous section we know that the Euler-Lagrange equations for this Lagrangian are

$$\nabla_{\mathbf{v}} \mathbf{v} = -\nabla e \tag{4.36}$$

$$\dot{\mathbf{v}}^i + \gamma^i_{j\nu} \mathbf{v}^j \mathbf{v}^\nu = -g^{ik} \frac{\partial e}{\partial x_k}; \quad \text{in coordinates,} \tag{4.37}$$

where $\gamma^i_{j\nu}$ are the Christoffel symbols in $\mathcal{S}$. In the next chapter we will work on the expression for a particular motion which will be understood as a Riemannian elastic wave equation.

Consider a set of transformations

$$x'_i = (id_{\mathcal{S}} + \varepsilon_1 A_{ij}) x_j, \tag{4.38}$$

$$t' = t + \varepsilon_2 Y(x, t), \tag{4.39}$$

where Y is a scalar function of position and $x \equiv x(t)$ is actually a curve in $\mathcal{S}$, say $x(t) = \phi(X, t)$ for $X \in \mathfrak{B}$ fixed. Approximating to 1^{st} order in ε_2 we get

$$x_j(t') - x_j(t) \approx \varepsilon_2 Y(x(t), t) \dot{x}_j(t) \tag{4.40}$$

$$x'_i(t') = (id_{\mathcal{S}} + \varepsilon_1 A_{ij})(x_j(t) + \varepsilon_2 Y(x(t), t) \dot{x}_j(t)) \tag{4.41}$$

$$x'_i(t') = x_i(t) + \varepsilon_2 Y(x(t), t) \dot{x}_i(t) + \varepsilon_1 A_{ij} x_j(t) \tag{4.42}$$

$$x'_i(t') = x_i(t) + \varepsilon \Psi_i(x(t), t). \tag{4.43}$$

Now, for the variation of the action

$$\delta S = \int_{t'_1}^{t'_2} \mathfrak{L}(x'(t'), \dot{x}'(t'), t') dt' - \int_{t_1}^{t_2} \mathfrak{L}(x(t), \dot{x}(t), t) dt,$$

and again, approximating to 1^{st} order, it is easy to get that for the action to be invariant, it must be satisfied the following equation

$$\sum_i \Psi_i \frac{\partial \mathfrak{L}}{\partial x_i} + (\dot{\Psi}_i - \dot{Y} \dot{x}_i) \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} + Y \frac{\partial \mathfrak{L}}{\partial t} + \dot{Y} \mathfrak{L} = 0. \tag{4.44}$$

Note that this equation provides in terms of the Lagrangian, the necessary and sufficient conditions for the action to be invariant under the transformations (4.38) and (4.39).
In terms of the motion of the body we get

$$\sum_i \Psi_i \frac{\partial \mathfrak{L}}{\partial \phi_i} + (\dot{\Psi}_i - \dot{Y} \mathbf{v}^i) \frac{\partial \mathfrak{L}}{\partial \mathbf{v}^i} + Y \frac{\partial \mathfrak{L}}{\partial t} + \dot{Y} \mathfrak{L} = 0. \tag{4.45}$$

Using the Lagrangian (4.35), we have

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{v}^i} = g_{ij}\mathbf{v}^j, \tag{4.46}$$

$$\frac{\partial \mathfrak{L}}{\partial \phi^i} = -\left(\frac{\partial e}{\partial \phi^i} + \frac{\partial e}{\partial g}\frac{\partial g}{\partial \phi^i}\right), \tag{4.47}$$

and then, Noether's condition (4.45) reads

$$-\Psi_i\left(\frac{\partial e}{\partial \phi^i} + \frac{\partial e}{\partial g}\frac{\partial g}{\partial \phi^i}\right) + \dot{\Psi}_i g_{ij}\mathbf{v}^j - \dot{Y} g_{ij}\mathbf{v}^i\mathbf{v}^j + Y\frac{\partial \mathfrak{L}}{\partial t} + \dot{Y}\mathfrak{L} = 0 \tag{4.48}$$

$$\dot{Y}(\mathfrak{L} - g_{ij}\mathbf{v}^i\mathbf{v}^j) - \Psi_i\left(\frac{\partial e}{\partial \phi^i} + \frac{\partial e}{\partial g}\frac{\partial g}{\partial \phi^i}\right) + \dot{\Psi}_i g_{ij}\mathbf{v}^j + Y\frac{\partial \mathfrak{L}}{\partial t} = 0 \tag{4.49}$$

$$-\dot{Y}\left(e + \frac{1}{2}g_{ij}\mathbf{v}^i\mathbf{v}^j\right) - \Psi_i\left(\frac{\partial e}{\partial \phi^i} + \frac{\partial e}{\partial g}\frac{\partial g}{\partial \phi^i}\right) + \dot{\Psi}_i g_{ij}\mathbf{v}^j + Y\frac{\partial \mathfrak{L}}{\partial t} = 0 \tag{4.50}$$

$$-\dot{Y}E(\phi) - \Psi_i\left(\frac{\partial e}{\partial \phi^i} + \frac{\partial e}{\partial g}\frac{\partial g}{\partial \phi^i}\right) + \dot{\Psi}_i g_{ij}\mathbf{v}^j + Y\frac{\partial \mathfrak{L}}{\partial t} = 0, \tag{4.51}$$

where $E(\phi)$ is the total energy functional.
Assume that the Euler-Lagrange equations are satisfied by ϕ, that is, ϕ is an extremizer of the action integral. One can prove that Noether's condition (4.45), reduces to

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{v}^i}(Yv^i - \Psi_i) - Y\mathfrak{L} = C \tag{4.52}$$

$$-Y(\mathfrak{L} - g_{ij}\mathbf{v}^i\mathbf{v}^j) - \Psi_i g_{ij}\mathbf{v}^j = C \tag{4.53}$$

$$YE(\phi) + \Psi_i g_{ij}\mathbf{v}^j = C, \tag{4.54}$$

where C is an arbitrary constant.

Proposition 4.5.1. *$E(\phi)$ is constant along the solutions of the Euler-Lagrange equations.*

Proof. By direct calculation we have

$$\frac{d}{dt}E(\phi) = \frac{1}{2}\frac{\partial g_{ij}}{\partial \phi^k}\dot{\phi}^k\dot{\phi}^i\dot{\phi}^j + g_{ij}\ddot{\phi}^i\dot{\phi}^j + \frac{\partial e}{\partial \phi^s}\dot{\phi}^s, \quad \text{since,}$$

$$\frac{\partial e}{\partial \phi^s} = -g_{ks}(\ddot{\phi}^k + \gamma^k_{ij}\dot{\phi}^i\dot{\phi}^j),$$

we get, $\frac{d}{dt}E(\phi) = 0$. □

With this proposition and equation (4.54), we get the following equations relating the symmetry operators as follows:

$$Y = \kappa - \frac{1}{E(\phi)}\Psi_i g_{ij}v^j, \text{ or} \tag{4.55}$$

$$\Psi_i v_i = C - YE(\phi) \tag{4.56}$$

where $\kappa = \frac{C}{E(\phi)}$.
This equations constrains the set $(\phi(X,t), \mathbf{v})$ and are used to obtain algebraic solutions

of the Euler-Lagrange equations after particular values of the operators Y and Ψ. Note that in the derivation of equation (4.43), we defined,

$$\Psi_i(\phi(X,t),t) \approx Y(\phi(X,t),t)\dot{\phi}^i(X,t) + A_{ij}\phi^j(X,t).$$

If we consider an orthotropic continuum, which possesses three orthogonal symmetry planes and taking time as absolute, we get

$$\Psi(\phi(X,t),t) = \frac{d}{dt}(\phi(X,t)) + \phi(X,t), \tag{4.57}$$

this means that for an orthotropic continuum, the space of transformations which leaves the action integral, associated to the Lagrangian (4.35) invariant, is a subspace of the space of linear differential operators on $\mathfrak{F}(\mathcal{S})$. Clearly, Ψ_{orth} is self-adjoint and continuous. If ϕ is a C^2 regular motion, then Ψ_{orth} is an element of the group of diffeomorphisms of $\mathfrak{F}(\mathcal{S})$. It is interesting to see that in the Euclidean case ($g_{ij} = \delta_{ij}$), equation (4.56) reduces to

$$(\dot{x})^2 + \dot{x}x = \gamma, \tag{4.58}$$

where $\gamma = C - E(x)$ is a constant.
This cases must be analyzed for different symmetries of a continuum, say isotropic, cubic, etc; to get relationships between the symmetry groups.

Chapter 5

Elastic Riemannian Wave Equation

5.1 Introduction

In this chapter we are going to use the Lagrangian proposed in the previous chapter to obtain the Euler-Lagrange equations for an elastic Riemannian body. This equation is going to be particularized to the Riemannian acoustic case, Chapter 3, the isotropic Riemannian elastic case, Section 4.2, and the Euclidean cases to prove the consistency of the equation.
In Section 5.3 and 5.4 we will follow an analogous procedure to the one done in [20], from which we will obtain the elastic OWWE equation, the adjoint form and subprincipal symbol of the OWWE tensor operator.

5.2 Euler-Lagrange Equations

Let $(\mathfrak{B}, G)$, $(\mathcal{S}, g)$ the body and configuration Riemannian manifolds and $\phi(X,t) : \mathfrak{B} \to \mathcal{S}$ be a C^2 regular motion. Recall that

$$\mathfrak{L}(x, \mathbf{v}, t) = \frac{1}{2} < \mathbf{v}, \mathbf{v} >_g - e(x, t, g), \tag{5.1}$$

is the Lagrangian on $T\mathcal{S}$ and that the Euler-Lagrange equations are given by

$$\sum_{k=1}^{m} \left(\frac{\partial \mathfrak{L}}{\partial (\phi^i_{;k})} \right)_{;k} = \frac{\partial \mathfrak{L}}{\partial \phi^i}, \tag{5.2}$$

according to equations (4.31) on Proposition 4.4.1. Parameter k is time and then we get

$$\frac{d}{dt} \left(\frac{\partial \mathfrak{L}}{\partial \dot{\phi}^i} \right) = g_{ij} \ddot{\phi}^j, \tag{5.3}$$

$$\frac{\partial \mathfrak{L}}{\partial \phi^i} = - \left(\frac{\partial e}{\partial \phi^i} + \frac{\partial e}{\partial g} \frac{\partial g}{\partial \phi^i} \right), \tag{5.4}$$

since $\frac{\partial e}{\partial g} = \frac{\sigma}{2\rho}$ by Proposition 4.2.13, we have the equation of motion:

$$g_{ij} \ddot{\phi}^j = - \left(\frac{\partial e}{\partial \phi^i} + \frac{\sigma^{ij}}{2\rho} \frac{\partial g_{ij}}{\partial \phi^i} \right). \tag{5.5}$$

Let the strain-energy functional be given by

$$e(\phi, t, g) = Trace_g\ [\varepsilon(C\varepsilon)], \tag{5.6}$$

where C is the stiffness tensor. Then equation (5.5) reads:

$$2\rho g_{ij}\ddot{\phi}^j + 2\rho\frac{\partial}{\partial\phi^i}(Trace_g\ [\varepsilon(C\varepsilon)]) + \left[\sigma^{ij}\frac{\partial g_{ij}}{\partial\phi^i}\right] = 0. \tag{5.7}$$

We call this equation, *The Elastic Riemannian Wave Equation.*
The term, $Trace_g\ [\varepsilon(C\varepsilon)]$, is seen in components as

$$Trace_g\ [\varepsilon(C\varepsilon)] = g^{im}g^{lj}\varepsilon_{ml}C_{ijkl}\varepsilon_{kl} \tag{5.8}$$

Let us take a motion ϕ which in coordinates $\{\theta\}$ on $\mathcal{S}$ is given by:

$$\phi^i(x) = x^i + u^i(x,t),$$

where $u = \sum u^i(x,t)\hat{e}_i$ is a displacement vector field. Recall that from equation (4.21), we have

$$\varepsilon_{ij}(x) = \frac{1}{2}\left(g_{in}\nabla_j u^n + g_{nj}\nabla_i u^n\right),$$

then, in terms of the displacement components and in local coordinates we have

$$2\rho g_{ij}\ddot{u}^j + 2\rho\frac{\partial}{\partial x^i}(g^{im}g^{lj}\varepsilon_{ml}C_{ijkl}\varepsilon_{kl}) + \left[\sigma^{ij}\frac{\partial g_{ij}}{\partial x^i}\right] = 0. \tag{5.9}$$

Now consider the Euclidean case, from which we have

$$\begin{aligned}
\sigma^{ij}\frac{\partial g_{ij}}{\partial x^i} &= 0,\\
Trace_g\ [\varepsilon(C\varepsilon)] &= \langle\varepsilon_{ij}, C_{ijkl}\varepsilon_{kl}\rangle_{\mathcal{R}^n}\\
\frac{\partial}{\partial x^i}Trace_g\ [\varepsilon(C\varepsilon)] &= \varepsilon_{ij}\frac{\partial}{\partial x^i}[C_{ijkl}\varepsilon_{kl}] + \mathcal{O}(2),
\end{aligned}$$

the elastic wave equation in the Euclidean case is

$$\begin{aligned}
P_{il}u^l &= 0,\\
P_{il} &= \delta_{il}\frac{\partial^2}{\partial t^2} + A_{il},\\
A_{il} &= \frac{\partial}{\partial x^j}(\cdot)\left[\frac{\partial}{\partial x^l}\left(C_{ijkl}\frac{\partial}{\partial x^k}(\cdot)\right)\right];
\end{aligned}$$

note that this equation is similar to equation (2.27), and we expect that the metodology used in [20], can be applied to it.
As another result from equation (5.9), consider a heterogeneous isotropic continuum, and we have the relations:

$$\begin{aligned}
C_{ijkl} &= \lambda g_{ij}g_{kl} + \mu g_{ik}g_{jl} + \mu g_{il}g_{kj};\\
g^{im}g^{lj}C_{ijkl} &= (\lambda+2\mu)\delta_k^m;\\
\varepsilon_{kl}\varepsilon_{ml} &= \frac{1}{4}(u_{k|l}+u_{l|k})(u_{m|l}+u_{l|m}),\\
&= \frac{1}{4}(u_{k|l}u_{m|l} + u_{k|l}u_{l|m} + u_{l|k}u_{m|l} + u_{l|k}u_{l|m}),
\end{aligned}$$

where $u_{i|j} = g_{i\nu}\nabla_j u^\nu$, and then,

$$\varepsilon_{kl}\varepsilon_{ml} = \frac{1}{4}\left(g_{k\nu}g_{m\nu}(\nabla_l)^2 + g_{k\nu}g_{l\nu}\nabla_l\nabla_m + g_{l\nu}g_{m\nu}\nabla_k\nabla_l + g_{l\nu}g_{l\nu}\nabla_k\nabla_m\right),$$

where $(g_{k\nu}g_{l\nu}\nabla_l\nabla_m)u^\nu := g_{k\nu}g_{l\nu}\nabla_l u^\nu\nabla_m u^\nu$; therefore

$$\begin{aligned} g^{im}g^{lj}C_{ijkl}\varepsilon_{kl}\varepsilon_{ml} &= \frac{1}{4}(\lambda+2\mu)\delta_k^m\left(g_{k\nu}g_{m\nu}(\nabla_l)^2 + g_{k\nu}g_{l\nu}\nabla_l\nabla_m\right. \\ &\quad \left.+g_{l\nu}g_{m\nu}\nabla_k\nabla_l + g_{l\nu}g_{l\nu}\nabla_k\nabla_m\right). \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x^i} &= \gamma_{ij}^n g_{in} + \gamma_{ii}^n g_{jn}, \\ \sigma^{ij} &= \lambda g^{ij}\nabla_l u^l + \mu g^{jl}\nabla_l u^i + \mu g^{il}\nabla_l u^j, \end{aligned}$$

we have after calculations and a rearrangement of the indices, that

$$\sigma^{ij}\frac{\partial g_{ij}}{\partial x^i} = \lambda\left[\gamma_{ij}^i + \gamma_{jj}^j\right]\nabla_l u^l + \mu\left[\gamma_{jj}^l + \gamma_{jj}^n g_{in}g^{jl} + \gamma_{ij}^n g_{jn}g^{il} + \gamma_{ij}^l\right]\nabla_l u^i.$$

Denote

$$A_{ji} = \delta_k^m\varepsilon_{kj}\varepsilon_{mj}, \tag{5.10}$$
$$B_{ijl} = \left[\gamma_{lj}^j + \gamma_{ll}^l\right]\nabla_i, \tag{5.11}$$
$$C_{ijln} = \left[g_{in}g^{jl}\left(\gamma_{ij}^n + \gamma_{jj}^n\right) + \left(\gamma_{ij}^l + \gamma_{ii}^l\right)\right]\nabla_l; \tag{5.12}$$

and it is clear that

$$A_{ji},\ B_{ijl},\ C_{ijln} : \mathfrak{F}(\mathcal{S}) \to \mathfrak{F}(\mathcal{S}).$$

With this notation we can write equation (5.9), as

$$2\rho g_{ij}\ddot{u}^j + 2\rho\frac{1}{4}(\lambda+2\mu)\frac{\partial}{\partial x^i}A_{ji}(u^i) + \lambda B_{ijl}(u^i) + \mu C_{ijln}(u^i) = 0;$$

multiplying by g^{ij} we get:

$$2\rho\ddot{u}^j + \frac{\rho}{2}(\lambda+2\mu)g^{ij}\frac{\partial}{\partial x^i}A_{ji}(u^i) + g^{ij}\left[\lambda B_{ijl}(u^i) + \mu C_{ijln}(u^i)\right] = 0 \tag{5.13}$$
$$2\rho\ddot{u}^j + \frac{\rho}{2}(\lambda+2\mu)\left(\nabla A_{ij}(u^i)\right)^j + g^{ij}\left[\lambda B_{ijl}(u^i) + \mu C_{ijln}(u^i)\right] = 0, \tag{5.14}$$

where $(\nabla A_{ij}(u^i))^j = (grad\ A_{ij}(u^i))^j$.

Definition 5.2.1. *Consider the maps $\widetilde{A},\ \widetilde{B},\ \widetilde{C} : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$, given by:*

$$\widetilde{A}(\mathbf{u}) = A_{ji}(u^i)\hat{e}_j \tag{5.15}$$
$$\widetilde{B}(\mathbf{u}) = B_{ijl}(u^i)\hat{e}_i \tag{5.16}$$
$$\widetilde{C}(\mathbf{u}) = C_{ijln}(u^i)\hat{e}_i, \tag{5.17}$$

where $\mathbf{u} = \sum u(x,t)^i\hat{e}_i$.

Now consider the equation

$$2\rho\ddot{\mathbf{u}}^j + \frac{\rho}{2}(\lambda+2\mu)\left(\nabla\widetilde{A}(\mathbf{u})\right)^j + g^{ij}\left[\lambda\widetilde{B}(\mathbf{u}) + \mu\widetilde{C}(\mathbf{u})\right] = 0. \tag{5.18}$$

Operators $\widetilde{A},\ \widetilde{B},\ \widetilde{C}$, are linear, continuous and self-adjoint. This equation says that for the proper treatment of Riemannian wave equation for an isotropic heterogeneous continuum, this equation must be written in terms of operators belonging to the space of diffeomorphism on $\mathfrak{X}(\mathcal{S})$, i.e, $\widetilde{A},\ \widetilde{B},\ \widetilde{C} \in Diff\ [\mathfrak{X}(\mathcal{S})]$.

5.3 Elastic OWWE on Riemannian Manifolds

Let $(\mathcal{S}, g)$ be the ambient Riemannian manifold and $\{\theta^a\}$ a coordinate system of diffeomorphisms. Denote $\theta(V) := \Omega \subset \mathcal{R}^n$ where $V \subset \mathcal{S}$ is open. Let $U \in C_0^\infty(\Omega)$ and define $\theta^* : C_0^\infty(\Omega) \to C^\infty(V)$ by the equation:

$$\theta^*_U := U \circ \theta, \tag{5.19}$$
$$\theta^*_U(x) = U[\theta(x)]. \tag{5.20}$$

Let $\phi(X,t)$ be a regular C^2 motion on $\mathcal{S}$ and consider the linear operator

$$A(x, D_x)_{ji} := C_0^\infty(V) \to C^\infty(V), \tag{5.21}$$
$$A(x, D_x)_{ji}(u^i(x,t)) = \frac{\partial}{\partial x^i}\left(g^{im}g^{lj}\varepsilon_{ml}C_{ijkl}\varepsilon_{kl}\right)(u^i(x,t)) \tag{5.22}$$
$$= \frac{\partial}{\partial x^i}\left[g^{im}(x)g^{lj}(x)C_{ijkl}(x)\left(u^m_{|l} + u^l_{|m}\right)\left(u^k_{|l} + u^l_{|k}\right)\right], \tag{5.23}$$

where $u^i(x,t)$ are the components of the displacement vector field.
Define also the linear operator $A_{loc}(z, D_z)_{JI}$ as:

$$A_{loc}(z, D_z)_{JI} : C_0^\infty(\Omega) \to C^\infty(\Omega), \tag{5.24}$$
$$A_{loc}(z, D_z)_{JI}(U^I) := A(x, D_x)_{ji}[(\theta^* U \circ \theta^{-1})^i], \tag{5.25}$$
$$A_{loc}(z, D_z)_{JI}(U(z,t)^I) = A(x, D_x)_{ji}[(\theta^* U \circ \theta^{-1})^i_{x,t}], \tag{5.26}$$

where $z = \theta(x)$ and $U^I(z,t) = \theta^I(u^i(x,t))$ are the local representations of the maps $u^i(x,t)$.
Our first goal is to find an operator $D(x, D_x)_{jh}$ such that

$$D(x, D_x)^{-1}_{hj} A(x, D_x)_{ji} D(x, D_x)_{ik} = diag(A(x, D_x)_h : h = 1, \ldots n)_{hk}.$$

As in Section 2.4, the principal symbol $A^{prin}_{ji}(x,\xi)$ is a positive and symmetric operator which can be diagonalized by an orthogonal operator, say $D^{prin}_{jh}(x,\xi)$ such that

$$D^{prin}_{hj}(x,\xi)^{-1} A^{prin}_{ji}(x,\xi) D^{prin}_{ik}(x,\xi) = diag(A^{prin}_h(x,\xi))_{hk}.$$

Since $A(x, D_x)$ is a differential operator on $\mathcal{S}$, Theorem 4.2 in [12], allows us to define the principal symbol $A^{prin}_{ji}(x,\xi)$ on the cotangent bundle $T^*\mathcal{S}$, i.e, $A^{prin}_{ji}(x,\xi) \in C^\infty(\mathcal{S} \times T^*_x\mathcal{S})$, which is a homogeneous polynomial of order m in ξ, i.e,

$$A^{prin}_{ji}(x, t\xi) = t^m A^{prin}_{ji}(x,\xi), \quad \text{for} \quad t > 0;$$

and is given by the relation:

$$A_{ji}^{prin}(x,\varphi'_x) = \lim_{\lambda\to+\infty} \lambda^{-2} e^{-i\lambda\varphi} A(x,D_x)_{ji}\cdot(e^{i\lambda\varphi}), \tag{5.27}$$

where $\varphi \in C^\infty(\mathcal{S})$.
Since $\nabla_l u^k$ is a linear differential operator with real coefficients on u^k, we find after calculations that the symbol $\sigma_{A_{ji}(x,\xi)}$ of the operator $A(x,D_x)_{ji}$ is given by

$$\sigma_{A_{ji}}(x,\xi)(u^n) = g^{im}(x) g^{lj}(x) C_{ijml}(x)\xi_i \left\{\left[\delta_n^m \xi_l + \delta_n^l \xi_m + \left(\gamma_{ln}^m + \gamma_{mn}^l\right)\right] u^n\right\}^2. \tag{5.28}$$

For the Euclidean case, i.e, the elastic case, the principal symbol of the operator $A_{ji}^{prin}(x,\xi)_{\mathcal{R}^N}$ is given by:

$$\sigma_{A_{ji}}(x,\xi)_{\mathcal{R}^N} = C_{ijml}(x)\xi_i\left[\xi_l u^m + \xi_m u^l\right]^2,$$

and it is a straightforward exercise to verify that for the isotropic symmetry and taking $N=3$ it is obtained the usual symbol with principal symbol

$$\sigma_{A_{ji}}^{prin}(x,\xi) = \begin{pmatrix} (\lambda+\mu)\xi_1^2 + \mu|\xi|^2 & (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_1\xi_3 \\ (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_2^2+\mu|\xi|^2 & (\lambda+\mu)\xi_2\xi_3 \\ (\lambda+\mu)\xi_1\xi_3 & (\lambda+\mu)\xi_2\xi_3 & (\lambda+\mu)\xi_3^2+\mu|\xi|^2 \end{pmatrix}.$$

Assume that $A(x,D_x)_{ji}$ diagonalize according to

$$D(x,D_x)_{hj}^{-1} A(x,D_x)_{ji} D(x,D_x)_{ik} = diag(A(x,D_x)_h : h=1,...n)_{hk}, \tag{5.29}$$

and define the following operators:

Definition 5.3.1. *Let $\mathbf{u} \in \mathfrak{X}(\mathcal{S})$, $D(x,D_x)_{jh}$ and $A_h(x,D_x)$ given by equation (5.29). The operators U_h, F_h and P_h are given by*

$$\begin{aligned} U_h &= D(x,D_x)_{hj}^{-1} u^j(x,t), & (5.30)\\ F_h &= D(x,D_x)_{hj}^{-1}\left[-\frac{\sigma^{ij}}{2\rho}\frac{\partial g_{ij}}{\partial x^i}\right], & (5.31)\\ P_h(x,D_x,D_t) &= g_{ij}\frac{\partial^2}{\partial t^2} + A_h(x,D_x). & (5.32) \end{aligned}$$

The system

$$P_h(x,D_x,D_t)U_h = F_h, \tag{5.33}$$

is equivalent to the system of Riemannian wave equations (5.7), since:

$$\begin{aligned} P_h(x,D_x,D_t)U_h &= g_{ij}\frac{\partial^2}{\partial t^2}U_h + A_h(x,D_x)U_h \\ &= g_{ij}D(x,D_x)_{hj}^{-1}\ddot{u}^j(x,t) + A_h(x,D_x)D(x,D_x)_{hj}^{-1}u^j(x,t), \\ &= g_{ij}D(x,D_x)_{hj}^{-1}\ddot{u}^j(x,t) + D(x,D_x)_{hj}^{-1}A(x,D_x)_{ji}u^i(x,t), \end{aligned}$$

where the last equation follows by equation (5.29), and the system (5.33) follows immediately.

Since the strain-energy functional depends only on the point values of the metric, see [33], the symbol of the multiplication operator $A_h(x, D_x)g^{ij}$ is $\sigma_{A_h}(x,\xi)g^{ij}$, see [13]. Then the system (5.33) can be written as:

$$\frac{\partial}{\partial t}\begin{pmatrix} g_{ij}U_h \\ g_{ij}\frac{\partial U_h}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -A_h(x, D_x)g^{ij} & 0 \end{pmatrix}\begin{pmatrix} g_{ij}U_h \\ g_{ij}\frac{\partial U_h}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 \\ F_h \end{pmatrix}. \tag{5.34}$$

Denote $A'_h(x, D_x) = A_h(x, D_x)g^{ij}$ and consider the matrix-valued pseudodifferential operators

$$V_h(x, D_x) = \begin{pmatrix} 1 & 1 \\ -iS_h(x, D_x) & iS_h(x, D_x) \end{pmatrix}, \tag{5.35}$$

$$\Lambda_h(x, D_x) = \frac{1}{2}\begin{pmatrix} 1 & iS_h(x, D_x)^{-1} \\ 1 & -iS_h(x, D_x)^{-1} \end{pmatrix}, \tag{5.36}$$

where the pseusodifferential operator S_h is the square root of $A'_h(x, D_x)$, i.e, $S_h(x, D_x) = \sqrt{A_h(x, D_t)g^{ij}}$. Note that

$$V_h(x, D_x)\begin{pmatrix} 0 & 1 \\ -A'_h(x, D_x) & 0 \end{pmatrix}\Lambda_h(x, D_x) = \begin{pmatrix} iS_h(x, D_x) & 0 \\ 0 & -iS_h(x, D_x) \end{pmatrix}.$$

Denote

$$U'_h = \begin{pmatrix} g_{ij}U_h \\ g_{ij}\frac{\partial U_h}{\partial t} \end{pmatrix},$$

$$\mathbf{A'}_h(x, D_x) = \begin{pmatrix} 0 & 1 \\ -A'_h(x, D_x) & 0 \end{pmatrix},$$

$$F'_h = \begin{pmatrix} 0 \\ F_h \end{pmatrix}$$

then we have the equation:

$$\frac{\partial}{\partial t}U'_h = \mathbf{A'}_h U'_h + F'_h. \tag{5.37}$$

Consider the transformation $U'_h = V_h(x, D_x)\nu_h$, from which equation (5.37), is transformed into the one-way wave system equations (5.38), on the Riemannian manifold $\mathcal{S}$:

$$\begin{aligned} V_h(x, D_x)\frac{\partial}{\partial t}\nu_h &= \mathbf{A'}_h(x, D_x)V_h(x, D_x)\nu_h + F'_h \\ \Lambda_h(x, D_x)V_h(x, D_x)\frac{\partial}{\partial t}\nu_h &= \Lambda_h(x, D_x)(x, D_x)\mathbf{A'}_h(x, D_x)V_h(x, D_x)\nu_h + \Lambda_h(x, D_x)F'_h \\ \frac{\partial}{\partial t}\nu_h &= \begin{pmatrix} iS_h(x, D_x) & 0 \\ 0 & -iS_h(x, D_x) \end{pmatrix}\nu_h + \Lambda_h(x, D_x)F'_h. \end{aligned} \tag{5.38}$$

Chapter 6

Conclusions

Wave propagation modeling is the first theoric step in exploration seismology, it provides the correct scenario and the equations for simulating the propagation of elastic and acoustic waves into the earth, which we can consider as a second step previous to imaging. The scenario is related to how we conceive the material continuum, as an Euclidean or Riemann space and then, we need to properly formulate the equations of propagating waves in any of those spaces.
We have investigated the wave propagation in a Riemannian space, since it is on these spaces where we can glue together the geometry and the mechanical properties of the continuum to get a set of equations and mathematical relationships that describe the wave propagation in a general setting from which the Euclidean formalism can be recovered.

In Chapter 3, we studied the acoustic Riemannian wave equation proposed by [22], [23], [25] as a pure eigenvalue equation. In Section 3.5 we derive stability and dispersion conditions 3.43, 3.44, 3.47, 3.51, from a finite difference scheme, and develop numerical simulations to compare with the work of Shragge [26]; the main result says that the limits obtained for the time sampling show that the computational cost of a propagation using the generalized acoustic wave equation is in general bigger than the same simulation with the usual acoustic wave equation (around 143%) bigger for a particular case shown. Other conclisions and results were listed on Section 3.5 and Section 3.6.

In Chapter 4, a Lagrangian density was proposed to obtain a Noether's condition for an elastic Riemannian body; the results were given in Section 4.5, particularly equations 4.44, 4.45, 4.54, 4.55, 4.56; which fully describe the group of symmetries in the elastic Riemannian formalism, for the Lagrangian 4.35. We also derive a constrain for the equations of motion for an elastic boby in Euclidean space, equation 4.58.
The results of this chapter must be the starting point to analyze the group of symmetries for different continuum in the context of Riemannian manifolds. It also should be investigated the definition of the stress-Energy-Momentum tensor via Noether's theorem and its properties such as symmetry and gauge-invariance.

In Chapter 5 we obtain the Riemannian equations of motion, 5.5, 5.7, via Euler-Lagrange equations on the manifold. We also worked the the equation in coordinates, 5.9 and the

equation of motion for tangent vector fields 5.18 from which we concluded that the proper treatment for a Riemannian isotropic and heterogeneous continuum, the vector fields of equation 5.18 must be diffeomorphisms on the space of vector fields.
We also proved that the spatial-geometry operator of the equation of motion, operator 5.23, is a pseudodifferential operator, we calculate its symbol in equation 5.28 and proved that it is diagonalizable where the diagonalizing operator is given by 5.36. This operator allowed us to obtain the one-way wave system of equations for the elastic Riemannian manifold, 5.38, which is the main result of the work.
Since equation 5.38, can be used to decompose the wavefield into upgoing and downgoing waves, we also need to obtain the up/down decomposition tensor and its adjoint form to get to an oscillatory integral representation of the solutions. A lot of research must be done in order to get the implementations of the results given here.

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