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An implicit class of continuous dynamical system with data-sample outputs: a robust approach

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This paper addresses the problem of robust control for a class of nonlinear dynamical systems in the continuous time domain. We deal with nonlinear models described by differential-algebraic equations (DAEs) in the presence of bounded uncertainties. The full model of the control system under consideration is completed by linear sampling-type outputs. The linear feedback control design proposed in this manuscript is created by application of an extended version of the conventional invariant ellipsoid method. Moreover, we also apply some specific Lyapunov-based descriptor techniques from the stability theory of continuous systems. The above combination of the modified invariant ellipsoid approach and descriptor method makes it possible to obtain the robustness of the designed control and to establish some well-known stability properties of dynamical systems under consideration. Finally, the applicability of the proposed method is illustrated by a computational example. A brief discussion on the main implementation issue is also included.

Keywords: robust control; DAE; semi-explicit DAE; attractive ellipsoid; invariant ellipsoid; Luenberguer observer; sample-data output; descriptor method; implicit systems.

1. Introduction

In the last 30 years, the dynamical behaviour of a wide number of constrained dynamical systems in numerous applications, such as economics, demography, mechanical systems, multibody dynamics, electrical networks, fluid mechanics, chemical engineering, control theory and many other areas, have been usually modelled via semi-explicit differential-algebraic equations (semi-explicit DAE) as a widely accepted tool (see Kunkel & Mehrmann (2006) for more applications) of simulation and whose general form appears as

$$\begin{aligned} \dot{x}_1 &= \zeta(x_1, x_2, t) \in \mathbb{R}^m \\ 0 &= \kappa(x_1, x_2, t) \in \mathbb{R}^{n-m}, \end{aligned} \tag{1.1}$$

where $(x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Here *algebraic* just means *nondifferential*. The main theory and numerical analysis of linear DAE with constant coefficients (LDAE-CC) is largely covered by Kunkel & Mehrmann (2006) and authors therein. Kunkel & Mehrmann (2006) explain that non-invertibility at a point (x_0, p_0) of derivative, $D_pF(x,p)$, of above semi-explicit DAEs in implicit form, $F(x,p) := (p_1 - \zeta(x_1, x_2), \kappa(x_1, x_2))^\top$, does not affect the surjectivity of $DF(x_0, p_0)$ and the rank of $D_pF(x,p)$ is constant, where $p = (p_1, p_2) \in \mathbb{R}^n$. Using the concept of *differentiation index*, which is the minimum number of times that all or part of $F(x,p)$ must be differentiated with respect to t in order to determine $\dot{x}$ as a continuous function of t and x, a semi-explicit DAE has an index 1 and eventually is reduced to an ordinary differential equation (ODE) on a manifold.

The functions $\zeta(t)$ and $\kappa(t)$ belonging to the given *Quasi-Lipschitz* (Q-L) classes, whose exact definition is given in the next section, is to be compatible with several widely used techniques of linear approximation related to plant models. Similar linearization-like ideas are common in the theoretical and numerical practice of control engineering (Khalil, 1996). This linearization-like approximation allows us to rewrite system (1.1) into a linear control problem. Linear control problems, called *descriptor systems*, use differential-algebraic systems in the form

$$\begin{aligned} E\dot{x} &= Ax + Bu + f(t) \\ y &= Cx + h(t), \end{aligned} \tag{1.2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{q\times n}$ are constant matrices, $f \in \mathscr{C}(\mathbb{I}, \mathbb{R}^n)$, $h \in \mathscr{C}(\mathbb{I}, \mathbb{R}^q)$ and both measurable functions for some interval $\mathbb{I} \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ represents the state, $u \in \mathbb{R}^m$ the input or control, and $y \in \mathbb{R}^q$ the output of the systemand are still a very active research area. All properties of the previous system can be determined by computing the *invariants* of the associated matrix pair (E, A) under equivalence transformations. Then original and transformed problems can be treated by purely algebraic techniques. One particular one-to-one transformation called *Weierstrass canonical form* (WCF) for regular pair matrix allows us to set a one-to-one correspondence between the corresponding solution sets. *Regularity* of a matrix pair is closely related to the solution behaviour of the corresponding DAE. This means that we can consider transformed problem instead of original problem with respect to solvability and related questions. For a general behaviour approach and its analytical treatment, see Kunkel & Mehrmann (2006). There exists a completely algebraic characterization when system is regular and consistent. Notice that a control problem as (1.2) is called consistent, if there exists an input function u, for which the resulting DAE is solvable. It is called regular, if for every sufficiently smooth input function u and inhomogeneity f, the corresponding DAE is solvable and the solution is unique for every consistent initial value.

The main assumption of classical optimal control theory is that a user possesses complete information on the model under consideration as well as on the environment in which this controlled model will evolve. When we have incomplete information on a dynamic model to be controlled, the main problem consists in designing an acceptable control that remains *close to the optimal or desired one* (having small sensitivity with respect to every unknown (unpredictable) factor from a given set of possibilities). In other words, the desired control should be robust with respect to unknown factors. In presence of uncertainties in the dynamic model, the attractive ellipsoid methodology (AEM) works for reach a suitable solution for a class of given models is to formulate a corresponding tracking control problem, where we are interested in the *best approximation* to a desired trajectory. In other words, we are interested in a zone stabilization or in the practical stability of the deviation of the trajectories of the given system from the desired one. The robust stabilization problem considered for different classes of nonlinear systems has been a hot topic of research over the past two decades (Poznyak *et al.*, 2014b; Utkin, 1992). The necessary assumptions for the tracking error dynamics for the system (1.2) are the following: the dynamic plant (1.2) is controllable and observable; the functions $\zeta(t)$ and $\kappa(t)$ may be unknown, but they belong to the given Q-L classes $\mathscr{C}_\zeta$ and $\mathscr{C}_\kappa$ of nonlinear functions, respectively; the unmeasured functions $f(t)$ and $h(t)$ are bounded; the control $u(t)$ is designed as a feedback (static or dynamic) of a given structure containing the set of parameters $\mathscr{P}$, that is, $u(t) = u(x(\tau)|_{0\leqslant\tau\leqslant t}, t, \mathscr{P})$; so that $u(t)$ depends on all measurable data $x(\tau)$ in the time interval $[0, t]$. If we have the nonzero terms $f(t)$ and $h(t)$, which are unmeasurable during the control process, then obviously, the application of the classical optimal control approach (as described above) is impossible. The situation looks much more difficult if the functions $\zeta \in \mathscr{C}_f$ and $\kappa \in \mathscr{C}_g$ describing the dynamic process are unknown *a priori*. In the control problem, formulated as a tracking problem, the set of considered control strategies is suggested to belong to a parameterized class of nonlinear (perhaps nonstationary) feedbacks $u(t) = u(x(\tau)|_{0\leqslant\tau\leqslant t}, t, \mathscr{P})$, whose parameters $\mathscr{P}$ are selected in such a way that all possible trajectories $x(t)$ of the closed controlled systems remain bounded and closed to the origin; taking into account that every set of bounded trajectories may be imposed within a convex bounded set, and particularly within an ellipsoid, the AEM suggests that we select the feedback parameters $\mathscr{P} = \mathscr{P}^*$ providing a minimal *size* of this ellipsoid containing all possible bounded trajectories of every dynamical system from the considered class of dynamics containing uncertain elements. In this case, we talk about zone convergence or *practical stability* (with a prescribed convex convergence zone) if the size of the convergence zone is of a predetermined value, so that the effectiveness of such robust control strategies is associated with the *size* of the corresponding attractive ellipsoid set. We study nonlinearly affine control systems in the presence of uncertainties and are interested in a constructive and easily implementable control strategy that guarantees, in a *practical* sense, some stability properties of the closed-loop realizations. In fact, we deal with a linear-type feedback control synthesis in the context of the above-mentioned nonlinear uncertain systems of an affine structure. The class of stabilizing feedbacks is given by the corresponding bilinear matrix inequalities (BMIs). If they are satisfied, then one may guarantee that all possible trajectories of the considered systems are bounded. Since bounded dynamics may be imposed inside an ellipsoid, defined by $\mathscr{E}(P) := \{z \in \mathbb{R}^n | z^\mathsf{T} P^{-1} z \leqslant 1\}$, where P is a symmetric positive definite $n \times n$ matrix, we associate the *best parameters* of the feedback with the minimal size of this ellipsoid.

Present contribution has the following structure: Section 2 is devoted to review the conventional theoretical results about LDAE-CC and AEM. In Section 3 we present problem formulation including the Luenberguer observer design and the optimization problem of the WCF of the initial problem. WCF stages from the initial problem formulation are also included in this section. Section 4 deals with practical stability analysis of transformed problem and presents the main analytical results. In Section 5 we present a numerical implementation of associated computational algorithm that is based on BMI solution. Section 6 summarizes the contribution. Finally, the important references are presented.

2. Preliminaries

The most important lemmas and definitions of the two main topics in this paper, LDAE-CC and AEM, are presented in this section. First topic regards necessary and sufficient conditions for the DAE solution. Second topic review allows us to identify AEM stages in order to ensure not only an admissible control strategy, numerically obtained but also the practical stability and robustness of DAE solution and perturbation rejection considering sample-data outputs.

2.1. *LDAE-CC*

According to LDAE framework in Kunkel & Mehrmann (2006), there exist some important theoretical results readers must know before dealing with main result of this contribution. These results are included in the following paragraphs.

The matrix pair (E, A) is called *regular*, where E, $A \in \mathbb{R}^{n\times n}$, if the so-called characteristic polynomial $p(\lambda) = \det(\lambda E - A)$ is not the zero polynomial. Then a matrix pair that is not regular is called singular. If the pair (E, A) is regular, the WCF of (1.2) is obtained, scaling (1.2) by a nonsingular matrix $\Pi \in \mathbb{R}^{n\times n}$ and the function x according to $x = \Psi\tilde{x}$ with a nonsingular matrix $\Psi \in \mathbb{R}^{n\times n}$. This introduced one-to-one relation, denoted by $(\sim)$, defines an equivalence relation and a correspondence between the corresponding solution sets. This means that we can consider the transformed problem instead of (1.2) with respect to solvability and related questions. Then we have the WCF of (1.2): $(E, A) \sim \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)$. Regularity of a matrix pair is necessary and sufficient for the property that for every sufficiently smooth inhomogeneity $f(\cdot)$, the DAE is solvable and the solution is unique for every consistent initial value. Then the WCF of (1.2) is solvable with a consistent initial condition. Moreover, every initial value problem with consistent initial condition is uniquely solvable.

In the control context, it is possible to modify system properties of (1.2) using feedbacks, in particular, to make non-regular systems regular or to change the index of the system.

2.2. *AEM*

The state equation parameterized by an *input parameter*

$$\begin{aligned} \dot{x} &= \phi(x(t), u(t), t) \\ y &= g(x(t), t), \end{aligned} \tag{2.1}$$

where $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are a suitable right-hand side, and the *parameter* $u(t)$ is chosen from a control set $U \subset \mathbb{R}^m$ of admissible control functions of type $u(x)$ such that the closed-loop system has a well-defined solution, and have been objects of the theory of ODEs for a long time (Lozada-Castillo *et al.*, 2014; Poznyak *et al.*, 2014a,b; Poznyak, 2015). We consider previous references to review stages of AEM. The robust AEM is restricted to a specific class of the Q-L functions $\phi(x)$. A vector function $\phi : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ is said to be of the class $\mathscr{C}(A, \delta_1, \delta_2)$ of Q-L functions if there exists a matrix $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and non-negative constants δ_1 and δ_2 such that for every $x \in \mathbb{R}^{\mathrm{n}}$, the following inequality holds: $\|\phi(x) - Ax\|^2_{Q_\phi} \leqslant \delta_1 + \delta_2\|x\|^2_{Q_x}$. This implies that the growth rates of $\phi(x)$ as $\|x\| \to \infty$ are not faster than linear (see Fig. 1, illustrating the single-dimensional case $n = 1$, $a > \sqrt{\delta_2} > 0$), where Q_ϕ, $Q_x \in \mathbb{R}^{n\times n}$, $Q_\phi = Q_\phi^\intercal \geqslant 0$, $Q_x = Q_x^\intercal \geqslant 0$. By $\|\cdot\|_{Q_j}$, $j = \phi, x$ (where Q_j is a given

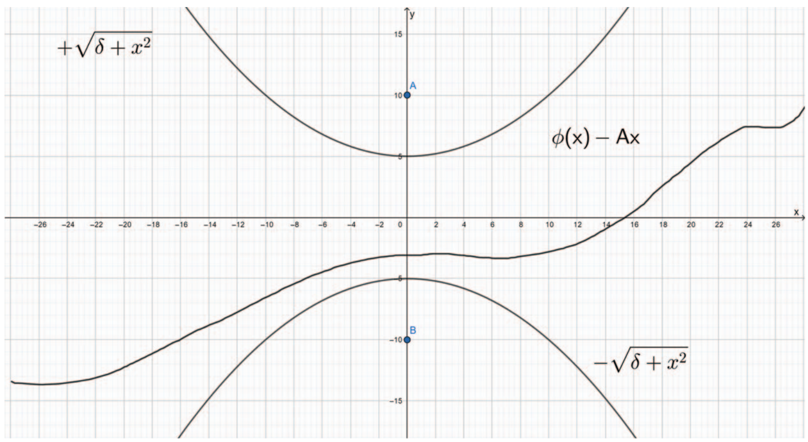

FIG. 1. Unidimensional Q-L function.

suitable symmetric positive-definite matrix), we denote a weighted Euclidean norm. We easily obtain an alternative description of the Q-L model as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu + \sigma(x(t), \phi(x)) \\ y &= g(x(t), t), \end{aligned} \tag{2.2}$$

where $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\sigma(x, \phi) := \phi(x) - Ax$. The condition of Q-L can be considered from two points of view: as a kind of a *linearization* procedure applied to a known function $\phi(\cdot)$ and as an *a priori* estimate for of the perturbation associated with a given system (2.2). In this paper, we will consider both of these as interpretations of the basic Q-L condition. Modern computational technologies make it possible to obtain an adequate and effective numerical implementation for the stability analysis with a concrete Lyapunov-based method. In AEM, Ω is a *positive invariant* for the closed-loop system $\dot{x} = \phi(x(t))$ if every solution of the Cauchy problem $\dot{x}(t) = \phi(x)$, $t > t_0$ and $x(t_0) = x_0$, where $x_0 \in \Omega$, satisfies condition $x(t) \in \Omega$ for all $t > t_0$. Roughly speaking, a set in the state space is said to be positively invariant if every trajectory initiated in this set remains inside the set at all future time. Ω is Lyapunov *asymptotically attractive* for the system above if every solution of the same Cauchy problem, where $x_0 \notin \Omega$, tends to Ω as t tends to infinity, i.e., $\rho(x(t), \Omega) \to 0$ if $t \to +\infty$. We call $\mathscr{E}$ an attractive ellipsoid for the closed-loop system $\dot{x}(t) = \phi(x(t))$ if it is a globally asymptotically attractive invariant set of a system $\dot{x}(t) = \phi(x(t), u(t))$. Note that the attractivity property mentioned above does not imply, in general, the Lyapunov asymptotic stability of the invariant set under consideration. Our aim is to generate a simple feedback-type control strategy $u(x)$ such that $\mathscr{E}$ is a globally asymptotically stable positively invariant set of minimal size (in some suitable sense) for the realization of $\dot{x}(t) = \phi(x(t), u(t))$. Lyapunov function method provide the main tools for stability and robustness analysis and the corresponding control design for nonlinear control systems. The function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be proper if it is continuously differentiable in $\mathbb{R}^n$, it is positive finite ($V(x) > 0$ for $x \neq 0$ and $V(0) = 0$) and it is radially unbounded ($\|x\| \to +\infty$ implies $V(x) \to +\infty$). Theorem 2.1 makes it possible to specify constructively an attractive invariant set not only for a concrete system $\dot{x}(t) = \phi(x(t), u(t))$, $t \geqslant 0$, with $x(0) = x_0 \in \mathbb{R}^n$, but also for the class (family) of corresponding dynamic processes that possess

Q-L right-hand sides. The corresponding robust and/or optimal control design schemes become BMI constraints in this case.This minimizing problem evidently includes some natural additional restrictions for the *free* parameters, namely for P and for the gain matrix K from control law $u = Kx$, where $K \in \mathbb{R}^{m\times n}$.

THEOREM 2.1 (Poznyak *et al.*, 2014b) Consider system $\dot{x}(t) = \phi(x(t), u(t))$, $t \geqslant 0$, with $x(0) = x_0 \in \mathbb{R}^n$ and Q-L right-hand side, and let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. If there exists a proper function $V : \mathbb{R}^n \to \mathbb{R}+ \cup \{0\}$ such that $\frac{\partial V(x)}{\partial x}(Ax+Bu(x)+w) < 0$ for all $x, w \in \mathbb{R}^n$ such that $\|w\|_{Q_f} \leqslant \delta + \|x\|_{Q_x}$ and $V(x) > 1$, then the set $\Omega = \{x \in \mathbb{R}^n : V(x) \leqslant 1\}$ is asymptotically attractive and the invariant set of the Q-L system with feedback control $u = u(x)$.

It is important to note that Theorem 2.1 uses the Lie derivative of function $V(\cdot)$; however, in DAE case, we will use the corresponding descriptor derivative reported in Fridman (2006).

3. Problem formulation

Consider the continuous-time dynamical system described by the following nonlinear DAE with constant coefficients, sampling-data outputs and initial condition:

$$E\dot{x}(t) = \phi(x(t)) + Bu(t) + \eta(t) \tag{3.1}$$

$$y(t) = Cx(t) + \xi(t), \quad x(0) = x_0, \; t \in \mathbb{R}^+, \tag{3.2}$$

$$\bar{y}(t) = \sum_{k=1}^{N} y(t_k)\chi\left[t_k, t_{k+1}\right)(t) \;\; N \in \mathbb{N}, \tag{3.3}$$

where $E \in \mathbb{R}^{n\times n}$ is a singular constant matrix, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{q\times n}$ are also constant. $x(t) \in \mathbb{R}^n$ and $u(x) \in \mathbb{R}^m$ denote n-dimensional descriptor variable and m-dimensional control input vector, respectively. $y \in \mathbb{R}^q$ is the output vector and $\bar{y}(t) = \sum_{k=1}^{N} y(t_k)\chi\left[t_k, t_{k+1}\right)(t)$ $N \in \mathbb{N}$ is the sample-data output measurable and available, where

$$\chi\left[t_k, t_{k+1}\right)(t) := \begin{cases} 1, & \text{if } t \in \left[t_k, t_{k+1}\right) \\ 0, & \text{otherwise} \end{cases}$$

define the characteristic function.

System (3.1)–(3.3) are under the following particular hypotheses:

- $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable nonlinear function, Q-L whose derivative is simply bounded.
- $\eta(t), \xi(t) \in \mathbb{R}^n$, $\eta \in \mathscr{C}(\mathbb{I}, \mathbb{R}^n)$, $\xi \in \mathscr{C}(\mathbb{I}, \mathbb{R}^q)$, measurable functions for some interval $\mathbb{I} \in \mathbb{R}_+$, are not only unknown and deterministic perturbation terms but also bounded. That is, $\eta(\cdot)$, $\xi(\cdot)$ are functions such that $||\eta(t)||_{Q_\eta} + ||\xi(t)||_{Q_\xi} \leqslant 1$, $\forall t \in \mathbb{R}^+$, where $Q_\eta, Q_\xi \in \mathbb{R}^{n\times n}$, $Q_\eta = Q_\eta^{\mathsf{T}} > 0$, $Q_\xi = Q_\xi^{\mathsf{T}} > 0$.
- $u(x) = Kx$ is the descriptor variable feedback input, where $K \in \mathbb{R}^{m\times n}$ is a constant gain matrix.

Control aim is to find a gain matrix, K, with respect to the descriptor variable feedback system (3.1)–(3.3), which guarantee practical stability of solution trajectories, $x(t)$, for all consistent initial condition. A general solution scheme of the solution idea for the initial valued problem, (3.1)–(3.3), is presented below. The methodology is based on the solution of equivalence between a new semi-explicit DAE system obtained through a particular one-to-one matrix transformation of the nonlinear control problem (3.1)–(3.3).

(A) Transform the nonlinear EDA into a linear EDA using a linearization-like technique of AEM.

(B) Obtain the feedback system using $u = Kx$.

*(C) Obtain canonical form of feedback system (optional).

*(D) Identify two subproblems generated from canonical form and the corresponding solution and necessary conditions for each one (optional).

(E) Construct the corresponding Luenberger observer system.

(F) State the optimization problem that guarantees practical stability of trajectories solution of Luenberger system in a ellipsoid region of minimal area.

(G) Solve numerically optimization problem, in terms of Linear Matrix Inequality (LMI) condition, generated from the main result of present contribution.

(H) Verify that the obtained control gain matrix K also guarantees practical stability of trajectories solution of nonlinear DAE system in a ellipsoid region of minimal area.

For a regular pair matrix (E,A), the WCF of system 3.5 is the simplest form of system 3.5; however, itdoes not constitute a necessary nor obligatory part of the solution methodology, represented by (C) and (D) stages in Section 3, for this contribution. Here the main result of present contribution is a new theorem, similar to Theorem 2.1, for the DAE case; however, descriptor derivative reported in Fridman (2006) is used instead of the Lie derivative. Section 3.1 presents description of the nonlinear DAE to semi-explicit DAE transformation (first four stages). Section 3.2 explains how the Luenberger observer model is included in order to estimate sample-data output. In Section 3.3 the control problem is formulated as an optimization problem. Finally Section 4 states through the main theorem of proposition the necessary LMI conditions that ensure numerical solution of optimization problem.

3.1. *DAE transformation*

WCF is obtained from (3.1) performing the following algebraic calculations:

(i) Transform the nonlinear EDA into a linear EDA using a linearization-like technique of AEM.

$$E\dot{x}(t) = Ax(t) + Bu(t) + \sigma(x(t), \eta(t)), \tag{3.4}$$

where $\sigma(x(t), \eta(t)) := \phi(x(t)) - Ax(t) + \eta(t)$ is a new perturbation term.

(ii) Obtain the feedback system using $u = Kx$.

$$E\dot{x}(t) = (A + BK)x(t) + \sigma(x(t), \eta(t)). \tag{3.5}$$

As we mentioned in Introduction and Section 2.1, regularity property is a regularity that is necessary and sufficient for the property, that for every sufficiently smooth inhomogeneity $\sigma(x(t), \eta(t))$, the differential-algebraic equation is solvable and the solution is unique for every consistent initial value (see Kunkel & Mehrmann (2006), Theorem 2.7 of page 16). However, if pair matrix (E, A) for system 3.5 is not regular, in terms of control, it is possible to modify system properties using proportional state or proportional output feedbacks. Thus, these feedbacks can be used to modify the system properties, in particular, to make non-regular systems regular or to change the index of the system. Q-L condition for system 3.5 ensures the existence of matrix A and therefore its linear form. In the numerical procedure summarized in Section 3 together, arbitrariness property of A matrix and feedback gain matrix selection will become an important solution strategy (see Kunkel & Mehrmann (2006), Theorem 2.56 in page 51).

*(iii) (Optional) Obtain canonical form of feedback system. If pair matrix (E, A) is regular, perform the variable change $x = \Psi z$ and pre-multiply by Π, at the same time, equation (3.4).

$$E_s\dot{z}(t) = (A_s + B_sK_s)z(t) + \sigma_s(z(t), \eta(t)), \tag{3.6}$$

where $E_s := \Pi E \Psi = \text{diag}\{I_\mu, N_\rho\}$, $A_s := \Pi A \Psi = \text{diag}\{J_\mu, I_\rho\}$, $B_s := \Pi B$, $K_s := K\Psi$, $\sigma_s(z(t),\eta(t)) := \Pi\sigma(z(t), \eta(t)) = \phi_s(z(t)) - A_s z(t) + \Pi\eta(t)$, $\phi_s := \Pi\phi$ and $\eta_s := \Pi\eta$. Equation (3.6) represents WCF of DAE (3.4). Note that dimensions of block matrices are μ and $\rho := n - \mu$, where $\mu, \rho \in \mathbb{N}$; N_ρ is a nilpotent matrix with an index of nilpotency $\nu \in \mathbb{N}$. N_ρ and J_μ are given in Jordan canonical form. Dimension of all matrices and vector in equation (3.6) remain due to appropriate dimension of transformation matrices.

*(iv) (Optional) Identify two subproblems generated from canonical form and the corresponding solution and necessary conditions for each one. Pre-multiply by $N_L = \text{diag}\{I_\mu, N_\rho^{\nu-1}\}$ equation (3.6) to write semi-explicit form of the original nonlinear DAE (3.1):

$$E_{Ls}\dot{z}(t) = (A_{Ls} + B_{Ls}K_s)z(t) + \sigma_{Ls}(z(t), \eta(t)), \tag{3.7}$$

where $E_{Ls} := N_L E_s = \text{diag}\{I_\mu, 0_\rho\}$, $A_{Ls} := N_L A_s = \text{diag}\{J_\mu, N_\rho^{\nu-1}\}$.

(v) Write a new Initial Value Problem (IVP) using the semi-explicit DAE form and define general hypothesis on this.

$$E_{Ls}\dot{z}(t) = A_{Ls}z(t) + B_{Ls}u(t) + \sigma_{Ls}(z(t), \eta(t)) \tag{3.8}$$

$$y(t) = C_s z(t) + \xi(t), \;\; z(0) = z_0, \; t \in \mathbb{R}^+, \tag{3.9}$$

$$\bar{y}(t) = \textstyle\sum_{k=1}^{N} y(t_k)\chi\left[t_k, t_{k+1}\right)(t) \;\; N \in \mathbb{N} \tag{3.10}$$

under the hypothesis:

- $\sigma_{Ls}(\cdot,\cdot) \in \mathscr{C}(\mathbb{I}, \mathbb{R})$ is an unknown deterministic and bounded perturbation function such that $||\sigma_{Ls}(z(t), \eta_{Ls}(t))||_{Q_\sigma} \leqslant ||\phi_{Ls}(z(t)) - A_{Ls}z(t)||_{Q_f} + ||\eta_{Ls}(t)||_{Q_\eta} \leqslant (1+\delta) + h||z(t)||_{Q_x}$, where $\mathbb{I}$ is an interval in $\mathbb{R}^+$, $Q_\sigma := \operatorname{diag}(Q_f, Q_\eta)$ and $C_s := C\Psi \in \mathbb{R}^{q\times n}$ and where Q_f, $Q_x \in \mathbb{R}^{n\times n}$, $Q_f = Q_f^{\mathsf{T}} > 0$, $Q_x = Q_x^{\mathsf{T}} > 0$.

- $u(t) = K_s z(t)$ is the linear descriptor variable feedback control.

Control aim now is to find a gain matrix, K_s, with respect to the descriptor variable feedback system (3.8)–(3.10), which guarantee practical stability of solution trajectories, $z(t)$, for all consistent initial condition. Aforementioned assumptions are all compatible with assumptions of (3.1)–(3.3). Notice that matrix A selection is equivalent to matrix J selection for regular pair matrix (E, A).

3.2. *Luenberger observer*

A state Luenberger observer will provide an estimate of the internal descriptor variable of the system (3.8)–(3.10) when it cannot be determined by direct observation.

REMARK 3.1 We want to remark the existence of controllability/observability condition for unperturbed implicit system in the form of Misrikhanov's observability condition in Misrikhanov & Ryabchenko (2008) and to extend the use of the state observer to fully reconstruct the system descriptor variable from its output measurements. According to the index concept section in Kunkel & Mehrmann (2006), present contribution supposes that the controllability/observability condition for the original nonlinear perturbed system (3.1)–(3.3) is equivalent to the solvability condition of the same system (listed on its corresponding assumption items). In addition, solvability condition $\|\sigma\|_{Q_\sigma} \leqslant (1+\delta) + h\|z(t)\|_{Q_z}$ of the nonlinear system (3.8)–(3.10) is an inherited property from the corresponding condition $\|\eta\|_{Q_\eta} + |\xi\|_{Q_\xi} \leqslant 1$ of nonlinear system (3.1)–(3.3). Then to ensure that system (3.8)–(3.10) is controllable/observable, condition $\|\sigma\|_{Q_\sigma} \leqslant (1+\delta) + h\|z(t)\|_{Q_z}$ must be held if condition $\|\eta\|_{Q_\eta} + |\xi\|_{Q_\xi} \leqslant 1$ is held for nonlinear system. See Theorem 4.1 proof.

Luenberguer observer will be computer implemented using the observer model of the dynamical system (3.8)–(3.10)

$$E_{Ls}\dot{\hat{z}}(t) = A_{Ls}\hat{z}(t) + B_{Ls}u(t) + L(\bar{y}(t) - C_s\hat{z}(t)), \tag{3.11}$$

where $L \in \mathbb{R}^{n\times q}$. The observer is called asymptotically stable if the observer error $e(t) := E_{Ls}\varepsilon(t)$ converges to zero when $t \to \infty$, where $\varepsilon(t) := z(t) - \hat{z}(t)$. Observer error satisfies the equation

$$E_{Ls}\dot{\varepsilon}(t) = (A_{Ls} - LC_s)\varepsilon(t) + \sigma(z, \eta) - L(\Delta y(t) + \xi(t)), \tag{3.12}$$

which is obtained from equations (3.8), (3.9)and (3.11) and considering addition of the two effective zero terms, $LC_sz(t) - LC_sz(t)$ and $y(t) - y(t)$. Feedback observer model is obtained using $u(t) = K_s\hat{z}(t)$,

$$E_{Ls}\dot{\hat{z}}(t) = (A_{Ls} + B_{Ls}K_s)\hat{z}(t) + L\Delta y(t) + LC_s\varepsilon(t) + L\xi(t), \tag{3.13}$$

where $\Delta y(t) := \bar{y}(t) - y(t)$. Expression (3.13) was obtained considering equation (3.9) and using the effective zero term $-Ly(t) + Ly(t) + LC_sz(t) - LC_sz(t)$ added in equation (3.11).

According to extended vector definition, $w(t) := \left(\hat{z}^{\mathsf{T}}(t), \varepsilon^{\mathsf{T}}(t)\right)^{\mathsf{T}} \in \mathbb{R}^{2n}$, expressions (3.12) and (3.13), which govern the observer model dynamics and estimation error dynamics, can be written in the the matrix form

$$E_L\dot{w}(t) = F(K_s, L)w(t) + G\psi(t) + M\Delta Y(t), \tag{3.14}$$

where E_L, F, G, $M \in \mathbb{R}^{2n\times 2n}$, $E_L := \mathrm{diag}\{E_{Ls}, E_{Ls}\}$, $\psi(t) := \left(\sigma_{Ls}^{\mathsf{T}}, \Xi^{\mathsf{T}}\right)^{\mathsf{T}}$, $\Xi := L\xi$, $\Delta Y(t) := (X^{\mathsf{T}}, X^{\mathsf{T}})^{\mathsf{T}}$, $X := L\Delta y$, $F(K_s, L) := \begin{pmatrix} A_{Ls} + B_{Ls}K_s & LC_s \\ 0_n & A_{Ls} - LC_s \end{pmatrix}$, $G := \begin{pmatrix} 0_n & I_n \\ I_n & -I_n \end{pmatrix}$ and $M := \begin{pmatrix} 0_n & I_n \\ I_n & -I_n \end{pmatrix}$. 0_n and I_n are zero and identity matrix in $\mathbb{R}^n$, respectively. Matrix expression (3.14) allows us to extend problem formulation of system (3.8)–(3.10) to the most general case of continuous-time dynamical system described by linear semi-explicit DAE and sampling-data outputs with a descriptor variable,

$$E_L\dot{w}(t) = F(K_s, L)w(t) + G\psi(t) + M\Delta Y(t) \tag{3.15}$$

$$y(t) = C_s z(t) + \xi(t), \;\; w(0) = w_0 \tag{3.16}$$

$$\bar{y}(t) = \sum_{k=1}^{N} y(t_k)\chi\left[t_k, t_{k+1}\right)(t) \;\; N \in \mathbb{N}, \tag{3.17}$$

where $w_0 = (z_0^{\mathsf{T}}, \varepsilon_0^{\mathsf{T}})^{\mathsf{T}}$ and $z_0 = z(0)$, $\varepsilon_0 = \varepsilon(0)$. Transformed IVP of previous section can be generalized into the next optimization problem using the invariant attractive ellipsoid concept and practical stability approach.

3.3. *Optimization problem formulation*

Optimization problem can be formulated as follows.

PROBLEM 3.1 Find gain matrices, K_s and L, with respect to the feedback system (3.15)–(3.17), which guarantee practical stability of its solution trajectories, $w(t)$, converging to the attractive ellipsoid, defined by $\mathscr{E}(P) := \{w \in \mathbb{R}^{2n} \mid w^{\mathsf{T}}Pw \leqslant 1, \;\; P \in \mathbb{R}^{2n\times 2n}, \;\; P > 0, P^{\mathsf{T}} = P\}$, of minimum area; they are equivalent to solve the next optimization problem:

$$\begin{aligned} &\text{minimize } \mathrm{tr}[P] \\ &\text{subject to } \; P > 0, \; P = P^{\mathsf{T}}, \; K_s \in \Upsilon, \; L \in \Omega, \end{aligned} \tag{3.18}$$

where $\Upsilon \subset \mathbb{R}^{m\times n}$ and $\Omega \subset \mathbb{R}^{n\times q}$ are the admissible control and observer sets that ensure invariance of attractive ellipsoid $\mathscr{E}(P)$, provided that $t \to \infty$ and all consistent initial condition w_0.

Analytical characterization of sets, Υ and Ω, allow to develop a feasible and easy-to-implement numerical algorithm that provides a computational approach in solving optimization Problem 3.1. This characterization will be developed considering the Lyapunov-like stability analysis approach and the use of BMI framework.

4. Practical stability

Main result of this contribution, in the form of the following theorem, allowsus to ensure practical stability of solution trajectories of the DAE system (3.15)–(3.17), under feasible initial conditions. Similar to Theorem 2.1, which requires the use of a *Lyapunov function*, the following theorem requires the use of a *storage function* definition, $V : \mathbb{R}^{2n} \to \mathbb{R}^+$, valuated along solution trajectories $w(t)$.

THEOREM 4.1 Let $\alpha, \delta > 0$, $\beta := 1 + \delta > 0$ and $V(w) := w^{\mathsf{T}} P w$, $P = P^{\mathsf{T}}$, $P > 0$, $P \in \mathbb{R}^{2n\times 2n}$. Derivative of storage function, $DV(w(t))$, is calculated using the so-called *descriptor methodology*, developed in Fridman (2006). Then $DV(w(t)) := \dot{V}(w(t)) + 2\langle[P_1 w + P_2 \dot{w} + P_3 \psi + P_4 \Delta Y], [-E_L \dot{w} + F(Ks, L)w(t) + G\psi(t) + M\Delta Y]\rangle$ and $P_1, P_2, P_3, P_4 \in \mathbb{R}^{2n\times 2n}$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{2n}$, are nonsingular constant matrices. If inequality

$$DV(w(t)) + \alpha V(w(t)) - \beta \leqslant W^{\mathsf{T}}(t) Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha) W(t)$$

holds, where $W(t) := (w^{\mathsf{T}}(t), \dot{w}^{\mathsf{T}}(t), \psi^{\mathsf{T}}(t), \Delta Y^{\mathsf{T}}(t))^{\mathsf{T}}$ is a new extended descriptor variable, $Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha)$ is a matrix function such that $Z \leqslant 0$, then it is clear that the following inequality $DV(t) \leqslant -\alpha V(t) + \beta$ holds, and therefore $\overline{\lim}_{t\uparrow\infty} V(t) \leqslant \frac{\beta}{\alpha}$, where $\overline{\lim}_{t\uparrow\infty} V(t) = \lim_{t_0\to\infty} \lim_{r\to 0} [\sup_{t\in S(t_0,r)} V(t)]$, and $S(t_0, r)$ denotes a vicinity with centre t_0 and radius r. Moreover and finally, $V(w(t))$ satisfy the next condition

$$V(t) \leqslant \frac{\beta}{\alpha} + (V(0) - \frac{\beta}{\alpha}) \exp(-\alpha t) \;\; \forall t \geqslant 0.$$

Proof. A sketch of the proof of theorem includes the so-called descriptor method (see Fridman (2006)) that allows us to estimate derivative of $V(w) = w^{\mathsf{T}} P w$ using

$$\begin{aligned} DV(w(t)) = \dot{V}(w(t)) + 2\langle\big[P_1 w(t) + P_2 \dot{w}(t) + P_3 \psi + P_4 \Delta Y\big], \\ [-E_L \dot{w}(t) + F(K_s, L) w(t) + G\psi(t) + M\Delta y]\rangle, \end{aligned} \tag{4.1}$$

where unknown matrices P_1, P_2, P_3, $P_4 \in \mathbb{R}^{2n\times 2n}$. Let $W(t) = (w^{\mathsf{T}}(t), \dot{w}^{\mathsf{T}}(t), \psi^{\mathsf{T}}(t), \Delta Y^{\mathsf{T}}(t))^{\mathsf{T}}$, then almost all terms of *adjoint form* defined by $\mathscr{F}(w(t)) := DV(w(t)) + \alpha V(w(t)) - \beta$. Right-hand side can be included into a matrix function in the next form $W^{\mathsf{T}}(t) \bar{Z}(P, P_1, P_2, P_3, P_4, K_s, L, \alpha) W(t)$, where

$$\begin{aligned}
&\bar{Z}_{11} := P_1^{\mathsf{T}} F + F^{\mathsf{T}} P_1 + \alpha P,\;\; \bar{Z}_{12} := P - P_1^{\mathsf{T}} E_L + F^{\mathsf{T}} P_2, \\
&\bar{Z}_{13} := P_1^{\mathsf{T}} G + F^{\mathsf{T}} P_3,\;\; \bar{Z}_{14} := P_1^{\mathsf{T}} M + F^{\mathsf{T}} P_4,\;\; \bar{Z}_{21} := P - E_L^{\mathsf{T}} P_1 + P_2^{\mathsf{T}} F, \\
&\bar{Z}_{22} := -P_2^{\mathsf{T}} E_L - E_L^{\mathsf{T}} P_2,\;\; \bar{Z}_{23} := P_2^{\mathsf{T}} G - E_L^{\mathsf{T}} P_3,\;\; \bar{Z}_{24} := P_2^{\mathsf{T}} M - E_L^{\mathsf{T}} P_4, \\
&\bar{Z}_{31} := G^{\mathsf{T}} P_1 + P_3 F,\;\; \bar{Z}_{32} := G^{\mathsf{T}} P_2 - P_3 E_L,\;\; \bar{Z}_{33} := P_3 G + G^{\mathsf{T}} P_3,\;\; \bar{Z}_{34} := P_3 M + G^{\mathsf{T}} P_4, \\
&\bar{Z}_{41} := M^{\mathsf{T}} P_1 + P_4 F,\;\; \bar{Z}_{42} := M^{\mathsf{T}} P_2 - P_4 E,\;\; \bar{Z}_{43} := M^{\mathsf{T}} P_3 + P_4 G,\;\; \bar{Z}_{44} := P_4 M + M^{\mathsf{T}} P_4.
\end{aligned}$$

If we add the effective zero term $\psi^{\mathsf{T}}(t) Q_3 \psi(t) - \psi^{\mathsf{T}}(t) Q_3 \psi(t)$, it is possible to use general hypothesis of (3.15)–(3.17) (where $\|\psi(t)\|^2_{Q_3} = \|\sigma_{Ls}(\eta, \Gamma w)\|^2_{Q_\sigma} + \|\xi(t)\|^2_{Q_\xi}$, $\|f_{Ls}(\Gamma w) - A_{Ls}\Gamma w\|^2_{Q_x} \leqslant \delta + h\|\Gamma w\|^2_{Q_x}$ and $\|\eta_{Ls}\|^2_{Q_\eta} + \|\xi\|^2_{Q_\xi} \leqslant 1$, with $\hat{z} := \Gamma w$ and $\Gamma := (I_n, 0_n)$) and to set an upper bound for the so-called *adjoint form* using the global matrix function

$$\mathscr{F}(w(t)) \leqslant W^{\mathsf{T}}(t) Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha) W(t),$$

where $\beta := 1 + \delta$ and new matrix function $Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha)$ has absorbed the terms

$$-\psi^{\mathsf{T}}(t) Q_3 \psi(t)$$

and

$$h\|\varGamma w(t)\|^2_{Q_x}$$

in $Z_{11} := \bar{Z}_{11} + h\varGamma^{\top} Q_x \varGamma$ and $Z_{33} := -Q_3$. In the rest of the cases, $Z = \bar{Z}$. Conditions of

$$Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha) \leqslant 0$$

guarantee that $V(w(\cdot))$ satisfy all conditions in Theorem 2.1. A detailed proof of the theorem can be found in Azhmyakov *et al.* (2013) and Juarez *et al.* (2012, 2013, 2011). □

Theorem 4.1 allows us to characterize in an easily implementable way the set $\varUpsilon$ of optimal control Problem 3.1 through solution of BMI condition $Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha) \leqslant 0$, under a fixed scalar parameter α. Optimal control Problem 3.1 can be rewritten in the following form.

PROBLEM 4.2 Find gain matrices, K_s and L, with respect to the feedback system (3.15)–(3.17), which guarantee practical stability of its solution trajectories, $w(t)$, converging to the attractive ellipsoid, defined by $\mathscr{E}(P) := \{w \in \mathbb{R}^{2n} \mid w^{\top} P w \leqslant 1, \ \ P \in \mathbb{R}^{2n\times 2n}, \ \ P > 0, P^{\top} = P\}$, of minimum area; they are equivalent to solve the next optimization problem

$$\begin{aligned} &\text{minimize } \operatorname{tr}[P] \\ &\text{subject to } P > 0, \ P = P^{\top}, \\ &Z(P, P_1, P_2, P_3, P_4, K_s, L, \alpha) < 0. \end{aligned} \tag{4.2}$$

Once BMI is solved, the following theorem will set the equivalence relation between solutions of transformed system (3.15)–(3.17) and system described by (3.1)–(3.3). The solution vector $(P^{opt}, P_1^{opt}, P_2^{opt}, P_3^{opt}, P_4^{opt}, K_s^{opt}, L^{opt}, \alpha^{opt})$, where $K_s = \varPsi K$, of convex Problem 4.2 under a fixed scalar parameter α, defines a suboptimal solution due to α-values quest and is limited to a finite set of $\mathbb{R}_+$.

THEOREM 4.3 If transformed optimization convex Problem 4.2 has a suboptimal solution $(P^{subopt}, P_1^{subopt}, P_2^{subopt}, P_3^{subopt}, P_4^{subopt}, K_s^{subopt}, L^{subopt})$, in the sense of minimum trace of ellipsoid $\mathscr{E}(P)$ for a fixed scalar parameter α, then 5-tuple $(P^{subopt}, P_1^{subopt}, P_2^{subopt}, P_3^{subopt}, P_4^{subopt}, K_s^{subopt}, L^{subopt}, \alpha)$ is also a suboptimal solution to IVP described by (3.1)–(3.3), with $K_s = \varPsi K$.

Proof. A detailed proof of Theorem is presented in Azhmyakov *et al.* (2013) and Juarez *et al.* (2012, 2013, 2011). □

Finally, transformed optimal Problem 4.2 sets a theoretical approach to deal with numerical treatment of IVP described by (3.1)–(3.3), just considering computational simulation of a BMI.

5. Numerical example

Here two numerical examples are presented. The first one is an academic example that exposes basic idea of optimization Problem 3.1 solution. The second one describes a chemical reactor where a first-order isomerization reaction takes place and the heat generated is removed from the system through an external cooling circuit. In both cases the well-known PENOPT toolbox from MatLab software is used to solve BMI, which includes Yalmip toolbox and PENBMI solver.

5.1. *Academic example*

Consider the optimization Problem 3.1 associated with dynamical system (3.15)–(3.17) with the following constant values: $E_{Ls} = \text{diag}\{1, 1, 1, 0\},\ C_s = (0, 1, 0, 0)$,

$$A_{Ls} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B_{Ls} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Previous values and full variable matrices $Ks \in \mathbb{R}^{2\times 4},\ L \in \mathbb{R}^4$, are used to construct block matrices, $E_L,\ F(K_s, L),\ G$ and M, of expression (3.14). Storage function, $V(w)$, has a symmetric variable $P \in \mathbb{R}^{8\times 8}$. $DV(x(t))$ calculation also has full variable matrices $P_i \in \mathbb{R}^{8\times 8}$, for $i = 1, 2$.

Numerical solutions of optimization Problem 3.1 satisfy all conditions of Theorem 4.1. The main condition that must be satisfied is BMI: $Z \leqslant 0$. Then quasi-optimal values of $K_s,\ L$ and P were used to proved practical stability of nonlinear initial system described by (3.1)–(3.3). Simulation details are presented.

$Z \in \mathbb{R}^{32\times 32}$ was a block matrix, where $\alpha = \beta = \delta = h = 1 \times 10^{-3},\ Q_x = 1 \times 10^{-3} * I_4$ and $Q_3 = 1 \times 10^3 * I_4$. MatLab, through PENOPT, solves BMI in an optimal way and found values of

$$K_s^{qopt} = \begin{pmatrix} -406.077 & -519.126 & -521.161 & 661.379 \\ -406.082 & -519.107 & -521.168 & 661.351 \end{pmatrix},$$

$L^{qopt} = 1 \times 10^{-6} \begin{pmatrix} -11.4312 & -7.46324 & -8.53132 & -9.11755 \end{pmatrix}^{\mathsf{T}}$ and

$$\text{eig}(P^{qopt}) \in [0.99994, 33469.4712] \subset \mathbb{R}_+.$$

The above-mentioned values were used in numerical simulation of IVP described by (3.1)–(3.3), where $\Pi = \Psi = I_4$, and therefore $E = E_{Ls}, B = B_{Ls},\ C = C_s, f(x) = \big(f_1(x)\ f_2(x)\ f_3(x)\ f_4(x)\big)^{\mathsf{T}}, f_1 = -x_1 - |x_2| + \sin(x_2) + 0.1x_3, f_2 = -0.1x_1 - x_2 + \cos(x_3), f_3 = -|x_1| - 0.1\cos(x_2) - x_3, f_4 = 0.1x_2$, initial condition $x_0 = (100, \text{-}553.6683, 200, 100)^{\mathsf{T}}$ and $\eta = (0.02sin(t), -0.05cos(t), -0.05cos(t), 0.03cos(t))^{\mathsf{T}}$,

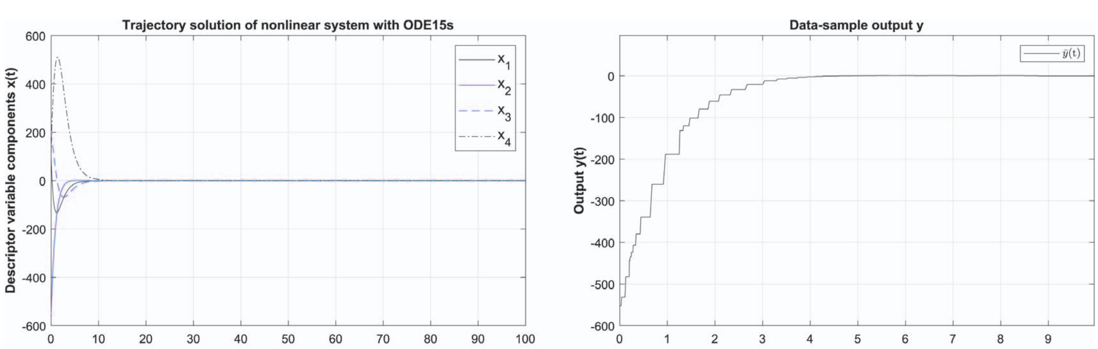

FIG. 2. (Left) (a) Descriptor variable components x(t) of nonlinear system. (Right) (b) Data-sample output y(t).

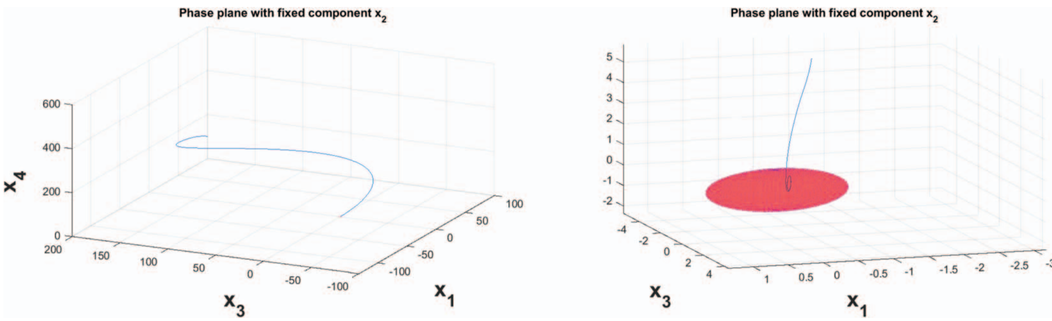

FIG. 3. (Left) a) Phase plane of trajectory solution emerging from initial conditions, where component x_2 is a fixed value. (Right) b) Zoom around origin on phase plane and Ellipsoidal region.

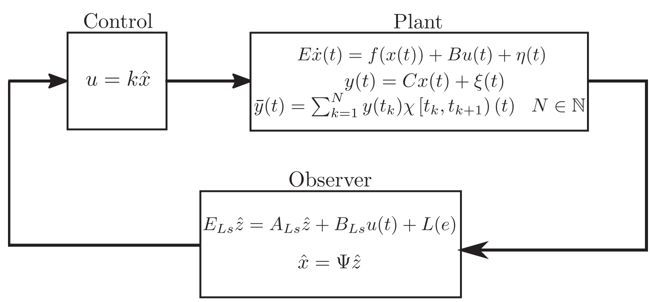

FIG. 4. Block diagram of the control loop.

$\xi = 0.1sin(t)$. The model shown in Fig. 4 is simulated in MatLab Simulink. Results reported here include time evolution of descriptor variable components, $x(t)$, in Fig. 2; phase portrait performed in x_1, x_2 and x_3 space that ellipsoid forms is also included, see Fig. 3.

5.2. *Continuous stirred-tank reactor example*

The relevant equations in a chemical reactor where a first-order isomerization reaction takes place and the heat generated is removed from the system through an external cooling circuit are

$$\begin{aligned}
C' &= -K_1 C + K_1 C_0 - R \\
T' &= K_1(T_0 - T) + K_2 R - K_3(T - T_C) \\
0 &= -K_3 exp^{-\frac{K_4}{T}} C + R \\
0 &= C - u.
\end{aligned} \tag{5.1}$$

Here C_0 and T_0 are the (assumed) known feed reactant concentration and feed temperature, respectively. C and T are the corresponding quantities in the product. R is the reaction rate per unit volume, T_C is the temperature of the cooling medium (which can be varied) and the K_i are constants. The more interesting case is where the last equation is a specified desired product concentration and we

TABLE 1 *Fixed parameters of the reactor*

$q_r = 0.08\ [m^3 \cdot min^{-1}]$	$V_r = 1.2\ [m^3]$
$h_1 = 4.8 \times 10^4 [kJ \cdot kmol^{-1}]$	$\rho_r = 985\ [kg \cdot m^{-3}]$
$c_{pr} = 4.05\ [kJ \cdot kg^{-1} \cdot K^{-1}]$	$U = 43.5\ [kJ \cdot min^{-1} m^{-2} K^{-1}]$
$A_r = 5.5[m^2]$	

want to determine the T_C (control) that will produce this C. In this case, we obtain a semi-explicit DAE system with state variables C, T, R, T_C, where $K_1 := \frac{q_r}{V_r}$, $K_2 := \frac{h_1}{\rho_r c_{pr}}$, $K_3 := \frac{A_r U}{V_r \rho_r c_{pr}}$, $K_4 := 13477$ K. Fixed parameters of reactor are shown in Table 1.

If we define $x_1 := C - C_0$, $x_2 := T - T_0$ and $x_3 := R$, $x_4 := T_c - T_0$, system (5.1) is written into the matrix form of equations (3.8)–(3.10): $E\dot{x}(t) = Ax(t) + Bu(t) + \sigma(x(t), \eta(t))$, where $E_{Ls} = \text{diag}\{1, 1, 0, 0\}$, $C_s = (1, 0, 0, 0)$,

$$A_{Ls} = \begin{pmatrix} -K_1 & 0 & -1 & 0 \\ 0 & -(K_1 + K_3) & K_2 & K_3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_{Ls} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

and

$$\phi_{Ls}(t) = \begin{pmatrix} 0 \\ 0 \\ -K_3 e^{-\frac{K_4}{x_2 + T_0}} (C + C_0) \\ C_0 \end{pmatrix}.$$

The aforementioned constant parameters and the full variable matrices $K_s \in \mathbb{R}^{2\times4}$, $L \in \mathbb{R}^4$, are used to construct block matrices, E_L, $F(K_s, L)$, G and M, of expression (3.14). Storage function, $V(w)$, has a symmetric variable $P \in \mathbb{R}^{8\times8}$. $DV(x(t))$ calculation also has full variable matrices $P_i \in \mathbb{R}^{8\times8}$, for $i = 1, 2$.

Numerical solutions of optimization Problem 3.1 satisfy all conditions of Theorem 4.1. The main condition that must be satisfied is BMI: $Z \leqslant 0$. Then quasi-optimal values of K_s, L and P were used to proved practical stability of nonlinear initial system described by (3.1)–(3.3). Simulation details are presented.

$Z \in \mathbb{R}^{32\times32}$ was a block matrix, where $\alpha = \beta = \delta = h = 1 \times 10^{-3}$, $Q_x = 1 \times 10^{-3} * I_4$ and $Q_3 = 1 \times 10^3 * I_4$. MatLab, through PENOPT, solves BMI in an optimal way and found values of

$$K_s^{qopt} = \begin{pmatrix} -736.842 & -924.657 & -819.611 & 569.390 \\ -736.243 & -924.701 & -819.238 & -569.351 \end{pmatrix},$$

$$L^{qopt} = 1 \times 10^{-5} \begin{pmatrix} -21.3124 & -15.63244 & -32.31325 & -33.75511 \end{pmatrix}^{\mathsf{T}}$$

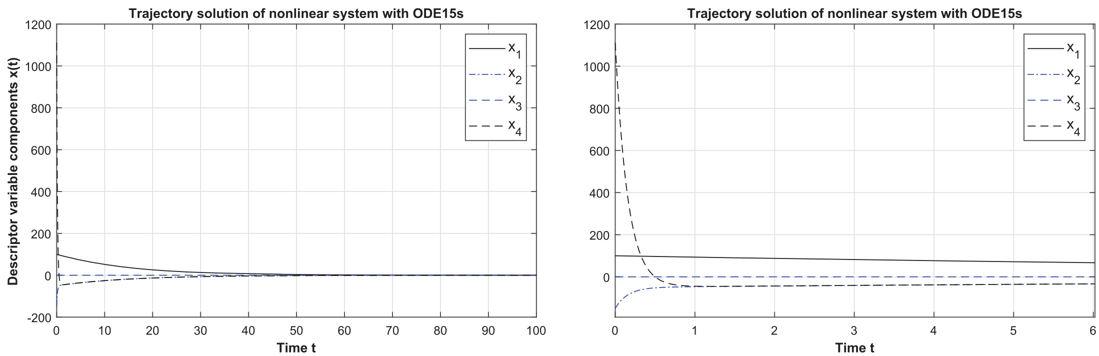

FIG. 5. (Left) a) State components of descriptor variable x(t) of nonlinear system and (Right) b) Zoom around first 6 s.

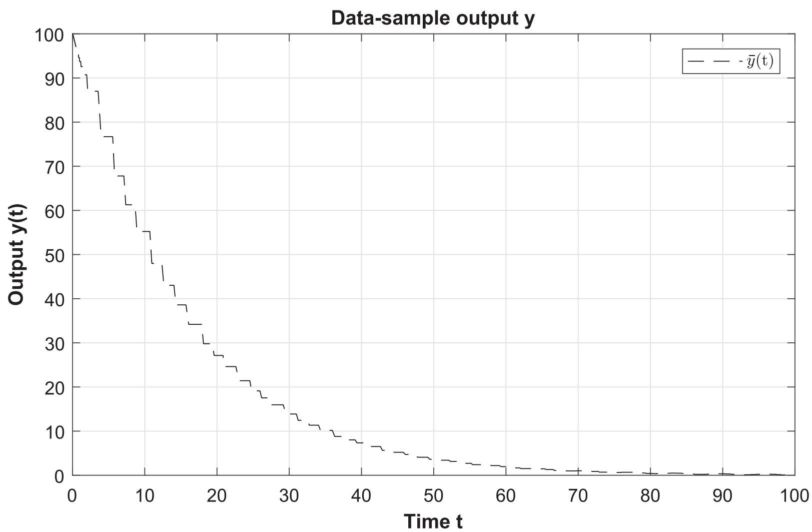

FIG. 6. Data-sample output y(t).

and $\mathrm{eig}(P^{qopt}) \in [0.99994, 35723.4218] \subset \mathbb{R}_+$. The above-mentioned values were used in numerical simulation of IVP described by (3.1)–(3.3). Since DAE initial condition must satisfy $F(x_0, \dot{x}_0, t_0) = 0$, then $x_0 = (100, -150, -2.84 \times 10^{-14}, 1.92 \times 10^3)^\intercal$, and

$$\eta = \left[0.001sin(t) \quad -0.05sin(2t) \ \ 0.003sin\left(\frac{t}{2}\right) \quad 0.001sin(t)\right]^\intercal$$

and $\xi = 0.1sin(t)$. The model shown in Fig. 4 is simulated in MATLAB Simulink. Results reported here include time evolution of descriptor variable components, $x(t)$, in Fig. 5; time evolution of the data-sample output $y(t)(x_1)$ in Fig. 6. Phase portrait performed in x_1, x_2 and x_3 space that ellipsoid forms is also included, see Fig. 7.

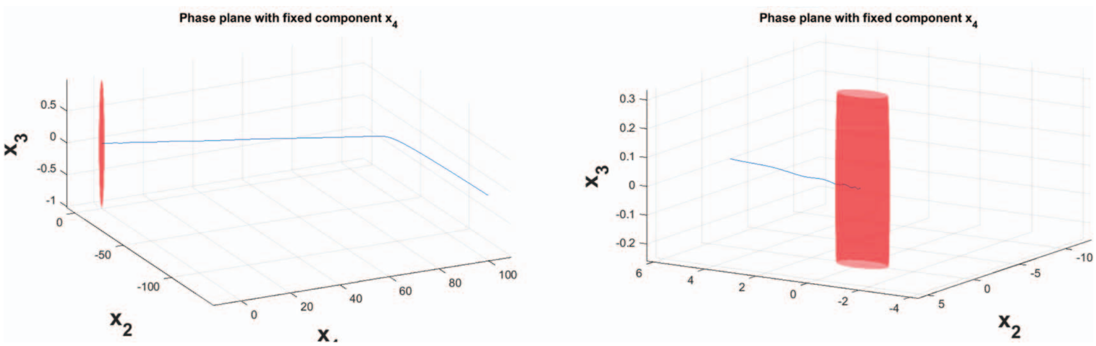

FIG. 7. (Left) a) Phase plane of trajectory solution emerging from initial conditions, where component x_2 is a fixed value. (Right) b) Zoom around origin on phase plane and ellipsoidal region.

TABLE 2 *Main features and characteristics of author contributions on DAE*

Contribution	System	Autonomous Design Complexity	Physical Implementation	Processing Time
Juarez *et al.* (2012)	Nonlinear	Nonlinear DAE None observer	Academic Example Two-Dimensional	Sedumi-Yalmip (Low) LMI
Juarez *et al.* (2011)	Linear	Linear DAE None observer	Academic Example Two-Dimensional	Sedumi-Yalmip (Low) LMI
Juarez *et al.* (2013)	Nonlinear	Semi-explicit DAE None observer	Academic Example Two-Dimensional	Sedumi-Yalmip (Medium) LMI
Azhmyakov *et al.* (2013)	Nonlinear	Implicit system None observer	Micro DC-Motor Three-Dimensional	Sedumi-Yalmip (Medium) LMI
Current study	Nonlinear	Semi-Explicit DAE Luenberguer observer	Academic Example Four-Dimensional	PENOPT (Low) BMI

6. Conclusion

This paper addressed the problem of robust control for a class of nonlinear dynamical system governed by DAE of an affine structure in the continuous time domain under the approach AEM using the linear feedback control $u = Kx$. DAEs were transformed into its WCF called *semi-explicit* LDAE-CC, in the presence of bounded uncertainties. The one-to-one transformation for the regular pair matrix (E, A) allowed us to set a one-to-one correspondence between the corresponding solution sets, so we considered transformed problem instead of original problem with respect to solvability and practical stability. Regularity of a matrix pair was ensured with election of matrix A. The combination of the modified invariant ellipsoid approach and descriptor method made it possible to obtain the robustness of the designed control and to establish some well-known stability properties of dynamical systems in terms of a BMI. With sub-optimal solution of above BMI we constrained all possible trajectories of the system as bounded inside an ellipsoid, $\mathscr{E}(P)$, of minimal size. Sub-optimal term is used here because the α-values quest is restricted to a finite interval of $\mathbb{R}_+$. This ellipsoid defined the zone convergence or *practical stability* of an effective robust control strategy. Finally, the applicability of the proposed method was illustrated by a computational example. Moreover, it is noteworthy that the problems of robust control for linear and nonlinear DAEs have been subjects of past studies by the authors, as is shown in Table 2. The current work is the more recent contribution of the authors in this research area.

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