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Error estimation and *h*-adaptive refinement in the analysis of natural frequencies

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Abstract

This paper deals with the estimation of the discretization error and the definition of an optimum *h*-adaptive process in the finite element analysis of natural frequencies and modes. Consistent and lumped mass matrices are considered. In the first case, the discretization error essentially proceeds from the stiffness modelization, so it is possible to apply the same error estimators than those considered in static problems. On the other hand, the error associated with the modelization of the inertial properties must be taken into account if lumped mass matrices are used. As far as *h*-adaptivity is concerned, it is usually interesting to obtain meshes with a specified error for each mode. However, traditional criteria for static problems consider only one load case. Defining the optimum mesh as the one that gets the desired error with the minimum number of elements, a method is proposed for the *h*-adaptive process taking into account a set of natural modes simultaneously. The proposed methods have been validated by applying them to bi-dimensional test problems. © 2001 Elsevier Science B.V. All right reserved.

Keywords: Error estimation; Natural frequencies; Free vibration analysis; *h*-adaptive refinement

1. Introduction

The essential aim of an *h*-adaptive finite element process is to obtain the simplest mesh that allows to achieve the discretization error required by the user. In order to get this objective is necessary to evaluate estimations of the discretization error, to define criteria that control the *h*-adaptive process, and to apply mesh generation procedures based on the prescribed element size in the domain considered.

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Nowadays, methods to estimate the discretization error for static problems have been developed and verified. With regard to vibration problems, the methods used in static problems can be valid if consistent mass matrices are used to model the inertial characteristics. However, such an extrapolation can be poorly accurate when lumped mass matrices are considered. Hager and Wiberg [1] make a revision of the works that deal with finite element solution of vibration problems. They present a finite element *h*-adaptive scheme to obtain the eigenvalue and eigenvector of a single vibration mode with high accuracy using consistent mass matrices.

Once the discretization error has been estimated, it is necessary to redefine the finite element mesh to get the required error. This basically consists of setting up an optimum mesh criterion. Traditionally, the optimum mesh has been defined as the one that distributes the absolute error uniformly among all the elements [2]. Recently, it has been proved that this criterion is equivalent to minimize the number of elements [3]. In general, these criteria have been developed considering only a single load case or analysis condition. Therefore, these criteria cannot be directly applied to the analysis of natural frequencies, since a set of modes have to be considered in order to define the optimum mesh associated with all of them. Among the few papers about the subject, Shepard [4] points out that the use of optimum mesh criteria for several load cases can reduce considerably the computational effort.

With regard to the adaptive mesh generation, it is necessary to consider either a process of local refinement or a redefinition of the global mesh. The former can be more efficient, but leads to a more rigid *h*-adaptive process (i.e. it is not possible to increase the element size when required). The latter can save this drawback, and it has been applied in this work to assure that the convergence is not affected by other factors. Thus the finite element mesh will be completely redefined in each step of the *h*-adaptive process [3].

The problem of undamped free vibrations, in a finite element discretization, corresponds to

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices respectively, and $\mathbf{U}$ is the nodal displacement vector. The solutions to the eigenvalue problem are the sets of natural frequencies ($\omega_{\mathrm{fe}(r)}$) and modes ($\boldsymbol{\phi}_{\mathrm{fe}(r)}$), where r stands for the mode number.

$$(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = \mathbf{0} \rightarrow \omega_{\mathrm{fe}(r)}, \boldsymbol{\phi}_{\mathrm{fe}(r)}. \tag{2}$$

Taking into account the transformation to generalized co-ordinates, the modal stiffness and the modal mass can be defined as

$$\left.\begin{aligned} \boldsymbol{\phi}_{\mathrm{fe}(r)}^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_{\mathrm{fe}(r)} &= k_{\mathrm{fe}(r)} \\ \boldsymbol{\phi}_{\mathrm{fe}(r)}^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_{\mathrm{fe}(r)} &= m_{\mathrm{fe}(r)} \end{aligned}\right| \quad \text{with} \quad \omega_{\mathrm{fe}(r)}^2 = \frac{k_{\mathrm{fe}(r)}}{m_{\mathrm{fe}(r)}}. \tag{3}$$

In general, the natural modes will be supposed to be normalized so that $m_{\mathrm{fe}(r)} = 1$.

It is also possible to define the corresponding modal stiffness and mass for the exact value of the natural frequencies ($\omega_{\mathrm{ex}(r)}$), which can be written as

$$\omega_{\mathrm{ex}(r)}^2 = \frac{k_{\mathrm{ex}(r)}}{m_{\mathrm{ex}(r)}} = \frac{k_{\mathrm{fe}(r)} - \Delta k_{(r)}}{m_{\mathrm{fe}(r)} - \Delta m_{(r)}}, \tag{4}$$

where $\Delta k_{(r)} = k_{\mathrm{fe}(r)} - k_{\mathrm{ex}(r)}$ and $\Delta m_{(r)} = m_{\mathrm{fe}(r)} - m_{\mathrm{ex}(r)}$ represent the finite element error in modal stiffness and modal mass respectively. Expanding this expression, it yields:

$$\omega^2_{\mathrm{ex}(r)} = \frac{k_{\mathrm{fe}(r)}}{m_{\mathrm{fe}(r)}} - \frac{\Delta k_{(r)}}{m_{\mathrm{fe}(r)}} + \left(\frac{k_{\mathrm{fe}(r)}}{m_{\mathrm{fe}(r)}}\frac{\Delta m_{(r)}}{m_{\mathrm{fe}(r)}}\right) + \cdots \tag{5}$$

and taking into account (3) and the normalization condition of the natural modes, it is obtained

$$\omega^2_{\mathrm{ex}(r)} - \omega^2_{\mathrm{fe}(r)} \approx -\Delta k_{(r)} + \omega^2_{\mathrm{fe}(r)}\,\Delta m_{(r)}. \tag{6}$$

In this way, the error in natural frequency between the exact solution and the finite element solution depends on both the error in the modal stiffness and the error in the modal mass. The accuracy of the modal mass is higher than that of the modal stiffness if consistent mass matrices are used in the finite element analysis. Therefore, the second term of (6) is expected to be smaller than the first, and so the error arises essentially from the modal stiffness term [5]. However, this conclusion can be false if lumped mass matrices are used, since both the error in modal stiffness and modal mass must be considered.

2. Estimation of the discretization error

The energy norm of the exact solution of a linear elastic problem is defined by

$$\|\mathbf{u}_{\mathrm{ex}}\| = \left[\int_V \boldsymbol{\sigma}^{\mathrm{T}}_{\mathrm{ex}}\boldsymbol{\varepsilon}_{\mathrm{ex}}\,\mathrm{d}V\right]^{1/2} = \left[\int_V \boldsymbol{\sigma}^{\mathrm{T}}_{\mathrm{ex}}\mathbf{D}^{-1}\boldsymbol{\sigma}_{\mathrm{ex}}\,\mathrm{d}V\right]^{1/2}, \tag{7}$$

where $\mathbf{u}_{\mathrm{ex}}$ is the exact displacement field, $\boldsymbol{\sigma}_{\mathrm{ex}}$ is the exact stress field, $\boldsymbol{\varepsilon}_{\mathrm{ex}}$ is the exact strain field, $\mathbf{D}$ is the stress-strain relation matrix and V is the volume of the domain on which the problem is prescribed. This norm represents the square root of twice the strain energy. For a finite element solution of the problem, it can be expressed as:

$$\|\mathbf{u}_{\mathrm{fe}}\|^2 = \int_V \boldsymbol{\sigma}^{\mathrm{T}}_{\mathrm{fe}}\mathbf{D}^{-1}\boldsymbol{\sigma}_{\mathrm{fe}}\,\mathrm{d}V = \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}, \tag{8}$$

where $\mathbf{u}_{\mathrm{fe}}$ is the finite element solution and $\boldsymbol{\sigma}_{\mathrm{fe}}$ is the stress field of the finite element solution. Taking into account the corresponding solution for each mode r, it yields

$$\|\mathbf{u}_{\mathrm{fe}(r)}\|^2 = \int_V \boldsymbol{\sigma}^{\mathrm{T}}_{\mathrm{fe}(r)}\mathbf{D}^{-1}\boldsymbol{\sigma}_{\mathrm{fe}(r)}\,\mathrm{d}V = \boldsymbol{\phi}^{\mathrm{T}}_{\mathrm{fe}(r)}\mathbf{K}\boldsymbol{\phi}_{\mathrm{fe}(r)} = k_{\mathrm{fe}(r)}. \tag{9}$$

Note that the modal stiffness is the squared energy norm of the finite element solution corresponding to the deformed shape defined by the associated mode. The energy norm of the discretization error in displacements is defined by

$$\|\mathbf{e}_{\mathrm{ex}(r)}\|^2 = \|\mathbf{u}_{\mathrm{ex}(r)} - \mathbf{u}_{\mathrm{fe}(r)}\|^2 = \int_V (\boldsymbol{\sigma}_{\mathrm{ex}(r)} - \boldsymbol{\sigma}_{\mathrm{fe}(r)})^{\mathrm{T}}\mathbf{D}^{-1}(\boldsymbol{\sigma}_{\mathrm{ex}(r)} - \boldsymbol{\sigma}_{\mathrm{fe}(r)})\mathrm{d}V \tag{10}$$

and the following equality holds asymptotically when the number of degrees of freedom increases

$$\|\mathbf{e}_{\mathrm{ex}(r)}\|^2 = \|\mathbf{u}_{\mathrm{ex}(r)} - \mathrm{u}_{\mathrm{fe}(r)}\|^2 = |\|\mathbf{u}_{\mathrm{ex}(r)}\|^2 - \|\mathbf{u}_{\mathrm{fe}(r)}\|^2|. \tag{11}$$

2.1. Consistent mass matrix

As stated above, the estimation of the kinetic energy is more accurate than that of the strain energy if consistent mass matrices are used [5]. Therefore, it is possible to reject the term corresponding to the error in modal mass in expression (6). This yields

$$\omega^2_{\mathrm{fe}(r)} - \omega^2_{\mathrm{ex}(r)} \approx \Delta k_{(r)} = k_{\mathrm{fe}(r)} - k_{\mathrm{ex}(r)} \tag{12}$$

Tong et al. [6] analyze the influence of the consistent and lumped mass matrices in the finite element calculation of the natural frequencies. In the case of consistent mass matrices with the appropriated order of integration and for problems without singularities, the expression is

$$0 \leqslant \omega^2_{\mathrm{fe}(r)} - \omega^2_{\mathrm{ex}(r)} \leqslant Ch^{2p}, \tag{13}$$

where h represents the element size, p the polinomial degree of the displacement interpolation functions and C a positive constant which does not depend on h. In this way, the natural frequency obtained by means of a finite element analysis is higher than the exact one, so Eq. (11) must be expressed as

$$\|\mathbf{e}_{\mathrm{ex}}\|^2 = \|\mathbf{u}_{\mathrm{ex}} - \mathbf{u}_{\mathrm{fe}}\|^2 = \|\mathbf{u}_{\mathrm{fe}}\|^2 - \|\mathbf{u}_{\mathrm{ex}}\|^2. \tag{14}$$

Combining (9) and (12), and considering (14), the error in natural frequency can be evaluated as

$$\mathrm{e}^2_{\omega\mathrm{ex}(r)} = \omega^2_{\mathrm{fe}(r)} - \omega^2_{\mathrm{ex}(r)} \approx \|\mathbf{u}_{\mathrm{fe}}\|^2 - \|\mathbf{u}_{\mathrm{ex}}\|^2 = \|\mathbf{u}_{\mathrm{ex}} - \mathbf{u}_{\mathrm{fe}}\|^2 = \|\mathrm{e}_{\mathrm{ex}}\|^2. \tag{15}$$

That is, the error in natural frequency can be approximated to the energy norm of the discretization error corresponding to the associated mode.

The Zienkiewicz–Zhu error estimator [2] has been considered in order to evaluate an estimation of the absolute error of discretization. It is based on the use of a smoothed stress field ($\boldsymbol{\sigma}^*$) in (10) instead of the exact stress field.

$$\mathrm{e}^2_{\omega\mathrm{es}(r)} = \|\mathrm{e}_{\mathrm{es}(r)}\|^2 = \int_V (\boldsymbol{\sigma}^*_{(r)} - \boldsymbol{\sigma}_{\mathrm{fe}(r)})^{\mathrm{T}}\mathbf{D}^{-1}(\boldsymbol{\sigma}^*_{(r)} - \boldsymbol{\sigma}_{\mathrm{fe}(r)})\mathrm{d}V. \tag{16}$$

The relative error is estimated as

$$\eta_{\mathrm{es}(r)} = \frac{\|\mathbf{e}_{\mathrm{es}(r)}\|}{\|\mathbf{u}_{\mathrm{es}(r)}\|} = \frac{\|\mathbf{e}_{\mathrm{es}(r)}\|}{\sqrt{\|\mathbf{u}_{\mathrm{fe}(r)}\|^2 - \|\mathbf{e}_{\mathrm{es}(r)}\|^2}} \tag{17}$$

Obviously, the error estimator will depend on the method used to smooth the stress field. In the present work, the simple method of stress averaging in nodes was used for linear elements and the superconvergent patch recovery method [7] was used for quadratic elements.

2.2. Lumped mass matrix

In the case of lumped mass matrices, a similar method to the proposed by Cook and Avrashi [8] was followed. These authors estimate the discretization error by considering an improved estimation of the natural frequencies $\omega^*_{\mathrm{fe}(r)}$. The discretization error in natural frequency is defined as

$$e^2_{\omega\mathrm{es}(r)} = |\omega^{*^2}_{\mathrm{fe}(r)} - \omega^2_{\mathrm{fe}(r)}|. \tag{18}$$

The calculation of $\omega^*_{\mathrm{fe}(r)}$ is based on improving the estimation of the modal stiffness and mass

$$\omega^*_{\mathrm{fe}(r)} = \sqrt{\frac{k^*_{\mathrm{fe}(r)}}{m^*_{\mathrm{fe}(r)}}} \tag{19}$$

As stated above, the estimation of the improved modal stiffness can be made on the basis of the energy norm and the corresponding estimation of its discretization error

$$k^*_{\mathrm{fe}(r)} = \|\mathbf{u}_{\mathrm{fe}(r)}\|^2 - \|\mathbf{e}_{\mathrm{es}(r)}\|^2. \tag{20}$$

The sign in this expression corresponds to the case in which the finite element solution overestimates the modal stiffness, as it happens when consistent mass matrices are used. Although this cannot be assured if lumped mass matrices are considered, the estimation of the modal stiffness will be correct in most of the cases.

The improvement in the estimation of the modal mass can be made by means of

$$m^*_{\mathrm{fe}(r)} = \boldsymbol{\phi}^{\mathrm{T}}_{\mathrm{fe}(r)}\mathbf{M}^*\boldsymbol{\phi}_{\mathrm{fe}(r)} \tag{21}$$

where M^* is the consistent mass matrix. Note that the consistent mass matrix does not appear in the eigenvalue problem, but only in the error estimation, so it is not necessary a global assembling.

2.3. Effectivity of the error estimator

The proposed error estimators have been applied to a set of bi-dimensional test problems in order to evaluate their effectivity. Linear and quadratic triangular elements are considered in the analysis.

The standard problem of a free beam is depicted in Fig. 1, in which 10 mode shapes are shown. The error estimators have shown a great efficiency for this problem. In order to evaluate the effectivity of the error estimator, the exact eigenvalues have been estimated through (13) and Richardson's extrapolation of two solutions obtained by means of a strong uniform refinement with quadratic triangular elements (14 162 and 54 978 d.o.f. respectively). These values are also shown in Fig. 1.

Fig. 2 shows the effectivity index of the error estimator in natural frequency vs. the number of degrees of freedom. The effectivity index is defined as the ratio of the estimated to the exact error. Consistent mass matrices and both linear and quadratic triangular elements have been considered. The sequence of meshes has been obtained through uniform refinement. As it can be seen in the figure, the error estimator shows a good behavior, since the effectivity index takes values very close to the unit, approaching more to this value as the mesh is refined. The only significant discrepancy

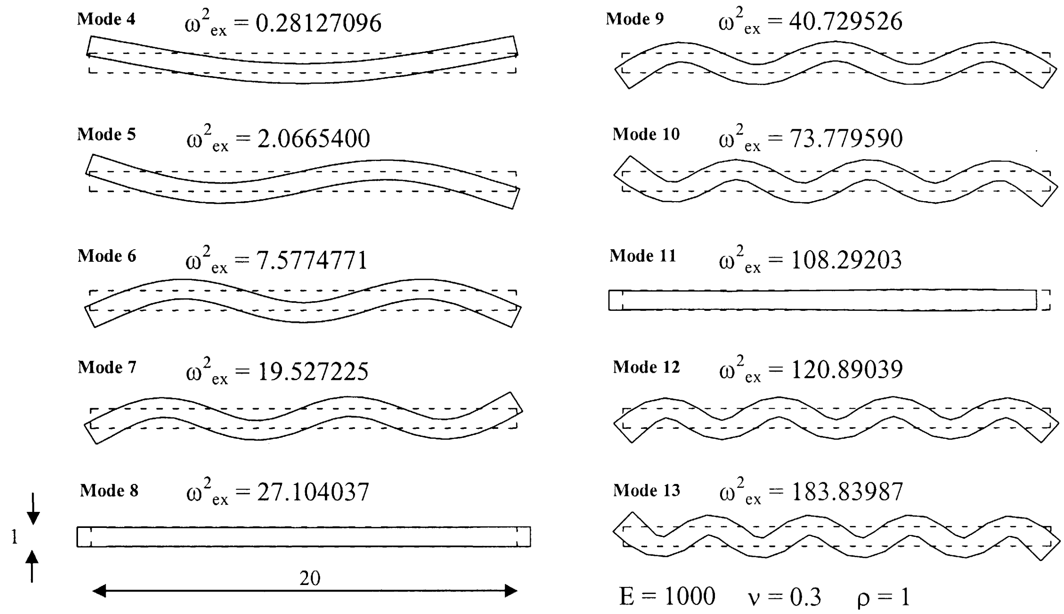

Fig. 1. Free beam problem: geometry, properties, eigenvalues and mode shapes.

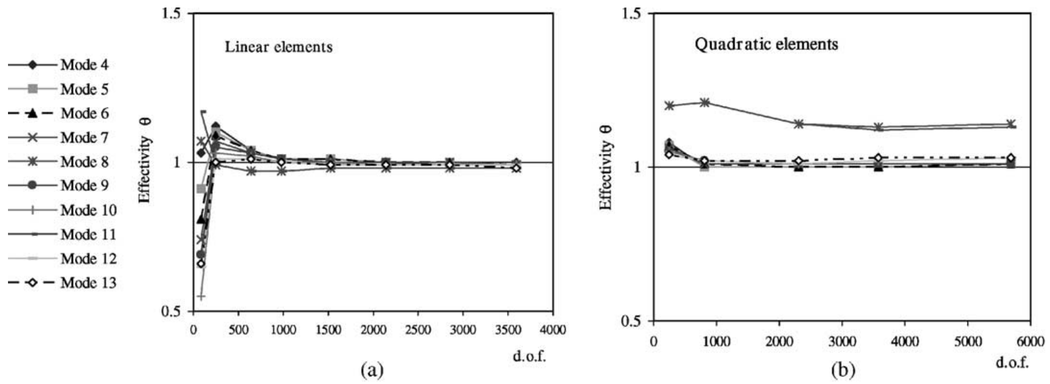

Fig. 2. Free beam: effectivity index with linear and quadratic triangular elements. Consistent mass matrix.

corresponds to the modes 8 and 11 in the analysis by quadratic elements. These cases are associated with axial modes whose discretization errors are very low (even for coarse meshes), so a major influence of numerical errors can exist in the analysis.

Lumped mass matrices have been taken into account as well. In this case the effectivity of the error estimator has been evaluated considering the improvement in the modal stiffness, the modal mass and both simultaneously. Therefore, the following improved natural frequencies have been defined

$$\omega^*_{U(r)} = \sqrt{\frac{k^*_{\mathrm{fe}(r)}}{m_{\mathrm{fe}(r)}}} \quad \omega^*_{V(r)} = \sqrt{\frac{k_{\mathrm{fe}(r)}}{m^*_{\mathrm{fe}(r)}}} \quad \omega^*_{UV(r)} = \sqrt{\frac{k^*_{\mathrm{fe}(r)}}{m^*_{\mathrm{fe}(r)}}}, \tag{22}$$

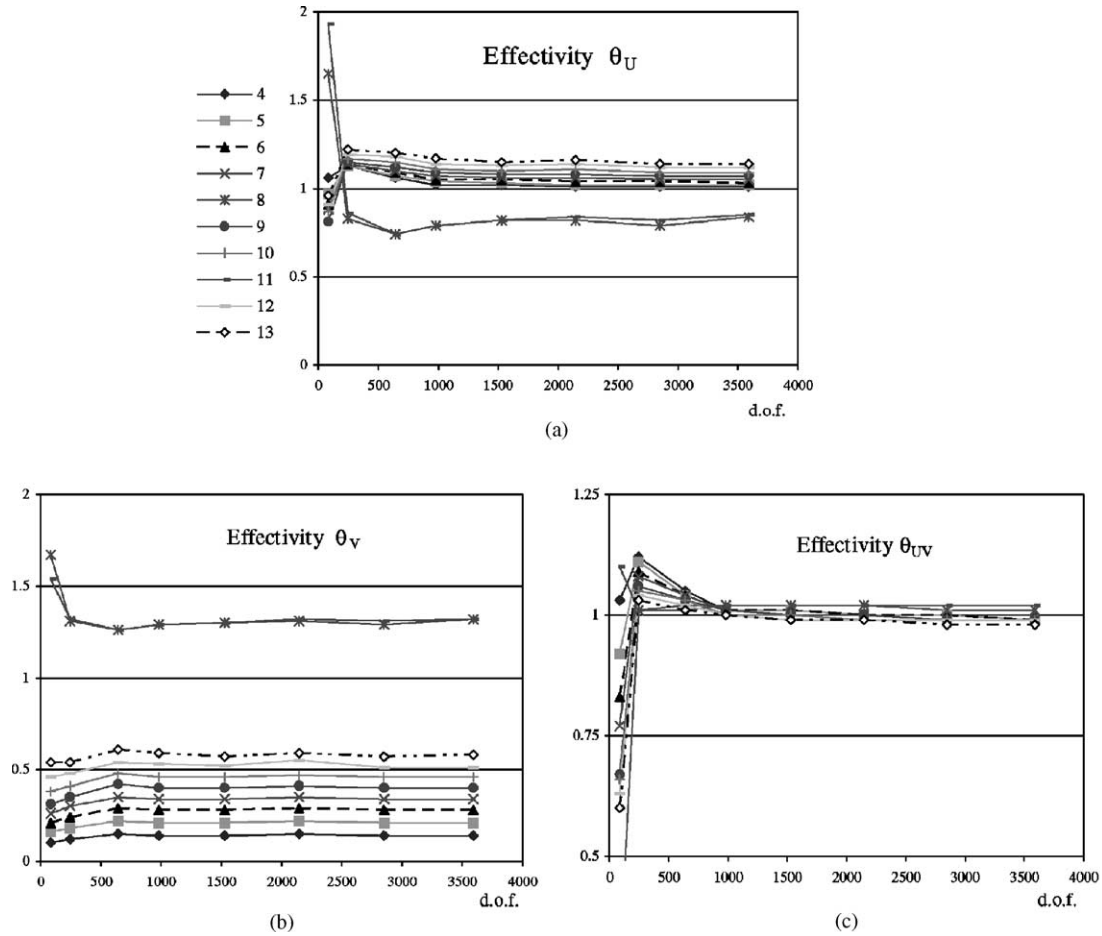

Fig. 3. Free beam: effectivity index with linear triangular elements. Lumped mass matrix.

with the corresponding effectivity index of the error estimator given by

$$\theta = \frac{|\omega_{(r)}^{*2} - \omega_{\mathrm{fe}(r)}^2|^{1/2}}{|\omega_{\mathrm{ex}(r)}^2 - \omega_{\mathrm{fe}(r)}^2|^{1/2}}. \tag{23}$$

These effectivity indexes are depicted in Fig. 3 for linear triangular elements. The values obtained by correcting the modal stiffness (θ_U) are acceptable in this case, but the results are worse if only the modal mass correction (θ_V) is considered. There is a significant improvement when both corrections are taken into account simultaneously (θ_{UV}).

It is worth to analyze the different methods of mass matrix lumping if quadratic elements are used. Fig. 4 shows the effectivity index of the estimated error in natural frequencies considering the HRZ lumping scheme [9]. The effectivity index is very poor when it is obtained by correcting the modal stiffness (θ_U), but it is acceptable when only the modal mass correction (θ_V) is taken into account. The best effectivity index is reached when both improvements are considered

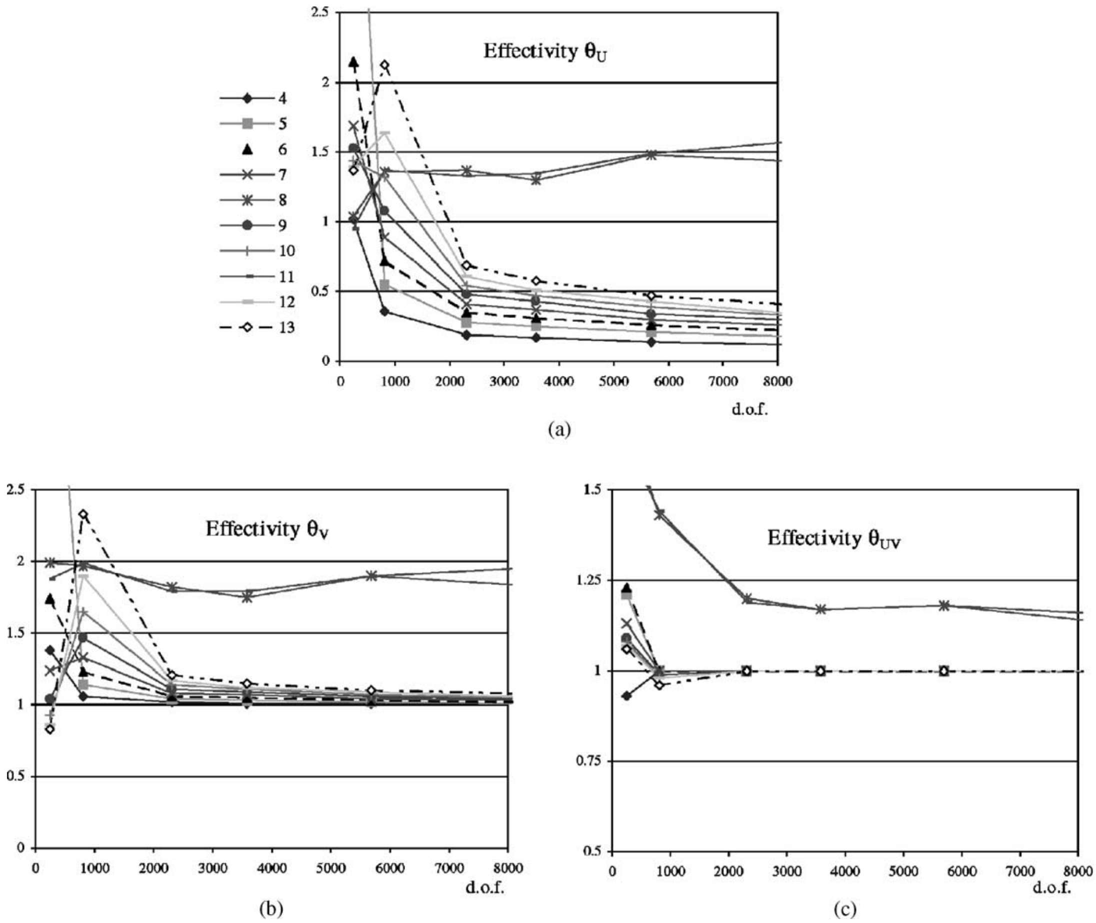

Fig. 4. Free beam: effectivity index with quadratic triangular elements. HRZ lumped mass matrix.

simultaneously (θ_{UV}). The two vibration modes for which the effectivity index does not converge to the unit are the axial modes commented previously.

Fig. 5 shows the effectivity index considering the optimum lumping mass scheme [9]. Unlike the former case, the effectivity index is acceptable when it is obtained by improving the modal stiffness (θ_U), it is very poor considering only the modal mass correction (θ_V) and it reaches the best behavior when both corrections are considered (θ_{UV}).

The proposed error estimator has been applied in several problems and it behaves in a similar way in most of them, with the effectivity index close to the unit.

3. Definition of the *h*-adaptive process

In absence of singularities the energy norm of the exact error is bounded by

$$e_{\omega(r)} \leqslant C_{(r)} h^p, \tag{24}$$

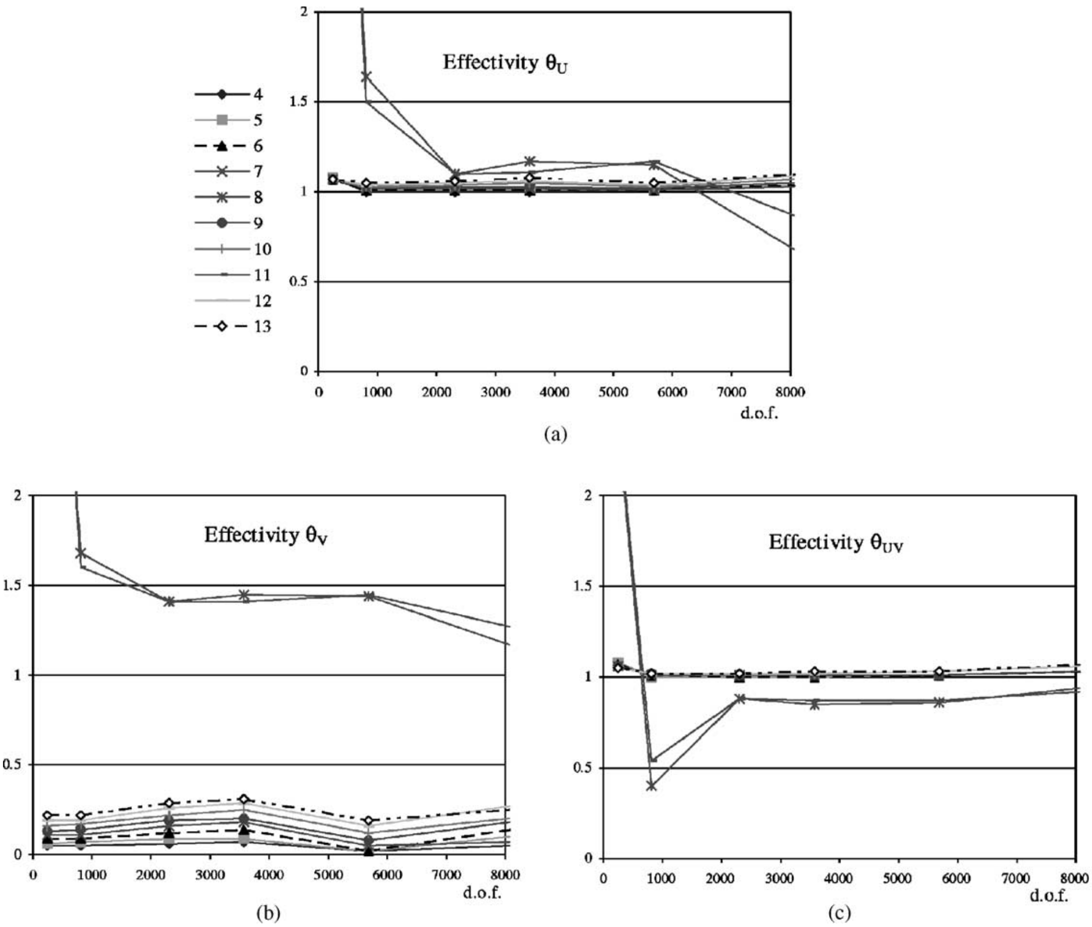

Fig. 5. Free beam: effectivity index with quadratic triangular elements. Optimum lumped mass matrix.

if consistent mass matrices are used in a uniform h-refinement. In this expression h is the element size, p is the polynomial degree of the displacement interpolation functions and C is a positive constant that does not depend on h. If lumped mass matrices are used, this equation is also valid without loss in the convergence rate [6].

Eq. (24) can be evaluated locally, considering the domain defined by one element. Assuming that it is possible to write this expression as a function of the estimated error, and taking into account two consecutive meshes of the h-adaptive process, the ratio of the local error in the new mesh to the previous mesh can be defined as

$$\frac{e^{(e)}_{\omega\mathrm{es}(r)}|_p}{e^{(e)p}_{\omega\mathrm{es}(r)}|_n} \approx \left[\frac{h^{(e)}_{\mathrm{p}(r)}}{h^{(e)p}_{\mathrm{n}(r)}}\right]^p = (r^{(e)}_{(r)})^p \tag{25}$$

where $e^{(e)}_{\omega\mathrm{es}(r)}|_p$ is the estimated error for the element e of the previous mesh, of size $h^{(e)}_{\mathrm{p}(r)}$, and $e^{(e)p}_{\omega\mathrm{es}(r)}|_n$ is the estimated error for all the elements of the new mesh (of size $h^{(e)p}_{\mathrm{n}(r)}$) contained in the element e of the previous mesh. This equation defines the local refinement ratio $r^{(e)}_{(r)}$ for each mode r.

The total error in the new mesh can be evaluated as a function of the estimated error in the previous mesh and the local refinement ratio $r^{(e)}_{(r)}$

$$e_{\omega\mathrm{es}(r)}|^2_{\mathrm{n}} = \sum_{e=1}^{N_n} e^{(e)}_{\omega\mathrm{es}(r)}|^2_{\mathrm{n}} = \sum_{e=1}^{N_p} e^{(e)p}_{\omega\mathrm{es}(r)}|^2_{\mathrm{n}} = \sum_{e=1}^{N_p} (e^{(e)}_{\omega\mathrm{es}(r)}|^2_{\mathrm{p}}(r^{(e)}_{(r)})^{-2p}), \tag{26}$$

where N_n and N_p are the number of elements in the new and previous meshes respectively.

On the other hand, the number of elements of the new mesh contained in the element e of the previous mesh, can be estimated for bi-dimensional problems as

$$N^{(e)p}_{\mathrm{n}(r)} \approx (r^{(e)}_{(r)})^2. \tag{27}$$

Thus the total number of elements in the new mesh can be estimated as

$$N_{\mathrm{n}(r)} \approx \sum_{e=1}^{N_p} (r^{(e)}_{(r)})^2. \tag{28}$$

3.1. Optimum mesh for one natural mode

Considering the optimum mesh as that which gets the required error with the minimum number of elements, we followed the scheme stated by Ladeveze et al. [10–12]. This criterion is equivalent to the uniform distribution of the absolute error in the elements of the new mesh [3,13,14].

Defining $e_{\omega(r)}|_{\mathrm{d}}$ as the absolute error desired for each mode in the new mesh, the problem of calculating the local refinement ratio can be stated as

$$\begin{aligned} &\text{Min.} \quad N_{\mathrm{n}(r)} = \sum_{e=1}^{N_p} (r^{(e)}_{(r)})^2 \\ &\text{s.t.} \quad \sum_{e=1}^{N_p} (e^{(e)}_{\omega\mathrm{es}(r)}|^2_{\mathrm{p}}(r^{(e)}_{(r)})^{-2p}) - e_{\omega(r)}|^2_{\mathrm{d}} = 0 \end{aligned}. \tag{29}$$

The solution to this problem can be expressed as [15]

$$(r^{(e)}_{(r)})^{2(p+1)} = \left[\frac{\sum_{e=1}^{N_p} e^{(e)}_{\omega\mathrm{es}(r)}|^{2/(p+1)}_{\mathrm{p}}}{e_{\omega(r)}|^2_{\mathrm{d}}}\right]^{(p+1)/p} e^{(e)}_{\omega\mathrm{es}(r)}|^2_{\mathrm{p}}. \tag{30}$$

The estimated errors can be evaluated in each element, so this expression allows to calculate the refinement in the previous mesh in order to get the desired error for each mode in the new mesh.

3.2. Optimum mesh for several natural modes

If several modes are considered, the problem that defines the optimum mesh consists of minimizing the number of elements in the new mesh, subject to the specified error constraint for each mode, that is

$$\begin{aligned} &\text{Min.} \quad N_{\mathrm{n}} = \sum_{e=1}^{N_p} (r^{(e)})^2 \\ &\text{s.t.} \quad \sum_{e=1}^{N_p} (e^{(e)}_{\omega\mathrm{es}(r)}|^2_{\mathrm{p}}(r^{(e)})^{-2p}) - e_{\omega(r)}|^2_{\mathrm{d}} = 0 \quad r = 1, \ldots, \mathrm{N_M} \end{aligned} \tag{31}$$

where N_{M} is the number of modes. The solution to this problem yields [15]

$$(r^{(e)})^{2(p+1)} = \sum_{r=1}^{N_{\mathrm{M}}} [\xi_{(r)} (r_{(r)}^{(e)})^{2(p+1)}], \tag{32}$$

where $\xi_{(r)}$ is called the global refinement contribution factor of the mode r. This expression allows to calculate the local refinement ratio for each element of the previous mesh. The global refinement contribution factor of each mode can be obtained from

$$\sum_{e=1}^{N_p} \left[e_{\omega \mathrm{es}(r)}^{(e)}|_{\mathrm{p}}^2 \left(\sum_{j=1}^{N_{\mathrm{M}}} \xi_{(j)} (r_{(j)}^{(e)})^{2(p+1)} \right)^{-p/(p+1)} \right] = e_{\omega(r)}|_{\mathrm{d}}^2 \quad r = 1, \ldots, N_{\mathrm{M}}. \tag{33}$$

In this way, the local refinement ratio can be evaluated as a function of the local refinement ratio obtained for each mode by means of (30), weighted by the corresponding global refinement contribution factor $\xi_{(r)}$.

The value of $\xi_{(r)}$ can be positive, negative or zero. A positive value of $\xi_{(r)}$ means that it is necessary to take into account the refinement of this mode to reach the required error. A zero value means that the mode does not participate in the refinement because it is well refined by other modes. A negative value of $\xi_{(r)}$ implies that other modes already refine the mesh in such a way that the error in the mode r is less than the required. Therefore this mode tends to unrefine the mesh so that the error takes the specified value. Mode shapes with a negative value of the global refinement contribution factor should be excluded from the system of equations (31) in order to obtain the optimum mesh.

3.4. Numerical verification

The proposed scheme to define the optimum mesh for several natural modes has been applied to different examples, considering linear and quadratic triangular elements. Fig. 6 shows one of them, whose three first modes are represented in Fig. 7. In order to evaluate the effectivity of the error estimator, the exact eigenvalues have been estimated through (13) and Richardson's extrapolation of two solutions obtained by means of a strong uniform refinement with quadratic triangular elements (20 951 and 46 739 d.o.f. respectively). These values are shown in Fig. 7.

With linear elements, the objective was to obtain a 10% discretization error for each mode by means of three consecutive h-adaptive refinements. Four h-adaptive processes have been defined. The first, second and third correspond to the processes that give the desired error considering each mode separately. The fourth corresponds to the proposed global refinement scheme. Fig. 8 depicts the sequence of meshes obtained with these processes and Table 1 shows the natural frequencies obtained with global refinement for each mesh and mode shape.

The convergence results obtained for these processes are shown in Fig. 9. If the refinement is made taking into account only the mode 1, the solution for this mode reaches the prescribed error whereas the other modes reach higher discretization errors. However, when the refinement is made according to the mode 2, the mode 3 also reaches the prescribed error (i.e. the mode 2 refines suitably the mode 3), but the mode 1 does not. If it is refined according to the mode 3, the mode 1 does not reach the prescribed error whereas the mode 2 approaches to the desired refinement. The

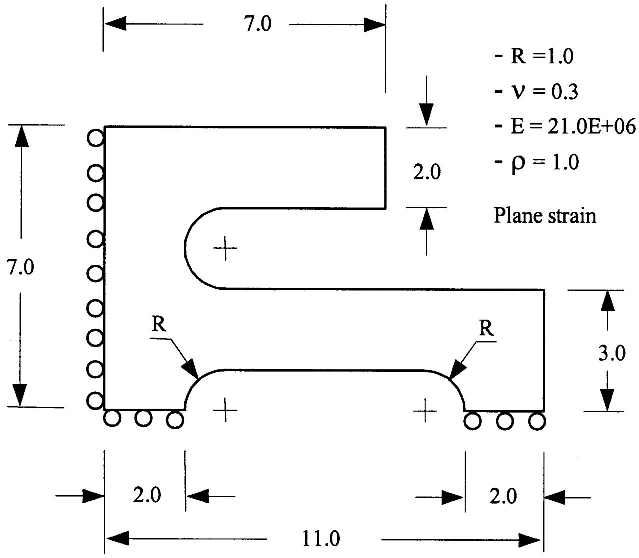

Fig. 6. Example problem considered in the *h*-adaptive refinement.

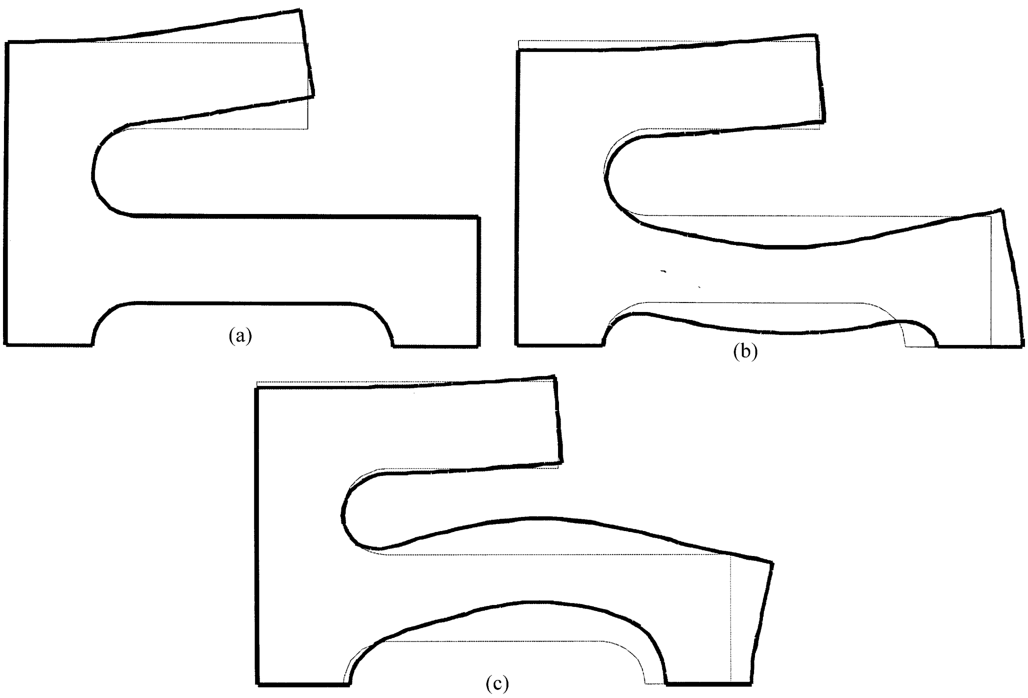

Fig. 7. Mode shapes and eigenvalues. (a) Mode 1, $\omega_{ex}^2 = 79715.173$; (b) mode 2, $\omega_{ex}^2 = 366119.33$; (c) mode 3, $\omega_{ex}^2 = 497756.37$.

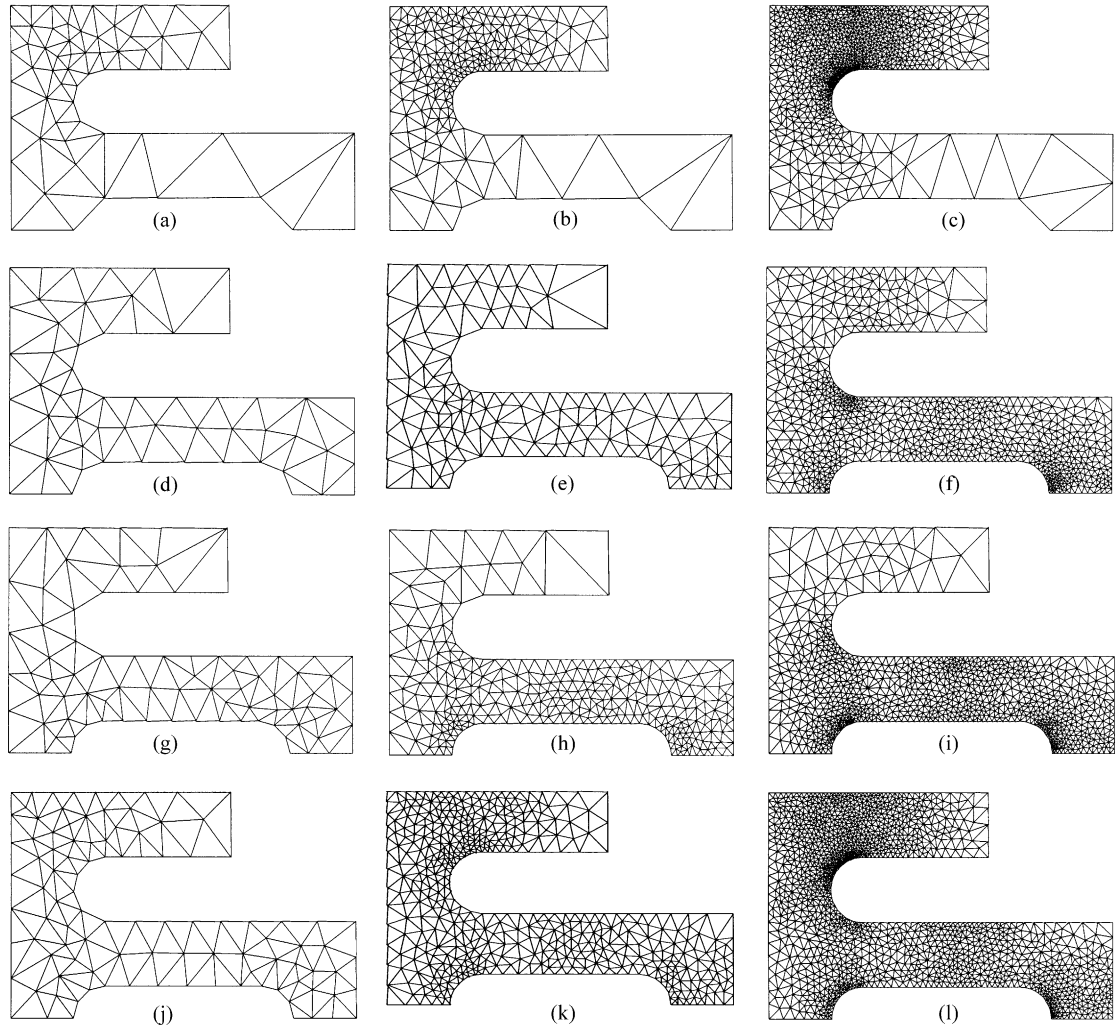

Fig. 8. Sequence of *h*-adaptive meshes with linear elements. 1st column: mesh 2; 2nd column: mesh 3; 3rd column: mesh 4. 1st row: mode 1 refinement; 2nd row: mode 2 refinement; 3rd row: mode 3 refinement; 4th row: global refinement.

Table 1
Sequence of natural frequencies with global refinement and linear elements

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Mode 1	184 339.598	99 872.647	82 614.452	80 402.062
Mode 2	463 694.96	419 013.26	379 932.49	369 978.38
Mode 3	906 587.76	614 801.69	512 638.56	501 916.69

refinement of all the modes as a whole is achieved with global refinement. The global refinement contribution factor of each mode in the *h*-adaptive sequence is detailed in Table 2. Only the modes 1 and 3 contribute to the first two steps of the *h*-adaptive process. However, the influence of the

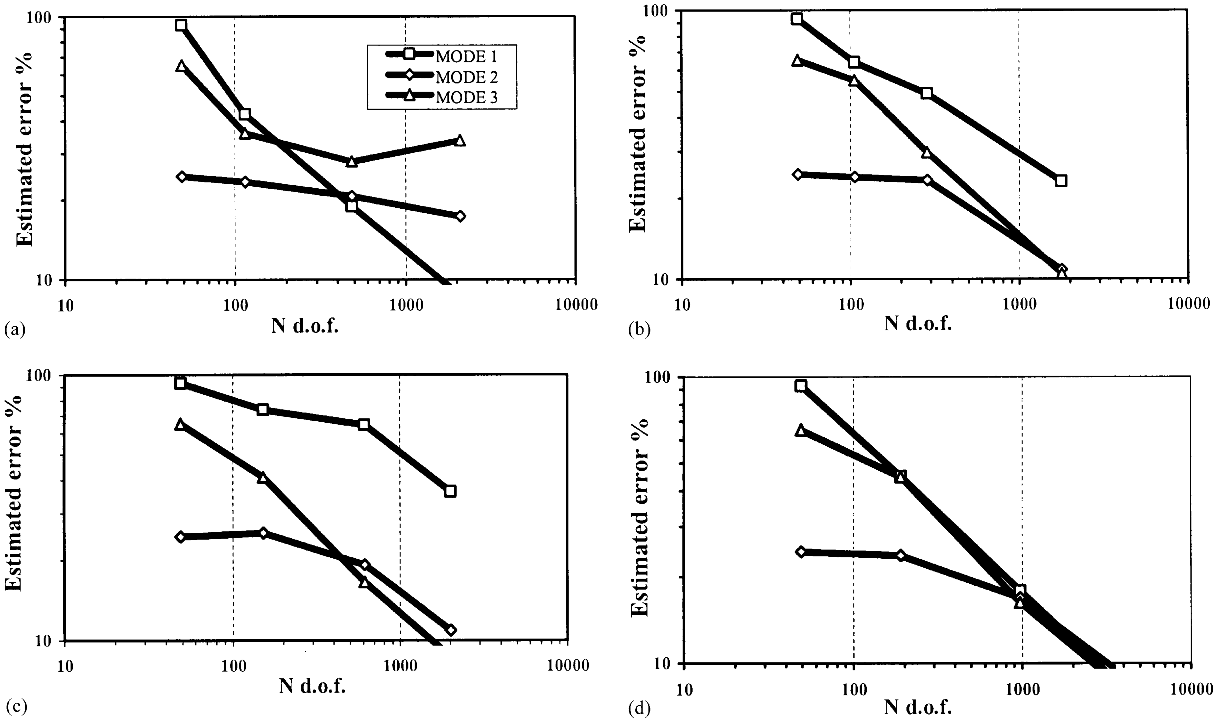

Fig. 9. Convergence of the estimated error for each refinement type. Linear elements. (a) Mode 1 refinement; (b) mode 2 refinement; (c) mode 3 refinement; (d) global refinement.

Table 2
Global refinement contribution factors. Linear elements

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Mode 1	0.71	0.84	0.85	0.83
Mode 2	0.00	0.00	0.39	0.79
Mode 3	0.72	0.86	0.45	0.00

mode 3 in the refinement tends to be cancelled at the expense of the influence of the mode 2 (as stated above, the mode 2 refines suitably the mode 3).

Fig. 10 shows the effectivity index of the error estimator in natural frequency. As it can be expected, it is possible that the error estimator loses its effectivity if the mesh is not adapted to one mode. This behavior is emphasized in the case of the mode 1 refinement, in which the effectivity index of the error estimator is close to 0.5 for modes 2 and 3.

The prescribed error was 1% in the case of quadratic triangular elements. Table 3 shows the sequence of natural frequencies obtained with global refinement and Fig. 11 depicts the evolution of the estimated error for each h-adaptive refinement considered. In this case, the discretization errors in the first three natural frequencies are similar for the first mesh. As in the case of linear

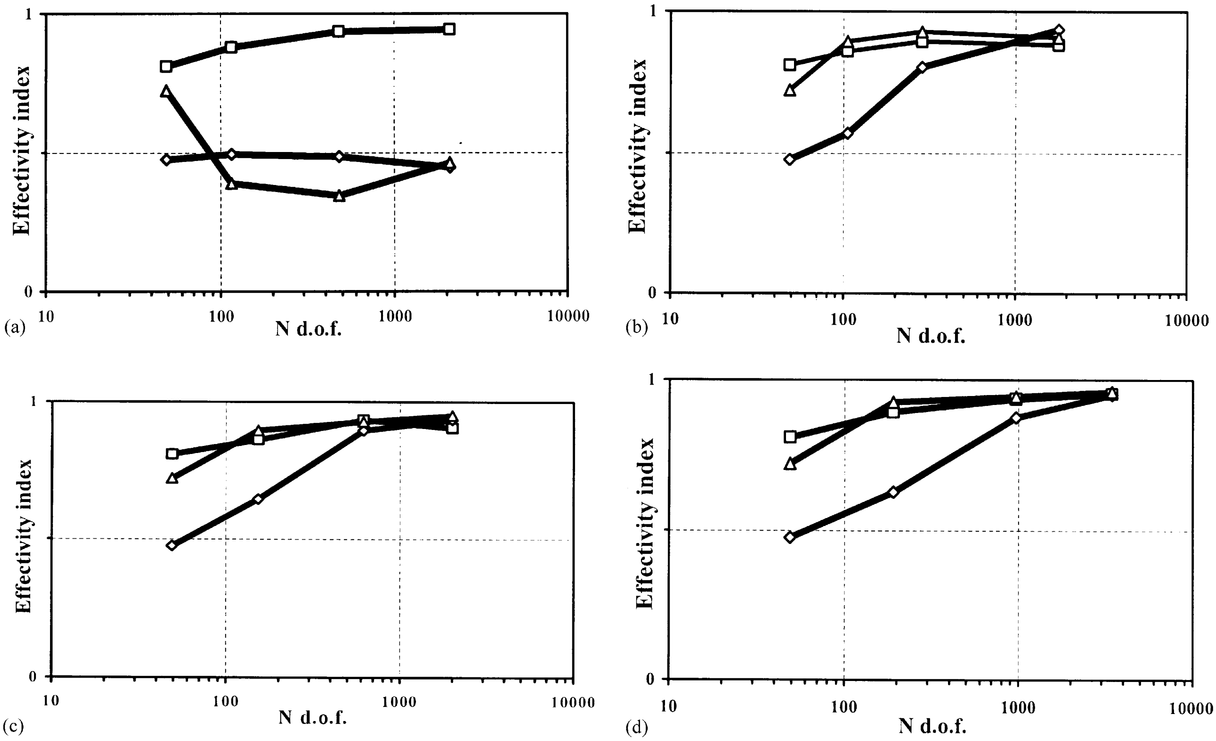

Fig. 10. Effectivity index of the error estimator. Linear elements. (a) Mode 1 refinement; (b) mode 2 refinement; (c) mode 3 refinement; (d) global refinement.

Table 3
Sequence of natural frequencies with global refinement and quadratic elements

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Mode 1	83 776.812	80 294.104	79 766.353	79 720.676
Mode 2	385 836.52	369 550.35	366 444.42	366 145.69
Mode 3	523 347.78	501 173.89	498 090.29	497 786.95

elements, the mode 1 refinement does not refine suitably the other modes, the mode 2 refinement refines well the mode 3, and the mode 3 refinement does not succeed in refining completely the mode 2. The proposed global refinement reaches the prescribed error for the three modes simultaneously. The global refinement contribution factor of each mode is summarized in Table 4, which shows the small contribution of the mode 3.

4. Conclusions

It has been shown an estimation of the discretization error in the analysis of natural frequencies, taking into account both consistent and lumped mass matrices. It is possible to apply conventional

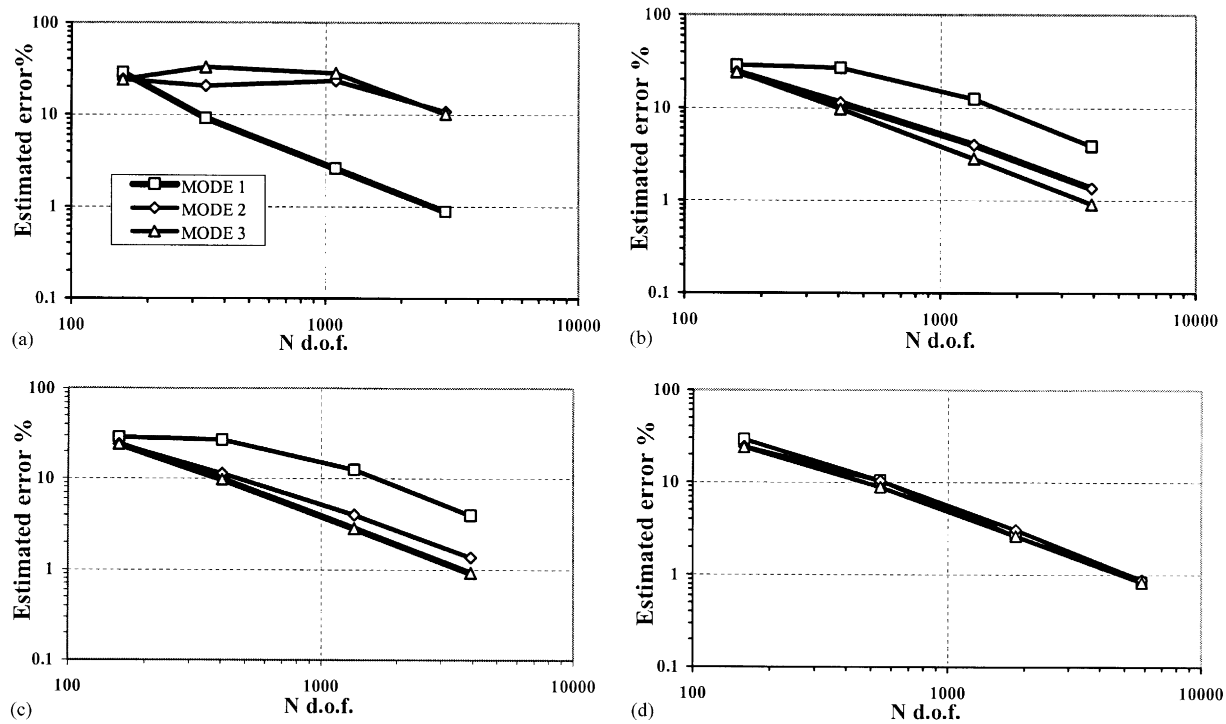

Fig. 11. Convergence of the estimated error for each refinement type. Quadratic elements. (a) Mode 1 refinement; (b) mode 2 refinement; (c) mode 3 refinement; (d) global refinement.

Table 4
Global refinement contribution factors. Quadratic elements

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Mode 1	0.66	0.64	0.63	0.63
Mode 2	0.15	0.58	0.74	0.74
Mode 3	0.65	0.24	0.06	0.06

techniques of error estimation if consistent mass matrices are used, such as the Zienkiewicz–Zhu estimator. In this case the effectivity index is similar to that obtained in static problems. Additional discretization errors appear in the modelization if lumped mass matrices are considered. Improved modal mass and stiffness have been considered in order to calculate the discretization error, and a reasonably effectivity index has been obtained for the majority of problems analyzed.

Considering the definition of optimum mesh for a given degree of interpolation as that which gives the required error with the minimum number of elements, it has been defined the h-adaptive process associated with the simultaneous consideration of several natural modes. This global refinement method can be defined on the basis of the participation of each mode to the global refinement through its contribution factor.

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