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Limits of quotients of bivariate real analytic functions ☆

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ABSTRACT

Necessary and sufficient conditions for the existence of limits of the form $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y)$ are given, under the hypothesis that f and g are real analytic functions near the point (a,b), and g has an isolated zero at (a,b). The given criterion uses a constructive version of Hensel's Lemma which could be implemented in a computer algebra system in the case where f and g are polynomials with rational coefficients, or more generally, with coefficients in a real finite extension of the rationals. A high level description of an algorithm for determining the existence of the limit as well as its computation is provided.

1. Introduction

Algorithms for computing limits of functions in one variable are studied in Gruntz (1996). Similar algorithms have been developed in Gruntz (1993, 1999). On the other hand, computational methods dealings with classical objects like power series and algebraic curves have been developed by several authors during the last two decades (Alonso et al., 1992b; Salvy and Shackell, 1999). In Alonso et al. (1992a) a symbolic computation algorithm for computing local parametrization of analytic branches and real analytic branches of a curve in n-dimensional space is proposed. Alternative approaches in the two-dimensional case are discussed in Cucker et al. (1989).

In this paper a theoretical method for determining the existence of limits of the form $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y)$, where f and g are real analytic functions in a neighborhood of (a,b), i.e., defined by power series which are absolutely convergent in an open disk centered at (a,b). Weierstrass' Preparation Theorem allows to reduce the problem to the case where f and g are monic

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polynomials in the variable y whose coefficients are real series in the variable x. Associated to f and g, a *discriminant real curve* is constructed using Lagrange Multipliers, with the property that the limit exists if and only if it exists along the branches of this curve. Hensel's Lemma, some Galois theory, and the theory of Puiseux series are utilized to parametrize these branches and select the *real ones*. The main result is summarized in Theorem 1. Proofs are provided in a constructive manner making it possible to implement the method in an algorithmic way when f and g are polynomials with rational coefficients, or more generally, with coefficients in a real extension field of $\mathbb{Q}$ (defined in Section 3.1). A high level description of this procedure is presented in the last section. One of the authors has written a rudimentary code in *Maple* that allowed him to compute a number of examples which already indicate the power of the method hereby developed. In many instances, the procedure is able to calculate some limits which apparently the latest version of *Maple* fails to compute.

As suggested by one of the referees, it is feasible that the algorithm here developed might be extended to the case where f and g are (more generally) polynomials in y with coefficients that are "effective power series" in x, i.e., power series encoded by some algorithm which for any given n one can compute in each case its n-th coefficient.

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2. Preliminaries

2.1. Some reductions

Let us denote by K either of the fields $\mathbb{C}$ or $\mathbb{R}$, and by $S = K\{x, y\}$ the ring of power series in the variables x, y with coefficients in K having positive radius of convergence. If $h(x, y)$ belongs to S, *the order of h in the variable y* is defined to be the smallest integer r such that $\bar{h}(y) = h(0, y)$ has the form $\alpha_r y^r + \alpha_{r+1} y^{r+1} + \cdots$ with $\alpha_r \neq 0$. (If $\bar{h}(y) = 0$ the order is defined to be $+\infty$.) It is not difficult to show that given $h_1, \ldots, h_n$ in $S - \{0\}$ there exists an integer $\nu \geqslant 1$ such that after the change of coordinates $x' = x + y^\nu$, $y' = y$, each series $h_i'(x', y') = h_i(x + y^\nu, y)$ is of finite order in the variable y'.

The existence of $\lim_{(x,y)\to(a,b)} f(x, y)/g(x, y)$ is obviously independent of the particular choice of local coordinates. Hence, there is no loss of generality in assuming that (a, b) is the origin, and that f and g have finite order in y. The following lemma (see Greuel et al., 2007) allows to reduce further to the case where f and g are monic polynomials in y. More precisely:

Lemma 1 *(Weierstrass). Let h be an element of $S = K\{x, y\}$ of order d in y. Then there exists a unique unit $u(x, y) \in S$ and unique series $a_1(x), \ldots, a_d(x)$ with coefficients in K and positive radii of convergence, such that $h(x, y) = u(x, y)(y^d + a_1(x) y^{d-1} + \cdots + a_d(x))$.*

Therefore we may assume that $f(x, y) = u(x, y) f_1(x, y)$ and $g(x, y) = v(x, y) g_1(x, y)$, where $u(x, y)$ and $v(x, y)$ are units and

$$f_1 = y^d + a_1(x) y^{d-1} + \cdots + a_d(x),$$
$$g_1 = y^b + c_1(x) y^{b-1} + \cdots + c_b(x)$$

are monic polynomials in $\mathbb{R}\{x\}[y]$.

Since units do not affect the existence of the limit, there is no loss of generality in assuming that u and v are equal to 1. Obviously, if the limit exists, it must be equal to $(\lim_{(x,y)\to(0,0)} u(x, y)/v(x, y)).(\lim_{(x,y)\to(0,0)} f_1(x, y)/g_1(x, y))$.

2.2. Discriminant variety

The following proposition provides a necessary and sufficient condition for the existence of $\lim_{(x,y)\to(0,0)} q(x, y)$. By q we will denote a C^1 function defined in the punctured disk $D_\rho^* = \{(x, y) \in \mathbb{R}^2\colon 0 < \sqrt{x^2 + y^2} < \rho\}$, and by $X(q) = \{(x, y) \in D_\rho^*\colon y\partial q/\partial x - x\partial q/\partial y = 0\}$ the associated *discriminant variety*.

Proposition 1. *Let $q(x, y)$ be as above. The limit $\lim_{(x,y)\to(0,0)} q(x, y)$ exists and equals $L \in \mathbb{R}$, if and only if for every $\epsilon > 0$ there is $0 < \delta < \rho$ such that for every $(x, y) \in X(q) \cap D^*_\delta$, the inequality $|q(x, y) - L| < \epsilon$ holds.*

Proof. The method of Lagrange multipliers applied to the function $q(x, y)$, subject to the condition $x^2 + y^2 = r^2$ where $0 < r < \rho$, says that the extreme values taken by $q(x, y)$ on each circle $C_r(0)$, centered at the origin and having radius r, occur among those points (x, y) of $C_r(0)$ for which the vectors $(\partial q/\partial x, \partial q/\partial y)$ and (x, y) are parallel, which amounts to $y\partial q/\partial x - x\partial q/\partial y = 0$. Let us assume that given $\epsilon > 0$ there exists $0 < \delta < \rho$ such that for every $(x, y) \in X(q) \cap D^*_\delta$ the inequality $|q(x, y) - L| < \epsilon$ holds. Let $(x, y) \in D^*_\delta$ and $r = \sqrt{x^2 + y^2}$. If $t_1(r), t_2(r) \in C_r(0)$ are such that $q(t_1(r)) = \min_{t \in C_r(0)} q(t)$, and $q(t_2(r)) = \max_{t \in C_r(0)} q(t)$, then

$$q\big(t_1(r)\big) - L \leqslant q(x, y) - L \leqslant q\big(t_2(r)\big) - L.$$

Since $t_1(r), t_2(r) \in X(q) \cap D^*_\delta$ we have

$$-\epsilon < q\big(t_1(r)\big) - L \quad \text{and} \quad q\big(t_2(r)\big) - L < \epsilon$$

and therefore $|q(x, y) - L| < \epsilon$.

The "only if" part is clear. □

Remark 1. It is important to observe that, as an immediate consequence of this proof, $X(q) \cap D^*_\delta \neq \emptyset$ for every $0 < \delta < \rho$.

We want to apply this proposition to $q = f/g$, where $f = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$, $g = y^b + c_1(x)y^{b-1} + \cdots + c_b(x)$, with each series $a_i(x)$, $c_j(x)$ convergent in D_ρ, for some $\rho > 0$, and g has an isolated zero at the origin. In this case $y\partial q/\partial x - x\partial q/\partial y$ can be written as h/g^2 where h is the polynomial

$$h = y\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right) - x\left(g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}\right). \tag{2.1}$$

Obviously $X(q) = \{(x, y) \in D^*_\rho\colon h(x, y) = 0\}$. This variety will accordingly be called the *discriminant variety* associated to f and g. It might happen that h is identically zero, and consequently the discriminant variety degenerates to the whole punctured disk D^*_ρ. This occurs precisely when $y\partial q/\partial x - x\partial q/\partial y = 0$ in D^*_ρ. Expressing q in polar coordinates, this equation amounts to saying that $\partial q(r, \theta)/\partial\theta = 0$ for all θ, which implies that q has the form $q(r, \theta) = q_0(r)$ with q_0 a C^1 function. But in this case $\lim_{(x,y)\to(0,0)} q(x, y)$ exists if and only if $\lim_{r\to 0} q_0(r)$ exists, and consequently the problem reduces to the case of one variable. Algorithms for computing limits in one variable are studied in Gruntz (1996).

For a suitable value of $n \geqslant 0$, the linear change of coordinates $\pi : (x, y) \to (x + ny, -nx + y)$ transforms h into a monic polynomial in y, $H(x, y) = h(x + ny, -nx + y)$ (see Greuel et al., 2007). Since $\pi^{-1}(X(q)) = \{(x, y) \in D^*_\rho\colon H(x, y) = 0\}$, the limit of f/g along the real branches of $X(q)$ can be computed as the limit of F/G along the real branches of $\pi^{-1}(X(q))$, where F, G denote the polynomials $(f \circ \pi)(x, y) = f(x + ny, -nx + y)$ and $(g \circ \pi)(x, y) = g(x + ny, -nx + y)$, respectively. Hence, for the problem of determining the real branches of the discriminant variety we may always assume that this variety is defined by a monic polynomial in y.

2.3. Hensel's Lemma

Our next goal is to parametrize the curve $H(x, y) = 0$. For this purpose we will use Hensel's Lemma (Eisenbud, 1994). Let us denote by k an arbitrary field, by R the ring of formal power series in the variable x with coefficients in k, $R = k[[x]]$, and by $k((x))$ its field of fractions. R is a local ring (R, m) whose maximal ideal is $m = (x)$. For each $P(x, y) \in R[y]$ that is monic in the variable y, let us denote by $\overline{P}$ its reduction modulo m, i.e., $\overline{P} = P(0, y)$. We will also denote $\overline{P}$ by the corresponding lower case letter p.

Lemma 2 *(Hensel's Lemma). Let $F(x, y)$ be an element of $R[y]$ that is monic in y, and let us assume that $\overline{F} = gh$ is a factorization in $k[y]$ whose factors are relatively prime, and of degrees r and s, respectively. Then there exist unique G and H in $R[y]$ with degrees r and s, respectively, such that*:

(1) $\overline{G} = g$ *and* $\overline{H} = h$,
(2) $F = GH$.

A constructive proof is based on the following elementary remark.

Remark 2. Let $p, q \in k[y]$ be relatively prime polynomials of degree r and s, respectively, and let f be a polynomial in $k[y]$ of degree $< r + s$. Then there exist unique polynomials $g, h \in k[y]$ such that $f = gp + hq$, and such that $\operatorname{degree}(g) < s = \operatorname{degree}(q)$ and $\operatorname{degree}(h) < r = \operatorname{degree}(p)$ (Mond and Saia, 2003).

We next prove Hensel's Lemma.

Proof of Lemma 2. F can be written as $F = f_0(y) + f_1(y)x + f_2(y)x^2 + \cdots$ with $f_0(y) = \overline{F}(y)$. We want to construct $G = g_0(y) + g_1(y)x + \cdots$, and $H = h_0(y) + h_1(y)x + \cdots$, where $g_0(y) = g(y)$ and $h_0(y) = h(y)$, and such that $GH = F$. Multiplying both expressions, and then comparing coefficients yields:

$$\begin{aligned} &h_0 g_1 + h_1 g_0 = f_1, \\ &h_0 g_2 + h_1 g_1 + h_2 g_0 = f_2, \\ &\quad\vdots \\ &h_0 g_k + h_1 g_{k-1} + \cdots + h_{k-1} g_1 + h_k g_0 = f_k. \end{aligned}$$

The first equation is solved by the remark above, which yields g_1 and h_1. The procedure can be repeated to solve recursively for the g_i's and the h_i's. □

In order to construct a parametrization of $H(x, y) = 0$, Puiseux series are used. We next review some basic facts.

2.4. Puiseux series

Let us denote by L the quotient field of fractions of $R = \mathbb{C}[[x]]$, which consists of Laurent series. Let $\bar{L}$ be an algebraic closure of L. For each positive integer n we will denote by $x^{1/n}$ a fixed n-th root of x in $\bar{L}$. It is clear that the n-th roots of x are

$$\theta x^{1/n}, \theta^2 x^{1/n}, \ldots, \theta^{n-1} x^{1/n}, x^{1/n}$$

where θ is any primitive n-th root of unity. It is easy to see that the polynomial $t^n - x$ is irreducible in $L[t]$ and therefore $L \subset L(x^{1/n})$ is an extension of degree n. Consider the directed system consisting of the positive natural numbers (partially) ordered by divisibility, i.e., $n \leqslant m$ if and only if $n|m$. The direct limit $\lim_{\to \mathbb{N}} L(x^{1/n})$ will be denoted by L^*. This limit can be identified with the field $\bigcup_n L(x^{1/n}) \subset \bar{L}$. Each element σ of L^* can therefore be written in the form $\sigma = \sum_{k \geqslant 1} c_k x^{q_k}$, with c_k nonzero in $\mathbb{C}$, and exponents $q_k \in \mathbb{Q}$ such that:

(1) $q_1 < \cdots < q_r < \cdots$,
(2) there is a positive integer b so that each exponent can be written as $q_i = a_i / b$, for some integer a_i.

The least exponent in the expression for σ, q_1, is called *the order of* σ. A well known theorem says that given a monic polynomial $H(x, y) = y^d + h_1(x)y^{d-1} + \cdots + h_d(x)$ in $R[y]$, there is an integer $r > 0$ such that H can be factored completely in $L(x^{1/r}) \subset L^*$ as

$$H(x,y) = \left(y - \sigma_1\left(x^{1/r}\right)\right)\cdots\left(y - \sigma_d\left(x^{1/r}\right)\right), \tag{2.2}$$

where each $\sigma_i(t)$ is an element of $\mathbb{C}[[t]]$, i.e., a formal power series. Moreover, it can be seen that these series have positive radius of convergence and therefore define holomorphic functions (Greuel et al., 2007). From this, a parametrization of the variety $X = \{(x,y) \in \mathbb{C}^2 \colon |x| < \rho,\ H(x,y) = 0\}$ can be obtained. Moreover, the proof of the existence of factorization (2.2) can be done in such a way (see Lemma 3 below) that it is in fact possible to determine which $\sigma_i(t)$'s have only real coefficients. These will be called throughout the *real series of the Puiseux factorization of H*. These in turn determine the real part of X, i.e., the trajectories through the origin which conform $X \cap \mathbb{R}^2$, a fact that will be proved in Proposition 2. Once this is proved, the following central result follows immediately.

Theorem 1. *Let $\sigma_1(z),\ldots,\sigma_l(z)$, $l \leqslant d$, be the real series in Eq. (2.2) which go through de origin (i.e., $\sigma_i(0) = 0$, for $i = 1,\ldots,l$). Then the limit $\lim_{(x,y)\to(0,0)} f(x,y)/g(x,y)$ exists if and only if $\lim_{t\to 0} f(t^r,\sigma_i(t))/g(t^r,\sigma_i(t)) = L_i$ exists, for $i = 1,\ldots,l$, and $L_1 = \cdots = L_l$.*

2.5. Newton's automorphism

For each rational number $q \neq 0$ there exists a homomorphism $\alpha_q : L^* \to L^*$, which sends x to x^q and fixes the subfield $\mathbb{C}$. This homomorphism is constructed by first defining a homomorphism from $\mathbb{C}[x]$ into L^* which sends x to x^q, then extending it to $\mathbb{C}[[x]]$, and then to the field of fractions $\mathbb{C}((x))$. Since L^* is an algebraic extension of L, this homomorphism extends to a homomorphism α_q from L^* to $\bar{L}$. It is clear that the image α_q lies inside L^*, and therefore one can regard α_q as an endomorphism of L^*. For $p \neq 0$ rational, let $\beta_{q,p} : L^*[y] \to L^*[y]$ be the extension of α_q obtained sending y to yx^p. It is clear that $\beta_{q,p}$ is invertible and its inverse is $\beta_{1/q,-p/q}$.

With these preliminaries we can now state the following fundamental theorem (see Mond and Saia, 2003).

Theorem 2. *Every polynomial $H = y^d + h_1(x)y^{d-1} + \cdots + h_d(x)$ with coefficients $h_i(x)$ in L^* can be factored into linear factors $H = (y - \sigma_1)\cdots(y - \sigma_d)$, with $\sigma_i \in L^*$. Even more, if each $h_i(x)$ belongs to $\mathbb{C}[[x]]$, then there exists a positive integer n such that all $\sigma_i \in \mathbb{C}[[x^{1/n}]]$.*

An easy consequence of this last theorem is the following:

Remark 3. If $H = y^d + h_1(x)y^{d-1} + \cdots + h_d(x)$ is a monic polynomial with coefficients in $\mathbb{C}[[x]]$, then there exists a power $r > 0$, and polynomials $g_1(x,y)$ and $g_2(x,y)$ in $\mathbb{C}[[x]][y]$, which are monic in the variable y and of degrees $d_1, d_2 < d$, such that $H(x^r, y) = g_1(x,y)g_2(x,y)$.

It can also be shown that Theorem 2 admits the following refinement (see Mond and Saia, 2003).

Theorem 3. *Let $H = y^d + h_1(x)y^{d-1} + \cdots + h_d(x)$ be a monic polynomial with coefficients in $\mathbb{C}[[x]]$ that is irreducible over $L = \mathbb{C}((x))[y]$. Then, if ω denotes a primitive d-th root of unity, there exists $\sigma(t) = \sum_{k=0}^{\infty} c_k t^k$ such that $H = (y - \sigma(\omega x^{1/d}))\cdots(y - \sigma(\omega^d x^{1/d}))$, where $\sigma(\omega^r x^{1/d}) = \sum_{k=0}^{\infty} c_k(\omega^r x^{1/d})^k$.*

The following result allows to identify the real series in the factorization given in Theorem 2.

Lemma 3. *Let $F = y^d + b_1(x)y^{d-1} + \cdots + b_d(x)$ be a monic polynomial of degree d in the variable y and whose coefficients are real power series, i.e., $b_i(x) \in \mathbb{R}[[x]]$. Then $\bar{F} = (y - r)^d$, with $r \in \mathbb{R}$, if and only if the factorization of F over L^* has the form*

$$F = \prod_{i=1}^{s} F_i \tag{2.3}$$

where

$$F_i = \left(y - \sigma_i\left(x^{1/d_i}\right)\right)\left(y - \sigma_i\left(\omega_i x^{1/d_i}\right)\right) \cdots \left(y - \sigma_i\left(\omega_i^{d_i-1} x^{1/d_i}\right)\right),$$

and each $\sigma_i(t) = \sum_{k=0}^{\infty} c_k t^k$ *is a real series,* ω_i *is a primitive* d_i*-th root of unity,* $\sum_{i=1}^{s} d_i = d$*, and* $\sigma_i(0) = r$*.*

Proof. Let us first consider the only if part of the equivalence. We proceed by induction on d. If the degree of F is $d = 1$, then $F = y - b_1(x)$, with $b_1(x) \in \mathbb{R}[[x]]$, is a factorization in Puiseux series and $b_1(0) = r$. If $d > 1$, by Remark 3, there exists an integer $N > 0$ such that $F(x^N, y) = G(x, y)H(x, y)$ where G and H are polynomials which are monic and of degrees $d_1, d_2 < d$ in the variable y. Thus $\overline{F} = F(0, y) = G(0, y)H(0, y)$. And this implies that $\overline{G} = (y - r)^{d_1}$, and $\overline{H} = (y - r)^{d_2}$. By Hensel's Lemma applied to the ring $\mathbb{R}[[x]][y]$, and in particular, by its claim about uniqueness, one obtains that G and H are polynomials with coefficients in $\mathbb{R}[[x]]$. By the induction hypothesis G and H can be factored in the form (2.3), so F can also be factored in this way. Conversely, if (2.3) holds, then it follows, by taking $N = d_1, \dots, d_s$, that

$$F\left(x^N, y\right) = \prod_{i=1}^{s} \left(y - \sigma_i\left(x^{e_i}\right)\right)\left(y - \sigma_i\left(\omega_i x^{e_i}\right)\right) \cdots \left(y - \sigma_i\left(\omega_i^{d_i-1} x^{e_i}\right)\right), \tag{2.4}$$

where $e_i = N/d_i$ and $\sigma_i(t)$ is a real series. By replacing x by 0 in (2.4) one obtains

$$\overline{F} = F(0, y) = \prod_{i=1}^{s} \left(y - \sigma_i(0)\right)^{d_i} = (y - r)^d. \qquad \square$$

Proposition 2. *Let* $H(x, y)$ *be a polynomial monic in* y *with coefficients in* $\mathbb{C}[[x]]$*. Let* $X = \{(x, y) \in \mathbb{C}^2\colon H(x, y) = 0\}$ *denote the algebraic set cut by* H*. Let* $H(x^r, y) = (y - \sigma_1(x)) \cdots (y - \sigma_d(x))$ *as in* (2)*, and let* $\mathfrak{D}_\rho \subset \mathbb{C}^2$ *denote an open disk of radius* ρ *centered at the origin where all the* σ_i*'s are absolutely convergent. Then the semi-algebraic sets* $X \cap \mathbb{R}^2$ *consists of the union of the curves* $\{(t^r, \sigma_i(t))\colon |t| < \rho\}$*, for each* σ_i *a real series, i.e., a series with real coefficients.*

Proof. The complex variety $X = \{(x, y) \in \mathbb{C}^2 : H(x, y) = 0\}$ is the union of the (complex) parametric curves $X_i = \{(z^r, \sigma_i(z))\colon z \in \mathfrak{D}_\rho\}$. Let us suppose that there is an infinite sequence z_n approaching the origin in $\mathbb{C}^2$ such that $(z_n^r, \sigma_i(z_n))$ is an infinite sequence in $X_i \cap \mathbb{R}^2$ ($X_i \cap \mathbb{R}^2$ has points with real coordinates infinitely close to the origin). Since z_n^r is real then $z_n = \varepsilon_n t_n$ where ε_n is an r-th root of unity. By passing to a subsequence we may assume $z_n = \varepsilon t_n$ for a fixed r-th root of unity ε. It is a well known fact that for a holomorphic function to be real valued on a real sequence approaching zero it must have a series expansion around the origin with only real coefficients. Hence the series $\varphi_i(t) = \sigma_i(\varepsilon t)$ must have real coefficients, and consequently $\sigma_i(t) = \varphi_i(\omega t)$, where $\omega = \varepsilon^{-1}$. By Theorem 3, the series φ_i must appear in the factorization $H(x^r, y) = (y - \sigma_1(x)) \cdots (y - \sigma_d(x))$ as one σ_j and henceforth $X_i \cap \mathbb{R}^2 = X_j \cap \mathbb{R}^2 = \{(t^r, \sigma_j(t))\colon |t| < \rho\}$. $\square$

Let now $F = y^d + b_1(x)y^{d-1} + \cdots + b_d(x)$ be a monic polynomial whose coefficients are in $\mathbb{R}[[x]]$ and let us denote its reduction modulo x by f. We can write

$$f = (y - r_1)^{d_1'} \cdots (y - r_s)^{d_s'} (y - c_1)^{d_1} (y - \bar{c}_1)^{d_1} \cdots (y - c_l)^{d_l} (y - \bar{c}_l)^{d_l},$$

where $r_1, \dots, r_s$ are the real roots of f, and $c_i, \bar{c}_i$ are the non-real ones. Let us define $f_i(y) = (y - r_i)^{d_i'}$ and

$$g_i(y) = (y - c_i)^{d_i} (y - \bar{c}_i)^{d_i} = \left(y^2 - \alpha_i y + \beta_i\right)^{d_i},$$

with α_i, β_i real. Hensel's Lemma provides us with a lifting of the factorization $f_1 \cdots f_s g_1 \cdots g_l$, of the form $F = F_1 \cdots F_s G_1 \cdots G_l$, i.e., $\overline{F}_i = f_i$ and $\overline{G}_i = g_i$. From the proof of Hensel's Lemma it follows that each F_i is a monic polynomial in the variable y with coefficients in $\mathbb{R}[[x]]$. Each G_i admits a factorization $\prod_{j=1}^{q_i} G_{ij}$ into irreducible factors in $\mathbb{C}[[x]][y]$, and, by Gauss' Lemma, also in $\mathbb{C}((x))[y]$

(see Lang, 1984). Notice that if G_{ij} has degree e_{ij}, then $\sum_{j=1}^{q_i} e_{ij} = 2d_i$. By Theorem 3 each G_{ij} can be factored as $G_{ij} = \prod_{k=1}^{e_{ij}} (y - \sigma_j(\omega_j^k x^{1/e_{ij}}))$, where ω_j is an e_{ij}-th primitive root of unity and $\sigma_j(t) \in \mathbb{C}[[t]]$. If we let $e_i = e_{i1}, \ldots, e_{iq_i}$, it is clear that

$$G_i\left(x^{e_i}, y\right) = \prod_{j=1}^{q_i} \prod_{k=1}^{e_{ij}} \left(y - \sigma_j\left(\omega_j^k x^{e_i/e_{ij}}\right)\right),$$

and as a consequence $G_i(0, y) = \prod_{j=1}^{q_i} (y - \sigma_j(0))^{e_{ij}}$. It follows that $\sigma_j(0) = c_i$ or $\sigma_2(0) = \bar{c}_i$. Therefore none of the σ_j is a real series. By Lemma 3, $F_k = \prod_{h=1}^{m} F_{kh}$, and

$$F_{kh} = \left(y - \sigma_{hk}\left(x^{1/d_{hk}}\right)\right)\left(y - \sigma_{hk}\left(\omega_{hk} x^{1/d_{hk}}\right)\right) \cdots \left(y - \sigma_{hk}\left(\omega_{hk}^{d_{hk}-1} x^{1/d_{hk}}\right)\right), \tag{2.5}$$

with $\sigma_{hk}(t)$ being a real series. The following theorem summarizes what has been achieved so far.

Theorem 4. *Let $F = y^d + b_1(x)y^{d-1} + \cdots + b_d(x)$ be a polynomial that is monic in the variable y and whose coefficients lie in $\mathbb{R}[[x]]$, and let f be its reduction modulo x. Then f can be written as*

$$f = (y - r_1)^{d_1'} \cdots (y - r_s)^{d_s'} (y - c_1)^{d_1} (y - \bar{c}_1)^{d_1} \cdots (y - c_l)^{d_l} (y - \bar{c}_l)^{d_l}.$$

Let

$$f_i(y) = (y - r_i)^{d_i'}, \qquad g_i(y) = (y - c_i)^{d_i}(y - \bar{c}_i)^{d_i} = \left(y^2 - \alpha_i y + \beta_i\right)^{d_i},$$

with r_i, α_i, β_i real. Hensel's Lemma gives a lifting $F = F_1 \cdots F_s G_1 \cdots G_l$ of the factorization of $f = f_1 \cdots f_s g_1 \cdots g_l$ with $\overline{F}_i = f_i$ and $\overline{G}_i = g_i$. Then, the only real series that occur in the Puiseux factorization of F are the ones that come from the linear factors f_i.

3. Computations over real extensions of $\mathbb{Q}$

3.1. Real embeddings

Let $\mathbb{Q} \subset E$ be any finite field extension of the rationals. E will be called a *real extension* if there is an embedding of fields $\lambda : E \to \mathbb{R}$. Let $g(z)$ be an irreducible polynomial in $E[z]$. The class of z in the quotient $F[z]/(g(z))$ will be called a *symbolic root* of g. Symbolic roots will be denoted by lower case letters, so that, for instance, $E(b)$ will denote the extension field $E(b) \simeq E[z]/(g(z))$. If E is a real extension, we speak of b a s a *real symbolic root over* E if any given embedding $\lambda : E \to \mathbb{R}$ can be extended to an embedding $\lambda' : E(b) \to \mathbb{R}$.

Let $\lambda : E \to \mathbb{R}$ be an embedding of a finite field extension of the rationals, and let b be a symbolic root over E. Since any extension of the rationals is principal, there exists b' such that $E(b) = \mathbb{Q}(b')$, an element that can be computed as a linear combination with rational coefficients of a finite set of generator of $E(b)$. From this, its minimal polynomial over $\mathbb{Q}$ can also be effectively determined. Sturm's Theorem (Basu et al., 2003) can then be used to *effectively* determine if the minimal polynomial of b' over the rationals has a real root, and consequently to determine if b is a real symbolic root over E.

Let $f(X, Y) = \sum_{\alpha,\beta} c_{\alpha,\beta} X^\alpha Y^\beta$ and $g(X, Y) = \sum d_{\gamma,\delta\gamma,\delta} X^\gamma Y^\delta$ be polynomials with coefficients in a real field extension E (we assume $c_{\alpha,\beta} \neq 0$ and $d_{\gamma,\delta} \neq 0$, for all (α, β), and all (γ, δ)). Let $H(x, y)$ be the polynomial that defines the discriminant variety, and let $H(x, y) = (y - \sigma_1(x^{1/r})) \cdots (y - \sigma_d(x^{1/r}))$ be its factorization into linear factors in $L^*[y]$. If $\sigma_1, \ldots, \sigma_l$ are the real series in this factorization such that $\sigma_i(0) = 0$, Theorem 2 implies that $\lim_{(x,y)\to(0,0)} f(x, y)/g(x, y)$ exists if and only if each limit $\lim_{t\to 0} f(t^r, \sigma_i(t))/g(t^r, \sigma_i(t)) = L_i$ exists, for $i = 1, \ldots, l$, and they are all equal. Now, each $\sigma_i(t)$, $i = 1, \ldots, l$, can be written as $\sigma_i(t) = e_i t^{e_i} + s_i(t)$, with $e_i \neq 0$ and $s_i(t)$ a real power series of strictly positive order. Hence $f(t^r, \sigma_i(t)) = \sum_{\alpha,\beta} c_{\alpha,\beta} t^{r\alpha} \sigma_i^\beta(t)$. Let $(\alpha, \beta), (\gamma, \delta)$ be chosen such that $r\alpha + \beta e_i$ and $r\gamma + \delta e_i$ takes the minimum possible value. We may write

$$f\left(t^r, \sigma_i(t)\right) = c_{\alpha,\beta} e_i^\beta t^{r\alpha + \beta e_i} \zeta_i(t), \qquad g\left(t^r, \sigma_i(t)\right) = d_{\gamma,\delta} e_i^\delta t^{r\gamma + \delta e_i} \eta_i(t),$$

where $\zeta_i(t)$ and $\eta_i(t)$ are power series with $\lim_{t\to 0}\zeta_i(t)=\lim_{t\to 0}\eta_i(t)=1$. Thus,

$$\lim_{t\to 0}\frac{f(t^r,\sigma_i(t))}{g(t^r,\sigma_i(t))}=\frac{c_{\alpha,\beta}e_i^{\beta}t^{r\alpha+\beta e_i}\zeta_i(t)}{d_{\gamma,\delta}e_i^{\delta}t^{r\gamma+\delta e_i}\eta_i(t)} \tag{3.1}$$

$$=\begin{cases}0, & \text{if } r\alpha+\beta e_i>r\gamma+\delta e_i,\\ e_i^{\beta-\delta}c_{\alpha,\beta}/d_{\gamma,\delta}, & \text{if } r\alpha+\beta e_i=r\gamma+\delta e_i,\\ \text{does not exist} & \text{if } r\alpha+\beta e_i<r\gamma+\delta e_i.\end{cases} \tag{3.2}$$

In any case, it is possible to know if it exists, and to determine its value, *by just knowing the least order term of each series* $\sigma_i(t)$.

But the equality $L_i=L_j$ *can be tested effectively*, since it amounts to checking if two real algebraic numbers in some finite extension of $\mathbb{Q}$ are equal.

3.2. Puiseux factorizations over real fields

The results established in the previous sections can be used to develop an *effective* procedure to determine the existence of $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y)$, as well as its computation, for the case where f and g are polynomials with rational coefficients. More generally, the algorithm can also be implemented for polynomials with coefficients in any real finite extension field E of the rationals.

Let $H(x,y)$ be a monic polynomial of the form $y^d+a_1(x)y^{d-1}+\cdots+a_d(x)$, where the $a_i(x)$'s are polynomials with coefficients in any finite extension field $\mathbb{Q}\subset E$, not necessarily real. We already know the components of the discriminant variety $X=\{(x,y)\in\mathbb{C}^2\colon H(x,y)=0\}$ can be parametrized as $X_i=\{(z^r,\sigma_i(z))\colon z\in\mathbb{C}\}$, where

$$H\big(x^r,y\big)=\big(y-\sigma_1(x)\big)\cdots\big(y-\sigma_d(x)\big) \tag{3.3}$$

and where the $\sigma_i(x)$'s are power series in $\mathbb{C}[\![x]\!]$ which can be determined by an inductive procedure, using Newton's automorphism followed by Hensel's Lemma. Without loss of generality we may assume that $a_d(x)\neq 0$, otherwise we may factor out a power of y to write H as a product of polynomials of lower degree and then apply the procedure to each factor. Let m_d denote the degree of $a_d(x)$.

We show next that the integer r in the above factorization satisfies $r\leqslant e^{d/e}$, where e denotes the base for the natural logarithms. Let an upper bar stand for reduction, mod a certain fixed power of x. We claim that by choosing $N=\lceil e^{d/e}m_d\rceil$ we may guarantee that if Eq. (3.3) holds mod x^{N+1}, then $\bar{a}_d(x)\neq 0$ in $\overline{H}(x^r,y)$, and $\overline{\sigma}_i(x)\neq 0$. The first assertion is clear. For the second, if for any σ_i we had $\overline{\sigma}_i(x)=0$, this would imply that $a_d(x)=\prod_{i=1}^d\sigma_i(x)$ would also be zero, which contradicts $a_d(x)\neq 0$.

Thus, as we saw in the previous section, since all that is needed to compute the limit is to determine the first nonzero term of each real series $\sigma_i(x)$, then, for this value of N we guarantee that mod x^{N+1}, the polynomials $\overline{\sigma}_i(x)$ (the truncated series, mod x^{N+1}) could also be used to compute the limit. For most practical purposes N may be taken much smaller than $e^{d/e}m_d$. Nevertheless, one needs this bound in order to guarantee that $\overline{\sigma}_i$ has the same leading term as σ_i.

In order to show that $r\leqslant e^{d/e}$ we first notice that if $H=y^d+a_2(x)y^{d-2}+\cdots+a_d(x)$ is irreducible in $\mathbb{C}[\![x]\!][y]$ then, by Theorem 3 the exponent r can be taken equal to d. Hence if $H=H_1\cdots H_s$ is a factorization into irreducible polynomials in $\mathbb{C}[\![x]\!][y]$, each H_i a monic polynomial of degree d_i in y, with $d_1+\cdots+d_s=d$, then r can be taken as $lcm(d_1,\ldots,d_s)$, the least common multiple of $d_1,\ldots,d_s$. Clearly, $lcm(d_1,\ldots,d_s)\leqslant d_1\cdots d_s$. On the other hand, $(d_1\cdots d_s)^{1/s}\leqslant(d_1+\cdots+d_s)/s$. Thus, $lcm(d_1,\ldots,d_s)\leqslant(d/s)^s$. It is not difficult to see that the maximum of the function $(d/x)^x$ on the interval $[1,d]$ equals $e^{d/e}$, from which the assertion follows.

3.3. Computation of limits

With this preliminaries it is not difficult to describe an effective procedure to determine the existence and computation of

$$\lim_{(x,y)\to(0,0)} f(x,y)/g(x,y)$$

where f and g have coefficients in a real extension E_0 of the rationals. A high level description of such an algorithm is indicated next.

Step $i=0$: The procedure starts with a given real extension E_0 of the rationals containing the coefficients of all the $a_i(x)$ in H, and a set $S_0=\{H_0\}$, where H_0 denotes the discriminant variety associated to f and g.

At the i-th step the following data is available:

I) A finite set of polynomials

$$S_i=\{H_k\colon H_k=y^{c_k}+b_2(x)y^{c_k-2}+\cdots+b_{c_k}(x),\ 1\leqslant c_k<d=\text{degree}(H_0)\}$$

with the property that $H(x^{r_i},y)=\prod_k H_k(x,y)$ holds mod x^{N+1}, for some positive integer $r_i\leqslant N=\lceil m_d e^{d/e}\rceil$.

II) A finite *real* extension field $E_i=\mathbb{Q}(a_1,\dots,a_{s_i})$ constructed by successively adjoining symbolic roots a_i of certain irreducible polynomials with coefficients in E_{i-1}.

From the i-th step to the $(i+1)$-st step the following new data is computed:

1. Each $H_k\in S_i$, if necessary, is first transformed via the automorphism $(\psi\circ\phi)(H_k)=x^{c_k u_{r_k}}G_k(x,y)$ (as in the proof of 2). Then the reduction mod x of G_k, $G_k(0,y)=g_k$, is factored as a product $g_k=g_{k,1}^{\alpha_1}\cdots g_{k,s_k}^{\alpha_{s_k}}$, where each $g_{k,j}\in E_i[y]$ is irreducible, and they are pairwise relatively prime (this can be done effectively, see Cohen, 1993).

2i. If $s_k>1$,by Hensel's Lemma there exist liftings G_{kj} of the G_k's such that $H_k(x^{r_k},y)=\prod_j G_{kj}(x,y)$, for some positive integer $r_k\leqslant c_k=$ degree of (H_k).

2ii. On the other hand, if $s_k=1$, a new symbolic root a_k is chosen, such that $g_k=(y-a_k)^{\alpha_1 m}g_{k,2}^{\alpha_1}(y)$, with $g_{i,2}(a_k)\neq 0$. If the minimal polynomial for a_k over E_i admits at least one real root, we adjoin a_k to the field E_i. We then take E_{i+1} to be the field obtained from E_i by adjoining all those roots, one for each index k for which $s_k=1$.

3. The set S_{i+1} is then constructed in the following way: for each index k for which $s_k>1$ we include in S_{i+1} all the polynomials G_{kj}. If $s_k=1$, and the minimal polynomial for a_k over E_i admits at least one real root, we also include the liftings G_{1k} of $(y-a_k)^{\alpha_1 m}$ and G_{2k}, the lifting of $g_{k,2}^{\alpha_1}(y)$. Finally, if the minimal polynomial for a_k over E_i does not admit any real root we include H_k.

By repeating this procedure a finite number of times, let's say $i=v$ times, we end up with a factorization $H(x^r,y)=\prod_l(y-\sigma_l(x))U(x,y)$, for some positive integer $r\leqslant N=\lceil m_d e^{d/e}\rceil$, where:

i) All the σ_i have coefficients in a finite extension E_v of $\mathbb{Q}$.

ii) There is at least one embedding $\lambda: E_v\to\mathbb{R}$ which restricted to E_0 is the identity Id_{E_0}.

iii) The polynomial $U(x,y)$ cannot be further reduced. That is, when the inductive step is applied to $U(x,y)$ the integer s_v in step 2ii is equal to 1, and the minimal polynomial for a_v over E_v *does not admits any real root.*

These two conditions imply that the polynomial $\lambda(U(x,y))$, obtained from $U(x,y)$ by replacing all its coefficients by their images via λ, has a Puiseux factorization which *does not involve any Puiseux real series.* Since the factorization of $H(x,y)$ in $L^*[y]$ is unique, then for any other embedding $\lambda': E_v\to\mathbb{R}$, such that $\lambda'_{|E_0}=\lambda_{|E_0}=\mathrm{Id}_{E_0}$, it is true that

$$H(x^r,y)=\prod_l\bigl(y-\lambda'(\sigma_l(x))\bigr)\lambda'\bigl(U(x,y)\bigr)=\prod_l\bigl(y-\lambda(\sigma_l(x))\bigr)\lambda\bigl(U(x,y)\bigr).$$

Hence the Puiseux real series are precisely all the series $\lambda'(\sigma_l(x))$ for *any* such embedding λ'.

5. From this, $\lim_{t\to 0}f(t^r,\sigma_i(t))/g(t^r,\sigma_i(t))=L_i$ is computed using Eq. (3.2), where all the σ_i now have coefficients in some finite extension $\mathbb{Q}\subset E_v$. Since all the L_i belong to E_v, it can be effectively determined if $L_i=L_j$, for each pair i,j. But this obviously implies that $\lambda'(L_i)=\lambda'(L_j)$ for any embedding λ'. Theorem 1 guarantees the existence of the limit.

3.4. An example

We illustrate the procedure described above by means of an example.

Let $f=y^2+x^2y+x^4$, $g=x^2+y^2$. The discriminant variety associated to f and g is given by the zeroes of

$$h=2yx^5+y^4x+2y^3x^3-y^3x-x^3y-(1/2)x^5+(1/2)y^2x^3.$$

After a rotation, given by the linear transformation $(x, y) \to (x + y, -x + y)$, h is transformed into a monic polynomial in y:

$$\begin{aligned} H &= y^6 + (2x + 1/4)y^5 + \big(x^2 - 5/4x - 1/2\big)y^4 - x^2y^3 \\ &\quad + \big(-x^4 - x^3\big)y^2 + \big(-5/4x^4 - 2x^5\big)y + 1/4x^5 - x^6 + 1/2x^4. \end{aligned}$$

Let

$$\begin{aligned} F(x, y) &= f(x + y, -x + y) \\ &= y^4 + (1 + 4x)y^3 + \big(1 + x + 6x^2\big)y^2 + \big(-2x - x^2 + 4x^3\big)y + x^2 - x^3 + x^4, \\ G(x, y) &= g(x + y, -x + y) = 2x^2 + 2y^2. \end{aligned}$$

Factoring H we obtain

$$H := (x + y)\big(x^2 + y^2\big)\big(y^3 + (x + 1/4)y^2 + \big(-x^2 - 3/2x - 1/2\big)y + \big(-x^3 + 1/4x^2 + 1/2x\big)\big).$$

The Puiseux factorization of $x^2 + y^2$ can be readily obtained as $(y + ix)(y - ix)$. Now let

$$H_1 = y^3 + (x + 1/4)y^2 + \big(-x^2 - 3/2x - 1/2\big)y + \big(-x^3 + 1/4x^2 + 1/2x\big).$$

Reducing mod x gives the polynomial $h_1 = y^3 + 1/4y^2 - 1/2y$ which factors as $y(y^2 + 1/4y - 1/2)$. Let a_1 be a (real) symbolic root of the quadratic factor. In $\mathbb{Q}(a_1)$, $h1$ factor as $y(y - a_1)(y + a_1 + 1/4)$. Hensel's Lemma gives liftings of each factor

$$\begin{aligned} G_1 &= y - x + 2x^2 - 4x^3 + \cdots, \\ G_2 &= y - a_1 + (-16/11a_1 + 9/11)x + (296/363a_1 - 326/363)x^2 \\ &\quad + (-6096/1331a_1 + 1900/1331)x^3 + \cdots, \\ G_3 &= y + 1/4 + a_1 + (13/11 + 16/11a_1)x \\ &\quad + (-400/363 - 296/363a_1)x^2 + (3424/1331 + 6096/1331a_1)x^3 + \cdots. \end{aligned}$$

Thus, the factorization of H into Puiseux series is given by

$$H = (y + x)(y - ix)(y + ix)G_1G_2G_3.$$

Clearly the only real series are

$$\begin{aligned} \sigma_0(x) &= -x, \\ \sigma_1(x) &= x - 2x^2 + 4/3x^3 + \cdots, \\ \sigma_2(x) &= a_1 - (-16/11a_1 + 9/11)x - (296/363a_1 - 326/363)x^2 + \cdots, \\ \sigma_3(x) &= -(1/4 + a_1) - (13/11 + 16/11a_1)x - (-400/363 - 296/363a_1)x^2 + \cdots. \end{aligned}$$

Since σ_0 and σ_1 are the only real series which go through the origin, we have to check if $\lim_{t\to 0} F(t, \sigma_i(t))/G(t, \sigma_i(t))$ exist, and they are all equal, for $i = 0, 1$. For this:

$$\begin{aligned} &\lim_{t\to 0} F\big(t, \sigma_0(t)\big)/G\big(t, \sigma_0(t)\big) = \lim_{t\to 0} \frac{4t^2}{4t^2} = 1, \\ &\lim_{t\to 0} F\big(t, \sigma_1(t)\big)/G\big(t, \sigma_1(t)\big) = \lim_{t\to 0} \frac{4t^4(3 - 12t + 42t^2 - 84t^3 + \cdots)}{4t^2(1 - 2t + 6t^2 + \cdots)} = 0. \end{aligned}$$

Thus, the limit does not exist.

The other two limits are computed thus

$$\lim_{t\to 0} F\big(t,\sigma_2(t)\big)/G\big(t,\sigma_2(t)\big) = \frac{3}{4} + \frac{3}{8}a_1,$$
$$\lim_{t\to 0} F\big(t,\sigma_3(t)\big)/G\big(t,\sigma_3(t)\big) = \frac{21}{32} - \frac{3}{8}a_1.$$

When a_1 is evaluated, it gives two real roots: $r_1 = -1/8 + 1/8\sqrt{33}$ and $r_2 = -1/8 - 1/8\sqrt{33}$. For $a_1 = r_1$ the last two limits are equal to $225/32 + 15/32\sqrt{33}$ and $225/32 - (15/32)\sqrt{33}$, respectively. For $a_1 = r_2$, these are equal to $225/32 - (15/32)\sqrt{33}$ and $225/32 + (15/32)\sqrt{33}$, respectively. Thus, the limit does not exist.

A rather rudimentary code in *Maple* 12 written by one the authors allowed him to compute a number of examples which show the potential power of this algorithm. For some limits (for instance $\lim_{(x,y)\to(0,0)}(x^4 - y^2 + 3x^2y - x^2)/(x^2 + y^2))$ this procedure shows that it exists, and it is equal to -1, a result that apparently the latest version of *Maple* 12 is incapable to compute.

References

Alonso, M.E., Niesi, G., Mora, T., Raimondo, M., 1992a. Local parametrization of space curves at singular points. In: Falcidieno, B., Herman, I., Pienovi, C. (Eds.), Computer Graphics and Mathematics. In: Eurographic Seminar Series. Springer-Verlag, pp. 61–90.

Alonso, M.E., Mora, T., Raimondo, M., 1992b. A computational model for algebraic power series. J. Pure Appl. Algebra 77, 1–38.

Basu, S., Pollack, R., Roy, M.-F., 2003. Algorithms in Real Algebraic Geometry. Springer-Verlag.

Cohen, H., 1993. A Course in Computational Algebraic Number Theory. Springer-Verlag.

Cucker, F., Pardo, L., Recio, T., Roy, M.F., Raimondo, M., 1989. On the computation of the local and global branches of real algebraic curves. In: Proc. AAECC 6. In: Lecture Notes in Comput. Sci., vol. 356. Springer-Verlag, pp. 161–181.

Eisenbud, D., 1994. Commutative Algebra with a View towards Algebraic Geometry. Springer-Verlag.

Greuel, M., Lossen, C., Shustin, E., 2007. Introduction to Singularities and Deformations. Springer-Verlag.

Gruntz, D., 1993. A new algorithm for computing asymptotic series. ISSAC, 239–244.

Gruntz, D., 1999. Computing limits in computing algebra systems. In: Wester, M.J. (Ed.), Computer Algebra Systems: A Practical Guide. John Wiley & Sons, Chichester, UK, pp. 153–170.

Gruntz, D., 1996. On computing limits in a symbolic manipulation system. ETH Diss 11432.

Lang, S., 1984. Algebra. Addison–Wesley Publishing Company Inc.

Mond, D., Saia, M. (Eds.), 2003. Real and Complex Singularities. Lecture Notes in Pure and Applied Mathematics, vol. 232. Marcel Dekker.

Salvy, B., Shackell, J., 1999. Symbolic asymptotics: Multiseries of inverse functions. J. Symbolic Comput. 27 (6), 543–563.