On the Biharmonic Equation

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This article provides a comprehensive introduction to the biharmonic equation, focusing on its origins in elasticity and fluid mechanics. We derive the equation from physical principles of linear deformations and Stokes flow, illustrating its applicability in modeling phenomena such as plate bending and stream functions in viscous media. Solutions are developed in polar and spherical coordinates with radial symmetry, including boundary conditions for spherical domains, as well as in general 2D Cartesian coordinates and the biharmonic wave equation for structural mechanics. Throughout, we highlight practical applications across engineering fields, showcasing the biharmonic equation's role in predicting stress, displacement, and flow patterns.

Biharmonic Equation | Radial Solutions | Separation of Variables

1. Introduction

The biharmonic equation $\nabla^4\phi = 0$ was first introduced in the context of elasticity theory, primarily for addressing the bending of elastic plates under various loading conditions. Originally formulated to model stress and deformation in materials [9], it has become essential in multiple domains beyond elasticity. Today, the biharmonic equation is applied in fluid mechanics, electromagnetism, and continuum mechanics.

The biharmonic equation allows general solutions through analytic functions or combinations of harmonic functions, though these forms are often impractical [13]. Specific explicit solutions exist for square and rectangular domains using Dirichlet and Neumann boundary conditions via separation of variables [2], as well as foundational ground state [4] and Green's function solutions in bounded elastic media [11]. Fourier transforms and operator semigroups further support dynamic and frequency-dependent problems.

Modern Applications include the biharmonic wave equation in diffraction issues [15], and complex geometries are increasingly approached with Physics-Informed Neural Networks (PINNs), which embed physical laws into neural networks for effective solutions in challenging settings [3].

This paper presents a state-of-the-art review on the biharmonic equation , aiming to mathematically develop and compile existing solutions while highlighting their practical applications.

The structure of this article is as follows: In Section 2, we derive the biharmonic equation from principles of linear deformations and Stokes flow. In Sections 3 and 4, we examine solutions with radial symmetry, presenting formulations in polar and spherical coordinates and including boundary conditions in the latter. Then, in Section 5, we address general solutions in 2D Cartesian coordinates, and in Section 6, we extend this approach to the biharmonic wave equation within a structural mechanics context. Finally, Section 7 presents the conclusions of this study.

2. The Laplace and Biharmonic Equation

A. The Laplace Equation. To introduce the biharmonic equation, we begin by discussing its predecessor, the Laplace equation. The Laplace equation is a second-order partial differential equation (PDE) that emerges in various scientific fields, such as electromagnetism, fluid dynamics, and heat conduction. In electromagnetism, it describes the electric potential in a charge-free region; in fluid dynamics, it models steady-state fluid flow; and in thermal conduction, it governs the temperature distribution under steady-state conditions.

Significance Statement

The biharmonic equation is fundamental in addressing diverse challenges in engineering and physics, particularly in modeling complex structural deformations, fluid flows, and vibrational behaviors. This article provides a rigorous yet accessible derivation and analysis of the biharmonic equation across multiple coordinate systems, emphasizing radial and spherical solutions. Practical applications, such as plate bending in structural mechanics, boundary-conditioned spherical domains, and stream functions in viscous flow, demonstrate the equation's extensive relevance. By integrating real-world examples, this work enhances comprehension of the biharmonic equation's utility in predicting and controlling physical phenomena in design, engineering, and materials science.

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Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a real-valued scalar field. The **Laplace equation** for ϕ is given by the identity:

$$\Delta\phi = \nabla^2\phi = 0, \tag{1}$$

where Δ denotes the Laplace linear operator (or the Laplacian), defined as the divergence $(\nabla\cdot)$ of the gradient (∇) of the scalar field ϕ. Specifically,

$$\nabla^2\phi = \nabla\cdot(\nabla\phi). \tag{2}$$

Instead, if we consider a vector field $\mathbf{u}$, the expression $\nabla^2\mathbf{u}$ is defined as the vector field formed by applying the laplacian in each coordinate of $\mathbf{u}$. In general orthogonal coordinates (u_1, u_2, u_3), with corresponding scale factors h_1, h_2, and h_3, the Laplacian can be expressed as [5]:

$$\begin{aligned}\nabla^2 = \frac{1}{h_1h_2h_3}\Bigg[&\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) \\ &+ \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\Bigg].\end{aligned} \tag{3}$$

The solutions ϕ to the Laplace equation, known as **harmonic functions** [12], play a central role in modeling equilibrium states and potential fields in physics and engineering. However, the problem formulation is incomplete without specifying the domain and boundary conditions. The well-posedness of the Laplace problem relies on defining the domain and imposing boundary conditions on it.

Let Ω be an open subset of $\mathbb{R}^{n}$*, where n is a positive integer. We say $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$ is a solution of the Laplace equation with Dirichlet boundary conditions if $\phi \in C^2(\Omega) \cup C(\bar{\Omega})$† and satisfies:

$$\begin{cases}\nabla^2\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega,\end{cases} \tag{4}$$

where $\partial\Omega$ denotes the boundary of Ω [1]. This formulation defines the Laplace equation's solutions as equilibrium states, which remain static over time. However, if we introduce time dependency, the Laplace equation transforms into the more general wave equation [7]:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{5}$$

where c represents the speed of propagation through the medium. This wave equation thus describes how disturbances in ϕ propagate and evolve within the domain Ω. In this context, the harmonic solutions of the Laplace equation can be viewed as snapshots of these time-evolving wave phenomena at each instant.

B. The Biharmonic Equation and the Biharmonic Operator.

The Laplace equation is commonly introduced in advanced mathematics courses for physicists, mathematicians, and engineers. Despite its widespread applications, there exists a more complex counterpart known as the *biharmonic equation*, defined as [10]:

$$\nabla^4\phi = 0. \tag{6}$$

This equation involves the biharmonic operator, which can be thought as applying the laplacian operator to itself, as:

$$\nabla^4\phi = \nabla^2\left(\nabla^2\phi\right) = \nabla\cdot(\nabla(\nabla\cdot(\nabla\phi))). \tag{7}$$

In three-dimensional Cartesian coordinates, it may be expanded as [13]:

$$\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial z^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + 2\frac{\partial^4\phi}{\partial y^2\partial z^2} + 2\frac{\partial^4\phi}{\partial x^2\partial z^2} = 0. \tag{8}$$

Similarly to Eq. (4), it is necessary to define the domain and boundary conditions for the problem of solving Eq. (6) to be well-posed. Formally, let Ω be an open subset of $\mathbb{R}^n$. We say $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$ is a solution of the biharmonic equation (or plate equation) with Dirichlet - Neumann boundary conditions if $\phi \in C^4(\bar{\Omega})$ and satisfies [1]:

$$\begin{cases}\nabla^4\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega.\end{cases} \tag{9}$$

On another hand, in analogy with steady solutions to the wave equation, the biharmonic equation can represent the steady-state solution to the more general biharmonic wave equation:

$$\nabla^4\phi - \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{10}$$

where a is a constant related to the propagation speed in the medium. The additional order of differentiation in the biharmonic equation, compared to the Laplace equation allows it to account for stresses and displacements in elasticity that cannot be described by the Laplace equation alone [9].

When using alternative coordinate systems, the biharmonic operator's form will vary accordingly, adapting to the coordinate system used. In the following subsections, we motivate the study of the biharmonic equation by deriving it in the context of deformations in a continuum and stokes flow respectively.

C. Deformations in Linear Continuous Media. In general, small deformations $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ of a linear elastic continuum are described by the Navier equations [8]:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}. \tag{11}$$

In this equation $\mathbf{u}$ represents the displacement vector field, ρ is the material density, and λ and μ are Lamé's constants [14]. If we consider a steady state (i.e., $\mathbf{u}_{tt} = 0$) with no body forces ($\mathbf{f} = 0$), where forces act solely on the surface, then $\mathbf{u}$ satisfies:

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} = 0. \tag{12}$$

Taking the Laplacian of both sides, we obtain:

*Generally $n = 2$ or 3.

†This means ϕ is twice continuously differentiable in Ω and continuous on the of closure Ω.

$$(\lambda+\mu)\nabla^2\left(\nabla(\nabla\cdot\mathbf{u})\right)+\mu\nabla^2(\nabla^2\mathbf{u})=0. \tag{13}$$

Finally, by assuming that the deformation field has no sources or sinks (i.e., $\nabla\cdot\mathbf{u}=0$), we arrive at the vector biharmonic equation:

$$\nabla^4\mathbf{u}=0. \tag{14}$$

This equation is called vectorial because each component of the field $\mathbf{u}$ satisfies the scalar biharmonic equation: $\nabla^4 u_i=0$.

D. Stokes Flow. Stokes flow, which is defined as highly viscous flow with negligible inertia effects, is described by [8]:

$$\nabla\cdot\mathbf{v}=0, \quad \nabla p=\mu\nabla^2\mathbf{v}, \tag{15}$$

where $\mathbf{v}$ is the flow velocity field, p is the pressure field, and μ is the fluid's dynamic viscosity. According to Helmholtz's decomposition theorem, the velocity field can be written as:

$$\mathbf{v}=\nabla\phi+\nabla\times\mathbf{A}, \tag{16}$$

in which ϕ is a scalar potential, and $\mathbf{A}$ is a vector potential [7]. In 2D, the vector potential $\mathbf{A}$ reduces to having only one component perpendicular to the plane, meaning $\mathbf{A}=(0,0,\psi)$, where ψ is called the stream function.

To obtain the biharmonic equation, we apply the curl to the momentum equation $\nabla p=\mu\nabla^2\mathbf{v}$:

$$\nabla\times(\nabla p)=\mu\nabla^2(\nabla\times\mathbf{v}).$$

Since the curl of a gradient is always zero, i.e.$\nabla\times(\nabla p)=0$, we get:

$$\mu\nabla^2(\nabla\times\mathbf{v})=0. \tag{17}$$

In 2D, the stream function ψ describes the flow, where $\nabla\times\mathbf{v}=-\nabla^2\psi$. Substituting this in Eq. (17), we obtain:

$$\mu\nabla^2(-\nabla^2\psi)=0, \tag{18}$$

which simplifies to the biharmonic equation for the stream function:

$$\nabla^4\psi=0. \tag{19}$$

Thus, we have shown that the stream function ψ satisfies the biharmonic equation in the case of 2D Stokes flow.

3. Biharmonic Equation in Polar Coordinates

In this section, we reframe the biharmonic equation in polar coordinates, which is useful for solving problems with circular symmetry[‡]. Given the definitions of polar coordinates:

$$x=\rho\cos\theta, \quad y=\rho\sin\theta, \tag{20}$$

we rewrite the Laplacian operator in these terms as [7]:

$$\nabla^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}. \tag{21}$$

Using this form of the Laplacian, we derive the biharmonic operator in polar coordinates, allowing us to express the biharmonic equation as:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)\right)\right)+\frac{2}{\rho^2}\frac{\partial^4\phi}{\partial\theta^2\partial\rho^2}-\frac{2}{\rho^3}\frac{\partial^3\phi}{\partial\theta^2\partial\rho}+\frac{1}{\rho^4}\frac{\partial^4\phi}{\partial\theta^4}+\frac{4}{\rho^4}\frac{\partial^2\phi}{\partial\theta^2}=0. \tag{22}$$

A. Radial Solution. For circularly symmetric problems, the solution to the biharmonic equation depends only on the radial coordinate ρ, meaning that $\phi(\rho,\theta)=\phi(\rho)$. In such cases, the biharmonic equation reduces to:

$$\frac{\partial^4\phi}{\partial\rho^4}+\frac{2}{\rho}\frac{\partial^3\phi}{\partial\rho^3}-\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho^3}\frac{\partial\phi}{\partial\rho}=0. \tag{23}$$

Assuming a solution of the form $\phi=\rho^k$, we calculate its successive derivatives to substitute into the radial equation. These derivatives are calculated as follows:

- The second derivative divided by ρ^2:

$$\frac{1}{\rho^2}\phi_{\rho\rho}=\frac{1}{\rho^2}k(k-1)\rho^{k-2}=k(k-1)\rho^{k-4}. \tag{24}$$

- The third derivative multiplied by $\frac{2}{\rho}$:

$$\frac{2}{\rho}\phi_{\rho\rho\rho}=\frac{2}{\rho}k(k-1)(k-2)\rho^{k-3}=2k(k-1)(k-2)\rho^{k-4}. \tag{25}$$

- The fourth derivative:

$$\phi_{\rho\rho\rho\rho}=k(k-1)(k-2)(k-3)\rho^{k-4}. \tag{26}$$

Substituting these results into equation Eq. (23), we obtain:

$$k\rho^{k-4}\left(k^3-6k^2+11k-6+2k^2-6k+1-k+1\right)=0, \tag{27}$$

which simplifies further to:

$$k^2\rho^{k-4}\left(k^2-4k+1\right)=0. \tag{28}$$

Solving for k, we find the roots:

$$k_1=0, \quad k_2=2. \tag{29}$$

Thus, the general solution, accounting for the root's multiplicity with $\ln\rho$ terms is:

$$\phi=C_1+C_2\ln\rho+C_3\rho^2+C_4\rho^2\ln\rho. \tag{30}$$

This approach allows us to solve the biharmonic equation explicitly in cases of radial symmetry, providing solutions that are applicable in circular or annular domains as we show in the following application.

B. An Application. An application of the polar solution to the biharmonic equation is found and adapted in [6] for a two-dimensional creeping flow problem, where the objective is to model the stream function ψ in a circular cavity subjected to a buoyancy force. The governing equation for ψ in this case is the biharmonic equation: $\nu\Delta^2\psi=\operatorname{rot}\boldsymbol{f}$, where ν is the kinematic viscosity, $\boldsymbol{f}$ represents the volumetric

[‡] Notice it is also applicable to cylindrical problems with no dependence on z.

body force, and rot $\boldsymbol{f} \equiv \partial f_y/\partial x - \partial f_x/\partial y$.

The boundary conditions for this problem specify that ψ remains constant along the wall, and the normal component of its gradient vanishes. This setup resembles the clamped plate bending problem, typically approached with Morley finite elements. Here, a buoyancy force $\boldsymbol{f} = x\hat{\mathbf{j}}$ is applied, as is common in the Boussinesq approximation for natural convection with a horizontal temperature gradient.

In the case of a circular cavity of radius a, the problem has an exact polynomial solution with circular streamlines, given by $\psi = \left(1 - (x^2 + y^2)/a^2\right)^2/64$. The implementation of this solution involves creating a triangular mesh for the cavity, using Morley elements to solve the biharmonic equation, assembling the resulting system, and enforcing boundary conditions.

The following figures illustrate both the stream function and the associated velocity vectors for this configuration.

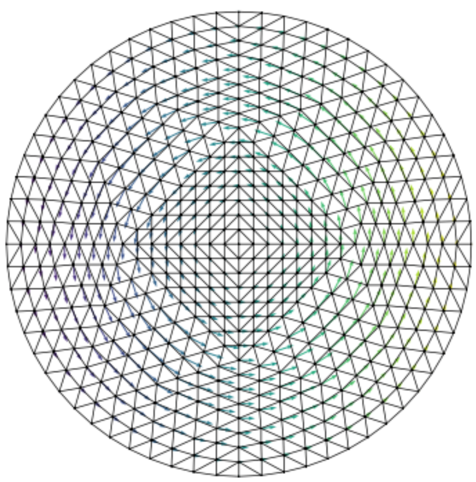

Fig. 1. Velocity field.

In the following section, we introduce spherical coordinates and present another application of the biharmonic equation.

4. Biharmonic Equation in Spherical Coordinates

Following our work with the biharmonic equation in polar coordinates, we now extend our approach to spherical coordinates. This extension is particularly useful for problems with radial symmetry in three-dimensional spaces, common in applications like fluid dynamics and heat transfer in spherical domains.

The Laplacian operator in spherical coordinates (r, θ, φ) is found from Eq. (8) as [7]:

$$\nabla^2 = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{31}$$

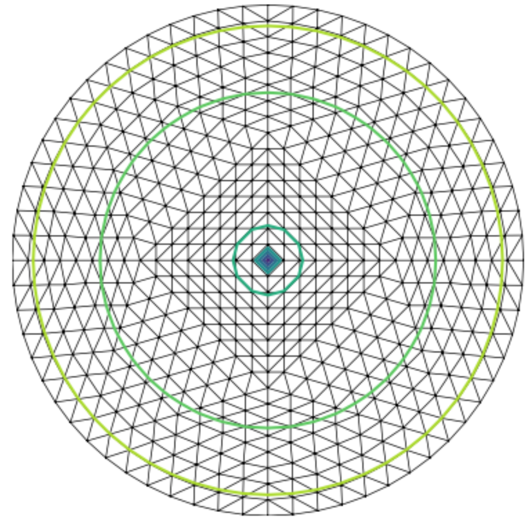

Fig. 2. Stream function and streamlines.

To obtain the biharmonic operator in spherical coordinates, we apply the Laplacian operator twice, as:

$$\nabla^4 = \nabla^2\nabla^2. \tag{32}$$

Thus, the biharmonic equation in spherical coordinates becomes:

$$\begin{aligned}\nabla^4 U = &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)\right)\right)\\ &+ \frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial^2}{\partial\theta\partial r}\left(r^2\frac{\partial U}{\partial r}\right)\right)\\ &+ \frac{1}{r^4\sin^2\theta}\frac{\partial^3}{\partial\varphi\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = 0.\end{aligned} \tag{33}$$

A. Radial Solution. For problems with radial symmetry, only the terms with radial dependence in Eq. (33) contribute to the solution, i.e., $U(r,\theta,\varphi) = U(r)$. Simplifying, we obtain:

$$\nabla^4 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)\right)\right) = 0. \tag{34}$$

Solving this equation requires integrating four times. The general solution in terms of r includes four integration constants and takes the form:

$$U(r) = -\frac{C_1 r}{2} + \frac{C_2 r^2}{6} + \frac{C_3}{r} + C_4. \tag{35}$$

B. A Computation Example. The following example models a physical situation often encountered in elasticity and membrane theory, where a circular, thin, elastic membrane—such as a drumhead or a diaphragm—is subject to specific boundary constraints. The conditions we impose here mirror real-world scenarios where the membrane is constrained to have:

1. No lateral movement at its center.

2. A fixed known displacement U_0 at the center.

3. A fixed edge at the boundary.

4. No shear stress along the edge

This four constrains allow us to determine the values of the constants in Eq. (35). In particular, these conditions can be mathematically expressed respectively as:

$$\frac{dU}{dr}(0) = 0, \quad U(0) = U_0, \quad U(R) = 0, \quad \frac{dU}{dr}(R) = 0. \tag{36}$$

Applying these boundary conditions, we find in Eq. (35), we find the constants to be:

$$C_1 = \frac{6U_0}{R^2}, \quad C_2 = \frac{12U_0}{R^3}, \quad C_3 = 0, \quad C_4 = U_0. \tag{37}$$

Substituting these constants into the general solution yields the particular solution:

$$U(r) = -\frac{3U_0}{R^2}r^2 + \frac{2U_0}{R^3}r^3 + U_0. \tag{38}$$

The plot in Fig. 3 shows this solution, where the central displacement of the membrane is set to $U_0 = 1$ cm and the radius R is 5 cm.

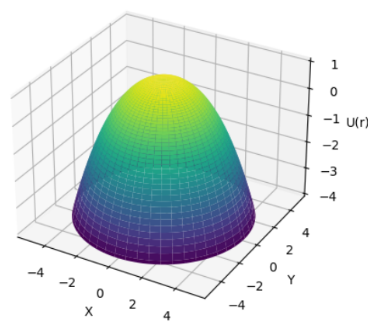

Fig. 3. Membrane Displacement Profile.

In the following section, we explore an innovative solution approach to the biharmonic equation in Cartesian coordinates by separation of variables, providing insights suitable for undergraduate students.

5. Biharmonic Equation in 2D Cartesian Coordinates

Starting from the Laplacian operator in Cartesian coordinates, we obtain the biharmonic equation in 2D Cartesian coordinates as follows:

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, \tag{39}$$

where $u(x, y)$ is the solution function. To solve this equation, we apply separation of variables, though with the added complexity of the mixed term $\frac{\partial^4 u}{\partial x^2 \partial y^2}$. We assume a separable solution of the form:

$$u(x, y) = X(x)Y(y). \tag{40}$$

Substituting Eq. (40) into Eq. (39) and simplifying gives:

$$\frac{X^{(4)}(x)}{X(x)} + 2\frac{X''(x)Y''(y)}{X(x)Y(y)} + \frac{Y^{(4)}(y)}{Y(y)} = 0. \tag{41}$$

We can rewrite this equation in functional form as:

$$E(x) + F(x)G(y) + H(y) = 0, \tag{42}$$

where $E(x)$, $F(x)G(y)$, and $H(y)$ represent each term in Eq. (41). By taking the derivative of Eq. (42) with respect to x, we find:

$$E'(x) + F'(x)G(y) = 0. \tag{43}$$

If $F'(x) \neq 0$, we can express this as:

$$\frac{E'(x)}{F'(x)} = -G(y), \tag{44}$$

implying that $G(y)$ must be a constant. Similarly, taking the derivative of Eq. (42) with respect to y yields:

$$F(x)G'(y) + H'(y) = 0, \tag{45}$$

which, using the same logic, implies that either $F(x)$ is a constant or $G'(y) = 0$. Assuming non-zero derivatives for X and Y (to retain full system dynamics), we conclude that one of the following must hold:

$$G(y) = K, \quad \text{or} \quad F(x) = K, \tag{46}$$

for some constant K. Without loss of generality, we assume $F(x) = K$[§]. Consequently, we have:

$$F(x) = \frac{X''(x)}{X(x)} = K = -\lambda^2. \tag{47}$$

This leads to the following ordinary differential equation (ODE) for $X(x)$:

$$X''(x) = -\lambda^2 X(x), \tag{48}$$

whose solution is:

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x). \tag{49}$$

Substituting Eq. (49) back into Eq. (41) produces:

$$\lambda^4 - 2\lambda^2\frac{Y''(y)}{Y(y)} + \frac{Y^{(4)}(y)}{Y(y)} = 0. \tag{50}$$

This can be rearranged to give the ODE:

$$Y^{(4)}(y) - 2\lambda^2 Y''(y) + \lambda^4 Y(y) = 0, \tag{51}$$

with characteristic equation:

$$r^4 - 2\lambda^2 r^2 + \lambda^4 = 0 \quad \Longleftrightarrow \quad \left(r^2 - \lambda^2\right)^2 = 0. \tag{52}$$

Factoring gives $(r - \lambda)^2(r + \lambda)^2 = 0$, yielding roots $r_1 = \lambda$ and $r_2 = -\lambda$, both with multiplicity two. According to ODE theory [7], the general solution to Eq. (51) is:

$$Y(y) = C_1 e^{\lambda y} + C_2 y e^{\lambda y} + C_3 e^{-\lambda y} + C_4 y e^{-\lambda y}. \tag{53}$$

Therefore, the general solution of the biharmonic equation in 2D Cartesian coordinates, which depends on the parameter λ[¶], is:

$$\begin{aligned} u_\lambda(x, y) = {} & [C_1\sin(\lambda x) + C_2\cos(\lambda x)] \\ & \times \left[C_1 e^{\lambda y} + C_2 y e^{\lambda y} + C_3 e^{-\lambda y} + C_4 y e^{-\lambda y}\right]. \end{aligned} \tag{54}$$

[§] This choice is generalizable due to the symmetry of the biharmonic equation: swapping x and y does not change the form of the equation.

[¶] The full general solution involves an integral over all λ: $u(x, y) = \int_0^\infty u_\lambda(x, y)\, d\lambda$.

6. Biharmonic Wave Equation in Cartesian Coordinates Applied to Structural Mechanics

This section models the vibration of a rectangular plate using the biharmonic wave equation in Cartesian coordinates, a scenario commonly encountered in structural mechanics. In this context, we consider an elastic plate undergoing small oscillations, where each point on the plate experiences a displacement due to applied forces.

Assuming the initial position of a point on the plate is $M(x, y, z)$ and its displaced position is $M'(x, y, z)$, the displacement vector $\vec{d}$ is given by:

$$\vec{d} = \overrightarrow{MM'} = (u(x, y, z), v(x, y, z), w(x, y, z)), \tag{55}$$

where u, v, and w represent the displacements along the x, y, and z directions, respectively.

The unit deformations for each coordinate are defined as:

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}. \tag{56}$$

In this scenario, the forces per unit surface acting on the plate, denoted σ_{ij}, represent stress components, with each subscript i and j referring to the coordinate directions of the plane on which the force acts. According to Hooke's law for adiabatic and isothermal deformations, we have:

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}, \tag{57}$$

where E is Young's modulus, ν is Poisson's ratio, and δ_{ij} is the Kronecker delta function, indicating zero stress components off the main diagonal when $i \neq j$.

To simplify the analysis, we assume:

- The material is ideal (linear and isotropic).
- The plate's thickness h is small relative to its other dimensions.
- The displacement w depends only on x, y, and t: $w = w(x, y, t)$.
- Shear stresses along xz and yz planes are zero: $\sigma_{xz} = \sigma_{yz} = 0$.
- Normal stress and strain in the z-direction are zero: $\sigma_{zz} = \epsilon_{zz} = 0$.

Based on these assumptions and following the formulation in [2], the governing equation for a freely oscillating plate is:

$$\frac{Eh^3}{12(1-\nu^2)}\nabla^4 w(x, y) + h\rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{58}$$

where ρ represents the plate's density. We assume no specific boundary conditions for now, but the general solution of Eq. (58) can be approached using separation of variables.

Assuming a solution of the form:

$$w(x, y, t) = W(x, y)T(t), \tag{59}$$

and substituting this expression into Eq. (58) gives:

$$\frac{D}{\rho h}\frac{\nabla^4 W}{W} = -\frac{1}{T}\frac{d^2 T}{dt^2} = \omega^2, \tag{60}$$

where ω^2 is the separation constant. This results in a simple harmonic solution for the time-dependent part:

$$T(t) = A\sin(\omega t + \phi_0). \tag{61}$$

For the spatial part, we define a new constant k^4 as:

$$k^4 = \frac{\omega^2 \rho h}{D}. \tag{62}$$

Thus, the spatial equation simplifies to:

$$(\nabla^4 - k^4)W(x, y) = 0. \tag{63}$$

We previously solved the case where $k = 0$. Applying similar logic and using a separation constant α^2, we find:

$$X(x) = B\sin(\alpha x + \delta). \tag{64}$$

The function $Y(y)$ satisfies a higher-order ODE with characteristic roots μ_i, assuming $k^4 > \alpha^4$:

$$\begin{aligned} \mu_1 &= \sqrt{\alpha^2 + k^2}, \quad \mu_2 = -\sqrt{\alpha^2 + k^2}, \\ \mu_3 &= i\sqrt{k^2 - \alpha^2}, \quad \mu_4 = -i\sqrt{k^2 - \alpha^2}. \end{aligned} \tag{65}$$

The general solution for $w(x, y, t)$ is then:

$$\begin{aligned} w(x, y, t) = {} & \sin(\omega t + \phi)\sin(\alpha x + \delta)\left[A_1\sin(\nu_1 y)\right. \\ & \left. + A_2\cos(\nu_1 y) + A_3\sinh(\nu_2 y) + A_4\cosh(\nu_2 y)\right], \end{aligned} \tag{66}$$

where:

$$\nu_1 = \sqrt{k^2 - \alpha^2}, \quad \nu_2 = \sqrt{k^2 + \alpha^2}. \tag{67}$$

The biharmonic wave equation applied to a vibrating plate, as derived here, is crucial for modeling and understanding structural dynamics in engineering. Plates undergoing vibration, such as those in building floors, bridges, and mechanical components, must meet stringent stability and resonance conditions. By understanding the behavior of such plates under vibrational stress, engineers can design structures that avoid resonance-related failures, ensuring both longevity and safety.

7. Conclusions

- The biharmonic equation has a wide range of applications across various engineering fields, including structural analysis, fluid mechanics, and elasticity theory. Its ability to model complex physical phenomena such as plate deflection, potential flow, and stress distribution in materials highlights its versatility and significance in solving real-world engineering problems.
- Overall, the article serves as a valuable resource for researchers, engineers, and students seeking to understand the theoretical foundations and practical implications of the Biharmonic equation. Its clear explanations, mathematical rigor, and illustrative examples make it accessible to a wide audience, facilitating further exploration and application of this powerful mathematical tool.

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