Rendiconti del Circolo Matematico di Palermo Series 2 (2024) 73:2709–2721
https://doi.org/10.1007/s12215-024-01038-y

Lie algebra classification, conservation laws and invariant solutions for the kind generalization of the Duffing-type equation

Oscar Londoño[1] · Danilo García[1] · Gabriel Loaiza[2] · Yeisson Acevedo[2]

Received: 3 May 2023 / Accepted: 13 April 2024 / Published online: 10 June 2024

Abstract

This paper makes significant contributions to the study of a generalized form of the Duffing-type equation. We derive the generating operators of the optimal system associated with this equation, enabling us to characterize an implicit solution. Additionally, we present a complete classification of group symmetries and obtain the Lagrangian for the equation. Our results include the classification of the Lie algebra and the optimal system, providing a thorough understanding of the equation's underlying structure. These contributions serve to enhance the current body of knowledge on the Duffing-type equation and provide useful insights for future research in this area.

Keywords Continuous group of Lie symmetries · Optimal system · Invariant solution · Lagrangian · Lie algebra representation

1 Introduction

We consider a member of the Duffing-type equation [1–3]

$$\frac{d^2y}{dx^2} + y^{2n+1} + \sum_{j=0}^{2n} p_j(x)y^j = 0, \quad n \geqslant 1, \tag{1}$$

Oscar Londoño, Danilo García, Gabriel Loaiza and Yeisson Acevedo contributed equally to this work.

✉ Yeisson Acevedo
yaceved2@eafit.edu.co

Oscar Londoño
o154278@dac.unicamp.br

Danilo García
d190684@dac.unicamp.br

Gabriel Loaiza
gloaiza@eafit.edu.co

1 Mathematics, UNICAMP, Campinas, Brazil

2 Applied Sciences and Engineering, EAFIT University, Medellín, Colombia

with $p_j(x+1) = p(x)$ and $p_j \in C^\infty$. Equation (1) describes a class of Hamiltonian systems with time-dependent potentials. In general, this equation is distinguished in a dynamical system concerning the jump resonance phenomenon. The qualitative aspects of this equation have been studied in [2–4].

It is important to note that these equations have numerous applications in physics, engineering, and applied mathematics, including the study of mechanical vibrations, oscillations in electrical systems, and nonlinear dynamics of structures. The Duffing-type equation is one of the simplest nonlinear equations exhibiting chaotic behavior, making it an ideal model for investigating complex phenomena in dynamical systems. Therefore, studying and finding solutions to Eq. (1) is not only of academic interest but also has significant practical implications in a variety of fields.

In line with the foregoing, in the present paper, we study the Eq. [1, 5, 6]:

$$y_{xx} + y^{2n+1} + f(x)y - g(x) = 0, \quad n \in \mathbb{R}, \tag{2}$$

which has been studied previously, particularly regarding the boundedness of solutions. In [1], Moser's twist theorem was employed to establish the boundedness of the solution. The objective of this work is to apply the Lie symmetry method [7–11] to the Eq. (2). Examples of this method can be found in [12–15].

The theory of Lie groups has emerged as a potent methodology for solving differential equations, leveraging its capacity to reveal inherent symmetries within the equations. The identification of these symmetries enables us to streamline the problem by reducing the number of variables involved, leading to more accessible solutions. Recent research papers [16–19] showcase the application of Lie symmetry group methods in solving differential equations. These studies highlight the pivotal role of Lie group theory in establishing a framework for comprehending the intrinsic structure of differential equations and identifying their symmetries. This, in turn, facilitates the development of efficient numerical methods for solving such equations. Currently, the theory of Lie groups has evolved into a crucial tool for mathematicians and scientists across diverse fields, spanning physics, engineering, and finance.

To conclude this introductory section, it is imperative to emphasize that, under specific limit regimes, the diffusion equation can be effectively approximated by the Duffing-type equation. The rationale behind this lies in the shared capability of both the diffusion equation and the Duffing-type equation to capture nonlinear phenomena. Through specific limits, the Duffing-type equation can be streamlined into a form closely resembling the diffusion equation. Consequently, leveraging the Duffing-type equation enhances our understanding of diffusion processes within nonlinear systems. Furthermore, its integration with the diffusion equation affords a sophisticated modeling approach for complex dynamical systems in physics and engineering, as highlighted in recent works [20–22].

The paper is structured as follows: In Sect. 2, a complete classification of Lie group symmetries is provided. Section 3 introduces the optimal system for (2). The solution obtained using this optimal system is explored in Sect. 4. Moving on, Sect. 5 demonstrates the derivation of a Lagrangian for equation (2) through the *Jacobi multiplier method*, followed by the classification of the symmetry Lie algebra.

2 Lie point symmetry group

In this section, we will determine the Lie symmetry group for Equation (2). Our primary objective is to identify a set of transformations that maintain the equation invariant under infinitesimal variations. The pivotal outcome of this section can be succinctly summarized as follows:

Proposition 1 *Equation* (2)*, with arbitrary functions f and g and constant n, does not possess a Lie point symmetry group. Complementary cases are presented in Table* 1.

Proof The one-parameter Lie group generating operators associated with Eq. (2) are:

$$y \longrightarrow y + \epsilon\omega(x, y) + O\left(\epsilon^2\right) \quad ; \quad x \longrightarrow x + \epsilon\psi(x, y) + O\left(\epsilon^2\right),$$

where ϵ represents the group parameter. The vector field associated with this group of transformations is $\Gamma = \psi(x, y)\frac{\partial}{\partial x} + \omega(x, y)\frac{\partial}{\partial y}$, with ψ, ω differentiable functions in $\mathbb{R}^2$. To determine the infinitesimals $\omega(x, y)$ and $\psi(x, y)$, we employed the second extension operator

$$\Gamma^{(2)} = \omega_{[xx]}\frac{\partial}{\partial y_{xx}} + \omega_{[x]}\frac{\partial}{\partial y_x} + \Gamma, \tag{3}$$

to the Eq. (2), we obtained the corresponding symmetry requirement

$$\psi(f_x y - g_x) + \omega\left[(2n+1)y^{2n} + f\right] + \omega_{[xx]} = 0, \tag{4}$$

where the coefficients in $\Gamma^{(2)}$ are $\omega_{[x]}$ and $\omega_{[xx]}$, respectively:

$$\begin{aligned} \omega_{[x]} &= D_x[\omega] - (D_x[\psi])y_x = \omega_x + (\omega_y - \psi_x)y_x - \psi_y y_x^2, \\ \omega_{[xx]} &= D_x[\omega_{[x]}] - (D_x[\psi])y_{xx}, \\ &= \omega_{xx} + (2\omega_{xy} - \psi_{xx})y_x + (\omega_{yy} - 2\psi_{xy})y_x^2 - \psi_{yy}y_x^3 \\ &\quad +(\omega_y - 2\psi_x)y_{xx} - 3\psi_y y_x y_{xx}. \end{aligned} \tag{5}$$

where $D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{y_x} + \cdots$ is the total differential (see [12]).

After applying (5) in (4) and substituting in the resulting expression y_{xx} by (2), we get:

$$\begin{aligned} &\psi\left[f_x y - g_x)\right] + \omega\left[(2n+1)y^{2n} + f\right] + \omega_{xx} + (2\omega_{xy} - \psi_{xx})y_x \\ &\quad +(\omega_{yy} - 2\psi_{xy})y_x^2 - \psi_{yy}y_x^3 + (\omega_y - 2\psi_x)\left(-y^{2n+1} - f(x)y + g(x)\right) \\ &\quad -3\psi_y y_x\left(-y^{2n+1} - f(x)y + g(x)\right) = 0. \end{aligned}$$

Analyzing the coefficients with respect to the independent variables y_x^3, y_x^2, y_x, and 1, we obtain the following system of determining equations:

$$\psi_{yy} = 0, \tag{6a}$$

$$\omega_{yy} - 2\psi_{xy} = 0, \tag{6b}$$

$$\psi_y\left(3y^{2n+1} + 3yf - 3g\right) + 2\omega_{xy} - \psi_{xx} = 0, \tag{6c}$$

$$\begin{aligned} &\omega\left(y^{2n} + 2ny^{2n} + f\right) + \psi\left(yf_x - g_x\right) + \omega_y\left(-y^{2n+1} - yf + g\right) \\ &\quad +\psi_x\left(2y^{2n+1} + 2yf - 2g\right) + \omega_{xx} = 0. \end{aligned} \tag{7}$$

Table 1 Infinitesimal generators of Eq. (2)

Case	n	f	g	Symmetries
$I.1.1)$	$n=1$	f *(arbitrary).*	$g(x)=0$	Trivial
$I.1.2)$	$n=1$	$f(x)=-\frac{c_2^{-2}}{2}\int\left(c_{2,xxx}c_2\right)dx+\left(k_2c_2^{-2}\right).$	$g(x)=0$	$\Gamma=c_2(x)\partial_x+\frac{y}{2}c_2(x)\partial_y$
$I.2)$	$n=1$	$f(x)=-\frac{g^{4/3}}{2k_3}\int\left(c_{2,xxx}g^{-2/3}\right)dx+\left(k_4g^{4/3}\right).$	$g(x)\neq 0,$ $c_2(x)=k_3(g(x))^{-\frac{2}{3}}$	$\Gamma=k_3(g(x))^{-\frac{2}{3}}\partial_x$ $-\frac{yk_3}{3}(g(x))^{-\frac{5}{3}}g_x\partial_y$
$II.1)$	$n=-\frac{1}{2}$	$f(x)=p_1(k_2-2k_1x)^{-2}$	$g(x)=(2k_1x-3k_2)^{-3}\left(\int\frac{-3c_{4,xx}dx}{(2k_1x-3k_2)^{-2}}-\frac{3p_1c_4(2k_1x-3k_2)^2dx}{(k_2-2k_1x)^2}\right)+k_4(2k_1x-3k_2)^{-3}$	$\Pi_1=-x\partial_x+y\partial_x,\quad \Pi_2=\partial_x$ and $\Pi_3=c_4(x)\partial_y$
$II.2.1.)$	$n=\frac{5}{2}$	$f=9k_4(2k_1x+3k_2)^{-2}$	$g(x)=\frac{k_4}{2}(2k_1x+3k_2)^{-2}+k_5.$	$\Pi_1=x\partial_x+(2y-6)\partial_y$ and $\Pi_2=\partial_x$
$II.2.2.1)$	$n=\frac{3}{2}$	$f=k_4(2k_1x+k_2)^{-2}$	$g=-k_1k_4(2k_1x+k_2)^{-2}\ln(2k_1x+k_2)^{-2}+k_5(2k_1x+k_2)^{-2}$	$\Pi_1=x\partial_x+(2y-1)\partial_y$ and $\Pi_2=\partial_x$
$II.2.2.2)$	$n\neq 1,-\frac{1}{2},\frac{5}{2},\frac{3}{2}$	$f=k_4(n-1)^2(k_1x+k_2(n-1))^{-2}$	$g=-\frac{k_1k_4(n-1)^3}{(3-2n)(k_1x+k_2(n-1))^2}+\frac{k_5}{(k_1x+k_2(n-1))^{-5+2n}}$	$\Pi_1=\frac{x}{(n-1)}\partial_x+\left(\frac{y(2n-1)}{2(n-1)}-1\right)\partial_y$ and $\Pi_2=\partial_x$

It is evident from (7) that if f and g are arbitrary, we have $\psi = 0$ and $\omega = 0$, implying the non-existence of the group of symmetries. By solving (6a), we have:

$$\psi(x) = yc_1(x) + c_2(x), \tag{8}$$

here, c_1 and c_2 are arbitrary functions. Substituting (8) into (6b), we obtain:

$$\omega(x, y) = y^2 c_{1,x} + yc_3(x) + c_4(x). \tag{9}$$

where c_3 and c_4 are arbitrary functions. From (8) and (9) into (6c) we get $y^{2n+1}(3c_1) + y\left(3fc_1 + 3c_{1,xx}\right) - 3gc_1 + 2c_{3,x} - c_{2,xx} = 0$, implying $c_1 = 0$ and $c_3(x) = \frac{c_{2,x}}{2} + k_1$, then using this in (8) and (9), we will obtain:

$$\psi(x) = c_2(x), \quad \omega(x, y) = y\left(\frac{c_{2,x}}{2} + k_1\right) + c_4(x). \tag{10}$$

From (10) into (7):

$$\begin{aligned} &y^{2n+1}\left(c_{2,x}(n-1) - k_1\right) + y\left(c_2 f_x + 2c_{2,x} f + \frac{c_{2,xxx}}{2}\right) \\ &\quad + y^{2n}\left((2n+1)(c_4 + k_1)\right) + c_4 f - c_2 g_x + g\left(\frac{-3c_{2,x}}{2} + k_1\right) + c_{4,xx} = 0. \end{aligned} \tag{11}$$

Given that y^{2n+1}, y^{2n}, y, and 1 are linearly independent, the coefficients in the expression (11) for these terms must be zero. This leads to the following system of equations:

$$c_{2,x}(n-1) - k_1 = 0, \tag{12a}$$

$$(c_4 + k_1)(2n+1) = 0, \tag{12b}$$

$$c_2 f_x + 2c_{2,x} f + \frac{c_{2,xxx}}{2} = 0, \tag{12c}$$

$$c_4 f - c_2 g_x + g\left(\frac{-3c_{2,x} + 2k_1}{2}\right) + c_{4,xx} = 0, \tag{12d}$$

In reference to (12a), we need to consider two cases: $n = 1$ and $n \neq 1$.

Case I. Suppose that (12a) leads to $n = 1$, which implies $k_1 = 0$. From (12b) we get $c_4(x) = 0$. From (12d) we have

$$c_{2,x} g + \frac{2c_2 g_x}{3} = 0. \tag{13}$$

Regarding (13), two cases must be considered $g = 0$ and $g \neq 0$.

Case I.1 Suppose that (13) leads to $g = 0$. From (12c) we have

$$c_2 f_x + 2c_{2,x} f + \frac{c_{2,xxx}}{2} = 0. \tag{14}$$

In reference to (14), we need to consider two cases: $c_2 = 0$ and $c_2 \neq 0$.

Case I.1.1 Suppose that (14) leads to $c_2 = 0$. From (10) we obtain $\psi(x) = 0$ and $\omega(x, y) = 0$. Hence the group of symmetries is trivial, then $g = 0$ and f is arbitrary function.

Case I.1.2 Suppose that (14) leads to $c_2 \neq 0$. This means that

$$f(x) = -\frac{c_2^{-2}}{2}\int \left(c_{2,xxx} c_2\right) dx + \left(k_2 c_2^{-2}\right).$$

Where k_2 is an arbitrary constant. From (10) we obtain $\psi(x) = c_2(x)$ and $\omega(x, y) = \frac{yc_2(x)}{2}$. Consequently, the generator of the group of symmetries is $\Gamma = c_2(x)\partial_x + \frac{y}{2}c_2(x)\partial_y$, with $g = 0$.

Case I.2 Suppose that (13) leads to $g \neq 0$, then $c_{2,x} + \frac{2c_2 g_x}{3g} = 0$. Solving for $c_2(x)$, we have $c_2(x) = k_3(g(x))^{-\frac{2}{3}}$, where k_3 is another arbitrary constant. From (12c) we get

$$f_x - \frac{4g_x}{3g} f = -\frac{c_{2,xxx}}{2k_3} g^{\frac{2}{3}}. \tag{15}$$

Solving in (15) for f, we have

$$f(x) = -\frac{g^{4/3}}{2k_3} \int \left(c_{2,xxx} g^{-2/3}\right) dx + \left(k_4 g^{4/3}\right).$$

Where k_4 is arbitrary constant. From (10) we obtain $\psi(x) = k_3(g(x))^{-\frac{2}{3}}$ and $\omega(x, y) = -\frac{yk_3}{3}\left(g^{-5/3} g_x\right)$. Therefore the generator of the group of symmetries is $\Gamma = k_3(g(x))^{-\frac{2}{3}}\partial_x - \frac{yk_3}{3}(g(x))^{-\frac{5}{3}} g_x \partial_y$, with $g \neq 0$.

Case II. Suppose in (12a) that $n \neq 1$. Then $c_2 = \frac{k_1}{n-1}x + k_2$. Where k_2 is arbitrary constant. From (12b):

$$(c_4 + k_1)(2n + 1) = 0. \tag{16}$$

Considering (16), we need to examine two cases: $n = -\frac{1}{2}$ and $n \neq -\frac{1}{2}$.

Case II.1 Suppose in (12b) that $n = -\frac{1}{2}$ then $c_2(x) = -\frac{2k_1 x}{3} + k_2$. From (12c) we get $f_x + \left(\frac{4k_1}{2k_1 x - k_2}\right) f = 0$, and solving for f, we obtain $f(x) = p_1(k_2 - 2k_1 x)^{-2}$. Where p_1 is abitrary constant. From (12d)

$$g_x + g\left(\frac{6k_1}{2k_1 x - 3k_2}\right) = -\frac{3c_{4,xx}}{2k_1 x - 3k_2} - \frac{3p_1 c_4 (k_2 - 2k_1 x)^{-2}}{(2k_1 x - 3k_2)}.$$

Solving for g, we get

$$g(x) = (2k_1 x - 3k_2)^{-3}\left(\int \frac{-3c_{4,xx} dx}{(2k_1 x - 3k_2)^{-2}} - \frac{3p_1 c_4 (2k_1 x - 3k_2)^2 dx}{(k_2 - 2k_1 x)^2}\right) + k_4 (2k_1 x - 3k_2)^{-3}.$$

Where k_4 is arbitrary constant. From (10) we obtain $\psi(x) = -\frac{2k_1 x}{3} + k_2$ and $\omega(x, y) = \frac{2yk_1}{3} + c_4(x)$. Hence the group of symmetries is $\Pi_1 = -x\partial_x + y\partial_y$, $\Pi_2 = \partial_x$ and $\Pi_3 = c_4(x)\partial_y$.

Case II.2 Suppose in (16) that $n \neq -\frac{1}{2}$, then $c_4(x) = -k_1$. From (12c):

$$f_x + \frac{2c_{2,x}}{c_2} f = 0. \tag{17}$$

Solving in (17) for f, we obtain $f = k_4(n-1)^2(k_1 x + k_2(n-1))^{-2}$. Where k_4 is arbitrary constant. From (12d):

$$g_x + g\left(\frac{k_1(5 - 2n)}{(k_1 x + k_2(n-1))}\right) = -\frac{k_1 k_4 (n-1)^3}{(k_1 x + k_2(n-1))^3}. \tag{18}$$

From (18) we have to consider two cases: $n = \frac{5}{2}$ and $n \neq \frac{5}{2}$.

Case II.2.1 Suppose at (18), that $n = \frac{5}{2}$ then $f = 9k_4(2k_1x+3k_2)^{-2}$ and $g_x = -\frac{27k_1k_4}{(2k_1x+3k_2)^3}$. Solving the above expression for g, we get $g(x) = \frac{k_4}{2}(2k_1x + 3k_2)^{-2}+k_5$. Where k_5 is arbitrary constant. From (10), we get $\psi(x) = \frac{2k_1x}{3}+k_2$ and $\omega(x, y) = y\left(\frac{4k_1}{3}\right) - k_1$. Hence the group of symmetries is $\Pi_1 = x\partial_x + (2y - 6)\partial_y$ and $\Pi_2 = \partial_x$

Case II.2.2 Suppose at (18), that $n \neq \frac{5}{2}$. Thus, if we want to solve (18) we have to consider two cases: $n = \frac{3}{2}$ and $n \neq \frac{3}{2}$.

Case II.2.2.1 Suppose at (18), that $n = \frac{3}{2}$, so $f = k_4(2k_1x + k_2)^{-2}$ and $g = -k_1k_4(2k_1x+k_2)^{-2}\ln(2k_1x + k_2)^{-2}+k_5(2k_1x+k_2)^{-2}$. Where k_5 is arbitrary constant. From (10), we get $\psi(x) = 2k_1x+k_2$ and $\omega(x, y) = 2k_1y-k_1$. Hence the group of symmetries is $\Pi_1 = x\partial_x + (2y - 1)\partial_y$ and $\Pi_2 = \partial_x$

Case II.2.2.2 Suppose at (18), that $n \neq \frac{3}{2}$, then solving for g we get $g = -\frac{k_1k_4(n-1)^3}{(3-2n)(k_1x+k_2(n-1))^2} + \frac{k_5}{(k_1x+k_2(n-1))^{-5+2n}}$ and $f = k_4(n - 1)^2(k_1x + k_2(n - 1))^{-2}$. From (10): $\psi(x) = \frac{k_1x}{n-1} + k_2$ and $\omega(x, y) = y\left(\frac{k_1}{2(n-1)} + k_1\right) - k_1$. Hence the group of symmetries is $\Pi_1 = \frac{x}{(n-1)}\partial_x + \left(\frac{y(2n-1)}{2(n-1)} - 1\right)\partial_y$ and $\Pi_2 = \partial_x$.

We have completed the proof of Proposition 1 by analyzing all the cases.

3 Optimal system

In the context of Lie symmetries, an optimal system is a set of generators that allow the identification of all symmetries and invariant solutions of a differential equation, which is invariant under a given Lie group. An optimal system is also the minimum set of generators needed to generate all symmetries and invariant solutions of the differential equation.

In other words, an optimal system allows for obtaining complete information about the symmetries and invariant solutions of a differential equation and is fundamental in the investigation of differential equations possessing Lie point symmetries. By using an optimal system, invariant solutions can be obtained more efficiently, which is particularly important in scientific and engineering applications where solving differential equations is a common task [8, 9].

Taking into account [23, 24], in this section, we are going to the optimal system from the previous continuous group of Lie symmetries for (2), using $II.2.2.2$) of the Table 1, a possible way to obtain invariant solutions is presented.

Initially, we compute the corresponding commutator table, which can be derived from the operator:

$$[\Pi_\alpha, \Pi_\beta] = \sum_{i=1}^{n}\left(\Pi_\alpha(\psi^i_\beta) - \Pi_\beta(\psi^i_\alpha)\right)\frac{\partial}{\partial x^i}, \tag{19}$$

where $\alpha, \beta = 1, 2$, with $i = 1, 2$, and the coefficients relate to the infinitesimal operators Π_α, Π_β are $\psi^i_\alpha, \psi^i_\beta$. Now using the operator (19) in the Lie symmetry group of (2), (see $II.2.2.2$ in Table 1), we obtain the commutator table for operators, namely:

Now, we will use the next self-adjoint operator by the continuous group of Lie symmetries of (2), and for this proposal, we will use the commutators table (see Table 2):.

Table 2 Commutators table of (2)

$[\Pi_\alpha, \Pi_\beta]$	Π_1	Π_2
Π_1	0	$\frac{-1}{n-1}\Pi_2$
Π_2	$\frac{1}{n-1}\Pi_2$	0

Table 3 Self-adjoint representation of (2)

$\mathrm{Ad}[\Pi_\alpha, \Pi_\beta]$	Π_1	Π_2
Π_1	Π_1	$e^{\left(\frac{1}{n-1}\right)\lambda}\Pi_2$
Π_2	$\Pi_1 - \left(\frac{1}{n-1}\right)\lambda\Pi_2$	Π_2

$$\mathrm{Ad}(e^{(\lambda\Pi)})G = \sum_{n=0}^{\infty} \frac{\lambda^i}{i!}(\mathrm{Ad}(\Pi))^i G,$$

where Π represents a symmetry, and G is a generic linear combination of symmetries.

Now, using the previous operator, we calculate Table 3, which presents the self-adjoint representation for the symmetries.

Proposition 2 *The following set of field vectors constitutes the optimal system associated with equation* (2)

$$\Pi_1,\ \Pi_1 + b_2\Pi_2\,,\ \textit{with}\ b_2 \neq 0.$$

Proof Consider the following linear combination of symmetries:

$$G = a_1\Pi_1 + a_2\Pi_2, \tag{20}$$

it generator was shown in $II.2.2.2$), and our goal is to simplify the a_i coefficients as much as we can by utilizing adjoint maps to G and referring to Table 3.

(1) Supposing $a_2 = 1$ in (20), we get $G = a_1\Pi_1 + \Pi_2$. Now, applying the self-adjoint operator to (Π_2, G) we obtain

$$G_1 = \mathrm{Ad}\left(e^{(\lambda_1\Pi_2)}\right)G = a_1\Pi_1 + \left(1 - \frac{\lambda_1}{n-1}\right)\Pi_2.$$

Since $n \neq 0, 1, -\frac{1}{2}, \frac{5}{2}, \frac{3}{2}$, we can use $\lambda_1 = n - 1$, consequently, Π_2 is eliminated, therefore $G_1 = a_1\Pi_1$. Finally, applying the adjoint operator to (Π_1, G_1) does not lead to any reduction, thus we obtain the first element of the optimal system:

$$G_1 = a_1\Pi_1.$$

This results in a simplification of the general element (20).

(2) Supposing $a_2 = 0$, $a_1 = 1$ in (20), we get $G = \Pi_1$. Using the adjoint operator again on (Π_1, G), we don't have any reduction, but using the adjoint operator on (Π_2, G) we finally obtain

$$G_2 = \mathrm{Ad}\left(e^{(\lambda_2\Pi_2)}\right)G = \Pi_1 - \frac{\lambda}{n-1}\Pi_2.$$

It is evident that this does not result in any reduction. Then, by utilizing $b_2 = \frac{-1}{(n-1)\lambda_2}$, with $\lambda_2 \neq 0$, we obtain another element of the optimal system

$$G_2 = \Pi_1 + b_2 \Pi_2\,, \text{ where } b_2 \neq 0.$$

This marks the conclusion of another simplification of the general element (20).

4 Invariant solutions by generators of the optimal system

In this section, we delineate the properties of all invariant solutions by considering certain operators that generate the optimal system outlined in Proposition 2. To achieve this objective, we employ the method of invariant curve condition [9] (as presented in section 4.3, p. 68−71). This method is mathematically expressed by the following equation:

$$0 = -y_x \psi + \omega = Q(x, y, y_x). \tag{21}$$

Utilizing the component Π_1 from Proposition 2 and II.2.2.2 of the Table 1, and applying the condition (21), we obtain the following result $Q = \omega_1 - y_x \psi_1 = 0$. This leads to the equation $\left(\frac{y(2n-1)}{2(n-1)} - 1\right) - y_x \left(\frac{x}{n-1}\right) = 0$. Solving this ordinary differential equation yields an implicit solution as shown in Table 4. A similar procedure with the element $\pi_1 + b_2 \Pi_2$, using $b_2 = 1$, provides another solution also presented in Table 4.

5 The Lagrangian and classification of Lie algebra

Now, we will determine the Lagrangian using Jacobi's Last Multiplier method, as presented by Nucci in [25]. For this purpose, we will calculate the inverse of the determinant, denoted as Δ^{-1}:

$$\Delta = \begin{vmatrix} x & y_x & y_{xx} \\ \Pi_{1,x} & \Pi_{1,y} & \Pi_1^{(1)} \\ \Pi_{2,x} & \Pi_{2,y} & \Pi_2^{(1)} \end{vmatrix} = \begin{vmatrix} x & y_x & y_{xx} \\ \frac{x}{(n-1)} & \frac{y(2n-1)}{2(n-1)} - 1 & \frac{2n-3}{2(n-1)} y_x \\ 1 & 0 & 0 \end{vmatrix},$$

where $\Pi_{1,x}, \Pi_{1,y}, \Pi_{2,x}$, and $\Pi_{2,y}$ are the components of the symmetries Π_1, Π_2 shown in the Proposition 1 case II.2.2.2 Table 1 and $\Pi_1^{(1)}, \Pi_2^{(1)}$ as its first prolongations. Then we get $\Delta = By_x^2 + H(x, y)$, where $B = \frac{2n-3}{2(n-1)}$ and $H(x, y) = \left(1 - \frac{y(2n-1)}{2(n-1)}\right)\left(-y^{2n+1} - f(x)y + g(x)\right)$, which implies that $M = \frac{1}{\Delta} = \frac{1}{By_x^2 + H}$.

Based on [25], it has been established that M can be expressed as $M = L_{y_x y_x}$, which in turn implies $L_{y_x y_x} = \frac{1}{By_x^2 + H}$. By performing double integration with respect to y_x, the Lagrangian can be derived as follows:

$$L(x, y, y_x) = \frac{y_x \tan^{-1}(Cy_x)}{D} - \frac{\ln(C^2 y_x^2 - 1)}{2CD} + y_x f_1(x, y) + f_2(x, y), \tag{22}$$

where $C = \frac{\sqrt{B}}{\sqrt{H}}$, $D = \sqrt{BH}$ and f_1, f_2 are arbitrary functions.

The study of Lie algebras serves various purposes. For example, in geometry, the classification of Lie algebras gives rise to several models, including the contact Lie algebra, quadratic Lie algebra, and others. In the Lie symmetry method, the group of transformations

Table 4 Solutions for (2)

Element	$Q(x, y, y_x) = 0$	Solution
Π_1	$\left(\frac{y(2n-1)}{2(n-1)} - 1\right) - y_x \left(\frac{x}{n-1}\right) = 0$	$\frac{k_6(2n-1)(2n-3)}{4} x^{\frac{2n-5}{2}} + \left(\frac{2(n-1)}{2n-1} + k_6 x^{\frac{2n-1}{2}}\right)^{2n+1} + f(x)\left(\frac{2(n-1)}{2n-1} + k_6 x^{\frac{2n-1}{2}}\right) - g(x) = 0;$ $n \neq 1, \frac{-1}{2}, \frac{5}{2}, \frac{3}{2};$ with $y = \frac{2(n-1)}{2n-1} + k_6 x^{\frac{2n-1}{2}}$, $f(x) = k_4(n-1)^2(k_1 x + k_2(n-1))^{-2}$, and $g(x) = -\frac{k_1 k_4 (n-1)^3}{(3-2n)(k_1 x + k_2(n-1))^2} + \frac{k_5}{(k_1 x + k_2(n-1))^{-5+2n}}$.
$\Pi_1 + \Pi_2$	$\left(\frac{y(2n-1)}{2(n-1)} - 1\right) - y_x \left(\frac{x}{n-1} + 1\right) = 0$	$\frac{k_7(2n-1)(2n-3)}{4} (n + x - 1)^{\frac{2n-5}{2}} + \left(\frac{2(n-1)}{2n-1} + k_7(n + x - 1)^{\frac{2n-1}{2}}\right)^{2n+1} + f(x)y - g(x) = 0;$ $n \neq 1, \frac{-1}{2}, \frac{5}{2}, \frac{3}{2};$ with $y = \frac{2(n-1)}{2n-1} + k_7(n + x - 1)^{\frac{2n-1}{2}}$, $f(x) = k_4(n-1)^2(k_1 x + k_2(n-1))^{-2}$, and $g(x) = -\frac{k_1 k_4 (n-1)^3}{(3-2n)(k_1 x + k_2(n-1))^2} + \frac{k_5}{(k_1 x + k_2(n-1))^{-5+2n}}$.

is determined by the infinitesimal generators, which constitute the Lie algebra. Thus, knowing the type of Lie algebra we are dealing with enables us to draw additional conclusions about the equation under consideration [26, 27].

The classification of finite-dimensional Lie algebras relies on Levi's theorem, which draws upon two fundamental classes of Lie algebras: solvable and semisimple. These classes encompass specific subclasses, such as the nilpotent Lie algebra within the solvable class. Understanding these classes is crucial for precisely classifying Lie algebras..

The definitions and propositions presented below can be found in [28–31].

Definition 1 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and k an arbitrary field for $\mathfrak{g}$. The coefficients C_{ij}^k are called structure constants if a base is chosen e_j, $1 \leq i \leq n$, in $\mathfrak{g}$ where $n = \dim \mathfrak{g}$ and $[e_i, e_j] = C_{ij}^k e_k$.

Proposition 3 *Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two n-dimensional Lie algebras with bases such that their structure constants are identical. Under this condition, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

We denote $\mathfrak{aff}(2)$ as the solvable 2-dimensional Lie algebra with the basis e_1, e_2, where the commutator of e_1 and e_2 is given by $[e_1, e_2] = e_1$.

Proposition 4 *The Lie algebra associated with the continuous group of Lie symmetries for Eq. (2) is two-dimensional and isomorphic to the solvable Lie algebra $\mathfrak{aff}(\mathbb{R})$.*

Proof Consider the following assignments: $e_1 := \Pi_1$ and $e_2 := -(n-1)\Pi_2$. Verifying that $[e_1, e_2] = e_1$, we can then use Proposition 3 to conclude that the statement is proven.

6 Conclusion

By applying the Lie symmetry group (see Proposition 1 case $II.2.2.2$), we computed the optimal system presented in Proposition 2. With the assistance of these operators, we derived two solutions (see Table 4), notably, these solutions have not yet been documented in the literature. Additionally, we obtained a Lagrangian for the Eq. (2) using the Jacobi multiplier method. This Lagrangian holds significance for exploring the dynamics of this equation from a variational standpoint.

The Lie algebra associated with the Eq. (2) is isomorphic to the Lie algebra $\mathfrak{aff}(\mathbb{R})$. Thus, we have successfully achieved the original objective. In future studies, it is worthwhile to consider the theory of equivalence groups for obtaining preliminary classifications associated with a complete classification of the Eq. (2). Additionally, future investigations could explore nonclassical symmetries.

Acknowledgements Y. Acevedo, G. Loaiza, and O.M.L Duque are grateful to EAFIT University, Colombia, for the financial support in the project "Study and applications of diffusion processes of importance in health and computation" with code 11740052022.

Funding Open Access funding provided by Colombia Consortium

Declarations

Conflict of interest The authors affirm that there are no potential conflicts of financial or personal interest that could have affected the results or interpretations presented in this study.

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