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The direct summand conjecture for some bigenerated extensions and an asymptotic version of Koh's conjecture

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Abstract This article deals with two different problems in commutative algebra. In the first part we give a proof of the direct summand conjecture for module-finite extension rings of mixed characteristic $R \subset S$ satisfying the following hypotheses: the base ring R is a unique factorization domain of mixed characteristic zero. We assume that S is generated by two elements which satisfy, either radical quadratic equations, or general quadratic equations under certain arithmetical restrictions. In the second part of this article we discuss an asymptotic version of Koh's conjecture. We give a model theoretical proof using "non-standard methods".

Keywords Ring extension · Splitting morphism · Discriminant · Ultraproduct · Non-principal ultrafilter

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1 Introduction

The Homological Conjectures have been a focus of research activity since Jean Pierre Serre introduced the theory of multiplicities in the early 1960s (Serre and Gabriel 1975), and since the introduction of characteristic prime methods in commutative algebra by Peskine, Szpiro, and Hochster, in the mid 1970s (Hochster 1975; Peskine and Szpiro 1973). These conjectures relate the homological properties of a commutative rings to certain invariants of the ring structure, as, for instance, its Krull dimension and its depth. They have been settled for equicharacteristic rings (i.e., rings for which the characteristic of the ring coincides with that of its residue field) but many remain open in mixed characteristic (Hochster 2007). Their validity in mixed characteristic, nonetheless, is known for rings of Krull dimension less than four.

Among these conjectures the *Direct Summand Conjecture* occupies a central place, implying, or actually being equivalent, to many of the other conjectures (Hochster 1983; Ohi 1996; Roberts 2002; McCullough 2011).

Direct Summand Conjecture (*DSC*) Let $R \subset S$ be a module-finite extension of noetherian rings (i.e., S, regarded as an R-module is finitely generated) where R is assumed to be regular. Then, the inclusion map $R \subset S$ splits as a map of R-modules. Or equivalently, there is a retraction $\rho : S \to R$ from S into R. By a retraction we mean an R-linear homomorphism satisfying $\rho(1) = 1$. The central role of the DSC in Commutative Algebra as well as its relation to the other homological conjectures is comprehensively explained in Hochster (1975).

The problem of showing the existence of a retraction $\rho : S \to R$ may be reduce to the case where R and S are complete local domains (Hochster 1975). Therefore, one may assume that $(R, \mathfrak{m})$ is in particular a unique factorization domain (UFD) (Eisenbud 1994, page 483). If the Krull dimension of S is d one can always choose a system of parameters for S contained in $\mathfrak{m}$. By a *system of parameters* in an arbitrary local ring $(S, \mathfrak{n})$ of Krull dimension d we mean a sequence of elements $\{x_1, \ldots, x_d\}$ such that the radical of the ideal they generate in S is precisely $\mathfrak{n}$, the unique maximal ideal of S (Eisenbud 1994, page 222). It can be proved that the DSC holds if and only if for any system of parameters in S, and any natural number $t > 0$, the *socle element* $(x_1 \cdots x_d)^t$ is not contained in the ideal in S generated by the $t + 1$ powers of the parameters. That is, if $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})S$. This last statement is known as the *Monomial Conjecture*. More precisely:

Monomial Conjecture (*MC*) Let $(S, \mathfrak{n})$ be any local noetherian ring of dimension d and let $\{x_1, \ldots, x_d\}$ be a system of parameters for S. Then, for any positive integer t, $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})S$. This conjecture is equivalent to the (DSC) (Hochster 1973). In low dimension, that is, for rings of Krull dimension ≤ 2, the DSC (and consequently the MC) follows as a consequence of the existence of a *normalization* for S: by mapping S into its normalization one may assume that S is a *normal* domain, and, for dimensional reasons, a Cohen–Macaulay ring (Eisenbud 1994, pages 118, 420). Then the Auslander–Buchsbaum formula (Eisenbud 1994, page 469) implies that S must have projective dimension zero. That is, S must be an R-free module, and consequently the inclusion map automatically splits.

On the other hand, the general equicharacteristic case, i.e., the case if which R contains a field of zero characteristic, is handled by elementary methods: the trace map form the fraction field of S to the fraction field of R provides a natural retraction. In fact, the inclusion map splits as a map of R-modules under the much weaker hypothesis of R being a normal domain (Hochster 1973, Lemma 2).

If R is equicharacteristic, but contains a field of characteristic $p > 0$, a classical argument given by Hochster (actually, a precursor of his Tight Closure Theory) shows the validity of the MC by a method in which the properties of the iterated powers of the Frobenius map are exploited in a clever way.

It should be remarked that the existence of *Big Cohen Macaulay Modules and Big Cohen Macaulay Algebras* for equicharacteristic rings immediately implies the validity of the (MC) and the (DSC) for rings containing a field (Hochster 2007; Hochster and Huneke 1992).

In the mixed characteristic case it is known that the DSC holds for regular rings R of Krull dimension ≤ 3. The dimension three case was proved quite recently by R. Heitmann, by means of a rather involved combinatorial argument (Heitmann 2002 and Roberts 2002). This is, undoubtedly, one of the most significant advances in commutative algebra of the last decades. As mentioned before, the DSC would follow from the existence of Big Cohen Macaulay Modules or Big Cohen Macaulay Algebras in mixed characteristic, an open problem in dimensions greater than three (Hochster 2007).

A natural generalization of the (DSC) was proposed by J. Koh in his doctoral dissertation (Koh 1983). Koh's question replaces the condition of R being regular for the weaker condition of S having finite projective dimension as a module over R.

Koh's Conjecture Let R be a noetherian ring, and let $R \subset S$ be a module-finite extension such that S, regarded as an R-module, has finite projective dimension. Then there is a retraction from S into R.

It is known that many theorems fail when one weakens the hypothesis of R being regular and replaces it by the condition that R is just noetherian, or even a Gorenstein complete local domain (see Definition 3), and one just imposes the condition that the corresponding R-modules have finite projective dimension (if R is regular this hypothesis is satisfied automatically, due to Serre's Theorem, Eisenbud 1994, Chapter 19). For instance, *the Rigidity of Tor* is no longer true in this context (Heitmann 1993). In a similar manner, the positivity of the intersection multiplicity $\chi_R(M, N)$ for modules M, N of finite projective dimension over R, when R is not regular, is no longer valid (Dutta et al. 1985).
Notwithstanding, Koh's conjecture is true for rings of equal characteristic zero. Unfortunately, it turned out to be false for equicharacteristic rings of prime characteristic as well as in the case of mixed characteristic (Vélez 1995).

The fact that this conjecture is true in characteristic zero suggests that it may be true "asymptotically". By an asymptotic version we mean the following: given any bound $b > 0$ for the "complexity" of the extension, a notion we will define in a precise manner in Sect. 5, the set S_b of prime numbers for which there are counterexamples whose characteristic lie in S_b must be *finite* (Theorem 11). We will prove this asymptotic form for rings that are localization at prime ideals of affine k-algebras, where k is a an algebraically closed field. We achieve this by first formulating *Koh's Conjecture* as a first order sentence in the language of algebraically closed fields. Then we give a proof via *Lefschetz's Principle*. A main reference for the model theoretical methods involved is Schoutens (2010). Also Schoutens (2000) may be consulted for a more succinct account of "nonstandard methods " in commutative algebra.

In the first part of this article we provide a proof of the DSC for module-finite extensions of rings $R \subset S$ satisfying certain conditions. We will assume R to be a UFD., where the most interesting case will be when R is a ring of mixed characteristic zero. On the other hand, S will be a module-finite extension generated as a R-algebra by two elements satisfying, either radical quadratic equations (Theorem 1), or satisfying general quadratic equations, under certain arithmetical restrictions (Theorem 3).

In the second part of this article we discuss an asymptotic version of Koh's Conjecture. We will develop a Model Theoretical approach using "non-standard methods" similar to those developed by Schoutens (2000). The main result of this section is Theorem 11.

All rings will be commutative, with identity element 1, *and all modules will be assumed to be unitary.*

2 The DSC for some radical quadratic extensions

2.1 Some reductions

Let R be a UFD, and let us denote by L its fraction field. Let S be a module-finite extension of R such that S is generated as an R-algebra by two elements s_1 and s_2 that satisfy monic polynomials $f_1(x)$ and $f_2(x)$ in $R[x]$, respectively.

Without loss of generality we may assume that $f_1(x)$ and $f_2(x)$ have degree greater than one. For if one of them, for instance $f_1(x)$, had degree one, then the element s_1 would be already in R. In this case $S = R[s_2]$. By mapping $C = R[x]/(f_2(x))$ onto S we can represent S as a quotient of the form C/J, where J is an ideal of C of height zero. This is because the Krull dimension of S and T is the same and equal to the Krull dimension of R, since both rings are module-finite extensions of R (Kunz 2012, Corollary 2.13, page 47).

Thus, J would be contained in some minimal prime P of C. Since $R[x]$ is also a UFD, P is generated by a monic prime factor $p(x)$ of $f_2(x)$; hence $J \subset (p(x))$. But in order to find a R-retraction from S into R it suffices to find a retraction "further above", $\rho : C/P \to R$. This is because the composition of the canonical map $S = C/J \to C/P$ with ρ would provide a retraction from S into R. But the map $R \to C/P \simeq R[x]/(p(x))$ splits, since $R[x]/(p(x))$ is free as an R-module, and consequently ρ can be taken as the projection onto R.

Let us then assume that the degrees of f_1 and f_2 are greater that one and henceforth S is minimally generated as an R-algebra by $s_1, s_2 \in S$. Set $T = R[x_1, x_2]/I$, with $I = (f_1(x_1),\ f_2(x_2))$, where $f_1(x_1)$ and $f_2(x_2)$ are monic polynomials for s_1 and s_2, respectively. It is easy to see that T is a free R-module because $T \cong R[x_1]/(f_1) \otimes_R R[x_2]/(f_2)$. In fact, an R-basis for T consists of monomials of the form

$$B = \left\{ \overline{x}_1^{d_1} \overline{x}_2^{d_2},\ \text{with } 0 \leq d_i < \deg f_i \right\}. \tag{1}$$

Let $\varphi : T \to S$ be the surjective R-homomorphism defined by sending x_i into s_i. Let J denote its kernel, so that $S \cong T/J$, where the height of J must be zero, and therefore J must be contained in some minimal prime of T.

The next lemma analyzes the case when T is a domain, i.e., when $J = (0)$. In this case the existence of a retraction $\rho : S \to R$ follows automatically, since $S = T$ is free as an R-module.

In what follows i and j denote a pair of indices $1 \le i,\ j \le 2$, with $i \ne j$.

Lemma 1 *Let R, T, f_1, f_2 be as above. Let $E_i = L[x_i]/(f_i)$, and $F_j = E_i[x_j]/(f_j)$. Then T is a domain if and only if both E_i and F_j are fields. That is, if and only if f_i is irreducible in $L[x_i]$ and f_j is irreducible in $E_i[x_j]$.*

Proof First, we observe that $L \otimes_R T \cong L[x_1, x_2]/I \cong E_i[x_j]/(f_j) = F_j$. On the other hand, the natural homomorphism $\mu : T \hookrightarrow L \otimes_R T$ is an injection. This is because T is a torsion free R-module, since R is a domain and T is a free module. Therefore, T is a subring of F_j, and if F_j is a field, T must be a domain. This gives the "only if" part of the lemma.

Conversely, let us assume that T is a domain. Arguing by contradiction let us suppose that either E_i or F_j is not a field. In the first case there are monic polynomials of positive degree, g_1 and g_2 in $L[T_i]$, such that $f_i = g_1 g_2$ with $\deg(g_s) < \deg(f_i)$. Now, let $\alpha \in R \setminus \{0\}$ be a common denominator for the coefficients of g_1 and g_2. The equality $\alpha^2 f_i = (\alpha g_1)(\alpha g_2)$ in $R[x_i]$ implies that αg_i are zerodivisors in T. Besides, $\alpha g_i = \alpha \overline{x}_i^{\deg(g_i)} + \cdots$ cannot be zero because g_i, written in the R-basis (1) of T has at least one coefficient different from zero. Therefore, T would not be a domain, a contradiction. Then we may assume that E_i is a field.

On the other hand, if f_j were reducible over $E_i[x_j]$ we could write $f_j = h_1 h_2$, where $h_1, h_2 \in E_i[x_j]$ are monic polynomials of degree less than $\deg(f_j)$. Let us choose $\widetilde{h}_1, \widetilde{h}_2$ in $L[x_1, x_2]$ such that $\psi(\widetilde{h}_s) = h_s$, $s = 1, 2$, where $\psi : L[x_i, x_j] \to E_i[x_j]$ is the natural homomorphism induced by the projection map $L[x_i] \to E_i$. In fact, we can choose each $\widetilde{h}_s$, considered as a polynomial in $(L[x_i])[x_j]$, such that each of its coefficients in $L[x_i]$ is a polynomial in x_i with degree less than $\deg(f_i)$. Hence, there exists $\widetilde{h}_3$ in $L[x_1, x_2]$ such that $f_j - \widetilde{h}_1 \widetilde{h}_2 = \widetilde{h}_3 f_i$. Choose any nonzero element $c \in R$ such that $c\widetilde{h}_r \in R[x_1, x_2]$, for $r = 1, 2, 3$. Then we have that $c\widetilde{h}_1 c\widetilde{h}_2 = c^2 f_j - c(c\widetilde{h}_3) f_i \in I$ and consequently the classes of $c\widetilde{h}_1$ and $c\widetilde{h}_2$ in T must be different from zero. Thus, T would not be a domain, which is a contradiction. This proves f_j must be irreducible over $E_i[x_j]$. □

Corollary 1 *Let R be a UFD where its field of fractions L has characteristic different from two. Assume that $f_i = x_i^2 - a_i$ are irreducible polynomials in $L[x_i]$, $i = 1, 2$. If $T = R[x_1, x_2]/(f_1, f_2)$ is not a domain then there exist nonzero elements c, d, u in R such that $a_1 = d^2 u$, $a_2 = c^2 u$, where c, d are relatively prime.*

Proof Since T is not a domain, by Lemma 1 we may assume without loss of generality that one of the polynomials f_i, for instance $f_2(x_2)$, is reducible in $E_1[x_2]$. But this is equivalent to saying that $f_2(x_2)$ has a root $e \in E_1$, that we may write as $e = e_1 + e_2\overline{x_1}$, where $e_1, e_2 \in L$, and $\overline{x_1}^2 = a_1$. Hence

$$a_2 = e^2 = (e_1^2 + e_2^2 a_1) + 2e_1 e_2 \overline{x_1}.$$

Then $a_2 = e_1^2 + e_2^2 a_1$ and $2e_1e_2 = 0$. But $\mathrm{char}(L) \neq 2$ implies $e_1e_2 = 0$. If $e_2 = 0$ then $a_2 = e_1^2$ and therefore $f_2 = (x_2 + e_1)(x_2 - e_1)$, which is a contradiction. Thus, $e_1 = 0$ and $a_2 = e_2^2 a_1$.

Now write $e_2 = e/d$, where $c, d \neq 0$ are relatively prime elements in R. So $d^2a_2 = e^2a_1$; but d^2 does not divide c^2 and consequently d^2 divides a_1. Hence, there is $u \in R$ such that $a_1 = d^2u$. Replacing a_1 in $d^2a_2 = e^2a_1$ gives the equation $d^2a_2 = c^2d^2u$. After dividing by $d^2 \neq 0$ we obtain $a_2 = c^2u$, which proves the corollary. □

Lemma 2 *Let R be a UFD, let $B = R[x, y]$, and let u, c, d be elements in R different from zero. Define $f_1 = x^2 - d^2u$, $f_2 = y^2 - c^2u$, and set $I = (f_1, f_2)$, the ideal in B generated by f_1 and f_2. Assume that $\{c, d\}$ is a regular sequence in R* (Eisenbud 1994, page 173). *Then the minimal prime ideals of I are $P_r = (f_1, f_2, f_{3,r}, f_{4,r})$, where $f_{3,r} = dy + (-1)^r cx$, $f_{4,r} = xy + (-1)^r cdu$, $r = 0, 1$.*

Proof Let us introduce a new variable z, and let $g(z) = z^2 - u$ in $R[z]$. Clearly g has no roots in R, since f_1 is irreducible. Therefore, g is also irreducible and consequently the ideal (g) is prime in $R[z]$; this is because R is a UFD. Define $\psi_r : B \to R[z]/(g)$ as the unique R-homomorphism that sends x into $d\overline{z}$ and y into $(-1)^{r+1}c\overline{z}$, where $\overline{z}$ denotes the class of z in $R[z]/(g)$. We prove that $\ker(\psi_r) = P_r$.

First, we observe that $P_r \subset \ker(\psi_r)$, since:

$$\begin{aligned}
\psi_r(f_1) &= d^2\overline{z}^2 - d^2u = d^2u - d^2u = 0,\\
\psi_r(f_2) &= c^2\overline{z}^2 - c^2u = 0,\\
\psi_r(f_{3,r}) &= \psi_r(dy + (-1)^r cx) = d(-1)^{r+1}c\overline{z} + (-1)^r cd\overline{z} = 0,\\
\psi_r(f_{4,r}) &= \psi_r(xy + (-1)^r cdu) = (-1)^{r+1}cd\overline{z}^2 + (-1)^r cdu = 0.
\end{aligned}$$

To prove that $\ker(\psi_r) \subset P_r$ we use the well known fact that in a polynomial ring in one variable with coefficients in a commutative ring the ordinary division algorithm holds, as long as the divisor is a monic polynomial. This justifies the following procedure:

Let $h(x, y)$ be a polynomial in $\ker(\psi_r)$. If we regard $h(x, y)$ as a polynomial in the variable y with coefficients in $R[x]$ then, dividing by $f_1 = x^2 - d^2u$ we can write $h(x, y)$ as

$$h(x, y) = f_1q(x, y) + q_1(y)x + q_0(y), \tag{2}$$

where $q(x, y) \in B$ and $q_0(y), q_1(y) \in R[y]$.

Similarly, after dividing $q_0(y)$ by $f_2 = y^2 - c^2u$ we may find $q_2(y) \in R[y]$, and a_1, a_2 elements in R such that

$$q_0(y) = f_2q_2(y) + (a_1y + a_2). \tag{3}$$

Similarly, there exists polynomials $q_3(y)$, $q_4(y)$ in $R[y]$, and $b_1, b_2, \in R$ such that

$$q_1(y)x = f_{4,r}q_3(y) + q_4(y) + b_1x + b_2. \tag{4}$$

Dividing $q_4(y)$ by f_2 we obtain:

$$q_4(y) = q_5(y) f_2 + e_1 y + e_2, \tag{5}$$

for certain polynomial $q_5(y)$, and certain elements e_1, e_2 in R.

Replacing Eqs. (3, 4) and (5) in (2) we can write $h(x, y)$ as

$$h(x, y) = f_1 q(x, y) + f_{4,r} q_3(y) + (q_5(y) + q_2(y)) f_2 + l(x, y),$$

where $l(x, y)$ is the linear polynomial

$$l(x, y) = v_1 y + b_1 x + v_2,$$

where $v_1 = e_1 + a_1$ and $v_2 = e_2 + a_2 + b_2$. From the condition $\psi_r(h) = 0$ we get:

$$\psi_r(h) = \psi_r(l) = v_1(-1)^{r+1} c\bar{z} + b_1 d\bar{z} + v_2 = 0.$$

Since $R[z]/(g)$ is a R-free module with basis $\{1, \bar{z}\}$ we then must have:

$$(-1)^{r+1} v_1 c + b_1 d = 0 \quad \text{and} \quad v_2 = 0. \tag{6}$$

But $\{c, d\}$ is a regular sequence in R, hence there must be some $u \in R$ such that $b_1 = uc$. Since R is a domain we obtain from (6) that $(-1)^r d = v_1$, and consequently

$$l(x, y) = v_1 y + b_1 x = u((-1)^r dy + cx) = u(-1)^r f_{3,r}.$$

This shows $h \in P_r$.

Clearly P_0 and P_1 are prime ideal of B because B/P_r is an integral domain isomorphic to $R[z]/(g)$.

Finally, let us show that P_0 and P_1 are the only minimal primes of I. Let Q be any other minimal prime over I. Then

$$c^2 f_1 - d^2 f_2 = -(dy - cx)(cx + dy) \in Q,$$

so $f_{3,1} = dy - cx \in Q$, or $f_{3,0} = cx + dy \in Q$. □

Proof Let us suppose $f_{3,1} \in Q$. Hence

$$-x(cx - dy) = dxy - cx^2 = dxy - cd^2 u = d(xy - cdu) \in Q.$$

Similarly,

$$y(cx - dy) = cxy - dy^2 = cxy - dc^2 u = c(xy - cdu) \in Q.$$

We claim that $f_{4,1} = xy - cdu$ is an element of Q. If we suppose otherwise, the elements c, d would be contained in Q. Then we would have that $x^2 = (f_1 + d^2 u)$

and $y^2 = f_2 + c^2u$ would also be contained in Q. Consequently, x, y would also be elements of Q. Henceforth, the ideal $(c, d, x, y)B$ would be contained in Q. But each generator of P_1 is in $(c, d, x, y)B$ and consequently $P_1 \subset Q$. But this must be a proper inclusion, since $x \notin P_1$ (this is because every monomial containing x in each one of the generator of P_1 is either quadratic, x^2 or xy, or linear of the form cx, but $x \notin (x^2, xy, cx)B$, since $1 \notin (x, y, c)B$). This contradicts the minimality of Q, and proves the claim.

Thus Q must contain $f_1, f_2, f_{3,1}, f_{4,1}$ and consequently $P_1 = Q$.

The second case, when $f_{3,0} = cx + dy \in Q$, can be treated in a similar fashion. When this occurs we obtain that $P_0 = Q$. □

Theorem 1 *Let $R \subset S$ be a module-finite extension of noetherian rings such that R is a UFD and such that the characteristic of the fraction field L of R is different from two. Suppose S is generated as an R-algebra by two elements $s_1, s_2 \in S$ satisfying monic radical quadratic polynomials $f_1 = x_1^2 - a_1$ and $f_2 = x_2^2 - a_2$, respectively. Then $R \subset S$ splits.*

Proof By the Going Up (Eisenbud 1994, page 129) there exists a prime ideal $Q \subset S$ that contracts to zero in R. Therefore, we may replace S by S/Q without altering the relevant hypothesis. As observed in Sect. 2.1, it suffices to find a retraction from S/Q into R. Hence, we may reduce to the case where S is a domain.

As observed in that same section we can write S as a quotient T/J, where $T = R[x_1, x_2]/I$, $I = (f_1, f_2)$, and $J \subset T$ is an ideal of height zero. Moreover, we can assume each f_i to be irreducible in $\in R[x_i]$, otherwise, as noticed in that section, the splitting of $R \subset S$ would immediately follow from the fact that S would be a free R-module.

On the other hand, if T is a domain then $J = 0$, and so $S = T$ would be a free R-free (of rank four) and $R \subset S$ would also split. If on the contrary, T is not a domain, then by Corollary 1 there exist c, d, u nonzero elements of R such that $a_1 = d^2u$, $a_2 = c^2u$, where c, d are relatively prime. This implies that $\{c, d\}$ is a regular sequence in R. Then Lemma 2 implies that the minimal primes of T are precisely P_0/I and P_1/I.Then $J \subset P_r/I$, for some $r = 0, 1$. Hence, if $\alpha : T/J \to T/(P_r/I)$ is the natural map and $\psi'_r : T/(P_r/I) \to R[z]/(z^2 - u)$ is the R-isomorphism induced by ψ_r (as in the proof of Lemma 2) and $\pi_1 : R[z]/(z^2 - u) \to R$ is the R-module projection on the first component, then a retraction for $R \subset S$ is given by $\rho = \pi_1 \circ \psi'_r \circ \alpha$. This proves the corollary. □

Theorem 2 *Let $R \subset S$ be a module-finite extension of noetherian rings such that R is a UFD, and such that the characteristic of the fraction field L of R is different from two. Let us assume that the element 2 is a unit in R, and that S is generated as an R-algebra by two elements $s_1, s_2 \in S$ satisfying monic quadratic polynomials $f_1 = x^2 - ax + b$ and $f_2 = y^2 - cy + d$, respectively. Then $R \subset S$ splits.*

Proof We can reduce to the radical quadratic case, as in Theorem 1, by "completing the squares". That is, define rings

$$T' = R[u, v]/(u^2 + b - a^2/4, v^2 + c - d^2/4)$$

and

$$T = R[x, y]/(x^2 - ax + b, y^2 - cy + d).$$

Let $\psi : T' \to T$ be the linear isomorphism that sends the variable u to $x - a/2$ and the variable v to $y - c/2$.

We already know that S is isomorphic to a quotient of T hence it is also isomorphic to a quotient of T'. Then, we may choose new generators of S as R-algebra satisfying monic radical quadratic equations: namely, the classes of $\overline{u}$ and $\overline{v}$. Thus, by Theorem 1, the extension $R \subset S$ must splits. □

3 The DSC for some nonradical quadratic extensions

Our next goal is to prove the following result.

Theorem 3 *Let R be a UFD such that such that the characteristic of the fraction field L of R is different from two. Let $R \subset S$ be a module-finite extension such that S is minimally generated as an R-algebra by elements $s_1, s_2 \in S$. Let us assume $f(s_1) = g(s_2) = 0$, where $f(x) = x^2 - ax + b$ and $g(y) = y^2 - cy + d$, for some $a, b, c, d \in R$. If* $\gcd(2, c) = 1$ *and $a^2 - 4b$ is square free, then $R \subset S$ splits.*

In order to prove this theorem we need the following lemma:

Lemma 3 *Let R be a UFD such that the characteristic of the fraction field L of R is different from two. Let $T = R[x, y]/(f(x), g(y))$, where $f(x) = x^2 - ax + b$ and $g(y) = y^2 - cy + d$, for some $a, b, c, d \in R$. Suppose that* $\gcd(2, c) = 1$, *that the discriminant of $f(x)$, $a^2 - 4b \neq 0$, is square free in R and that $f(x)$ is irreducible. If T is not a domain, then there exists $e \in R$ such that $(c \pm ae)/2 \in R$. In this case the minimal primes of T are $P_1 = (\overline{h}_1)$ and $P_2 = (\overline{h}_2)$, where $\overline{h}_1$ and $\overline{h}_2$ are the classes in T of the polynomials $h_1(x, y) = y - ex - (c - ae)/2$ and $h_2(x, y) = y - ex - (c + ae)/2$.*

Proof If we assume that T is not a domain, then, by Lemma 1, g must be reducible in $E[y]$, where E denotes the field $L[x]/(f(x))$. It is clear that E, as a field, is isomorphic to the extension field $L(u^{1/2})$, for $u = a^2 - 4b$. Therefore, $g(y)$ has a root $\gamma = \alpha + \beta u^{1/2}$ in E, since $T \cong E[y]/(g(x))$ is not a domain. But one can verify directly that the conjugate $\overline{\gamma} = \alpha - \beta u^{1/2}$ is also a root of g. Thus, $g = (y - \gamma)(y - \overline{\gamma})$. By comparing coefficients we get $\alpha^2 - \beta^2 u = d$ and $c = 2\alpha$. Hence, $4d = c^2 - 4\beta^2 u$.

Let us write $\beta = q/r$, for $q, r \in R$ such that $\gcd(q, r) = 1$. From $4d = c^2 - 4\beta^2 u$ we obtain that $4r^2 d = r^2 c^2 - 4q^2 u$. Then, $4(r^2 d + q^2 u) = r^2 c^2$. This implies that $4 \mid r^2 c^2$; but $\gcd(2, c) = 1$, therefore $4 \mid r^2$, and so $2 \mid r$. Write $r = 2t$, for some $t \in R \setminus \{0\}$. Thus,

$$4(r^2 d + q^2 u) = 4t^2 c^2 \tag{7}$$

After dividing by 4 Eq. (7) we obtain: $4t^2 d + q^2 u = t^2 c^2$. Or, equivalently, $t^2(c^2 - 4d) = q^2 u$. From this, it follows that $t^2 \mid q^2 u$, which in turn implies $t^2 \mid u$, because

$\gcd(t, q) = 1$. But u is square free, therefore t must be a unit. Then we may assume that $q/t \in R$. If we let $e = q/t$, then the equation $t^2(c^2 - 4d) = q^2 u$ can be rewritten as:

$$c^2 - 4d = e^2(a^2 - 4b). \tag{8}$$

We will now prove that $2 \mid (c \pm ae)$. In fact, suppose that $2 = \prod p_i^{n_i}$, and that $c + ae = \prod p_i^{m_i}$ and $c - ae = \prod p_i^{k_i}$ are factorizations into powers of prime elements. By allowing some exponents to be zero we may assume each product involves the same primes. We shall see that $n_i \le \min(m_i, k_i)$, for all i. From (8) it follows that

$$((c - ea)/2)((c + ea)/2) = d - e^2 b \in R.$$

This implies that $2n_i \le m_i + k_i$, since $(c - ea)(c + ea)/2^2$ belongs to R. Arguing by contradiction, suppose there is j such that $n_j > \min(m_j, k_j)$. Without loss of generality, we may assume $m_j = \min(m_j, k_j)$. Hence, $n_j \le k_j$, otherwise $2n_j > m_j + k_j$, which is a contradiction. Therefore, $p_j^{n_j} \mid (c - ae)$, and consequently

$$p_j^{n_j} \mid (c - ae) + 2ae = c + ae$$

which means that $n_j \le m_j$. Thus, $n_j \le \min(m_j, k_j)$. Summarizing, $n_i \le \min(m_i, k_i)$ for all i. Hence, $2 \mid (c \pm ae)$.

Now, let us see that $P_1 = (\overline{h}_1)$ and $P_2 = (\overline{h}_2)$ are the minimal primes of T. Using the fact that $((c - ea)/2)((c + ea)/2) = d - e^2 b$. We see by a direct computation that

$$h_1 h_2 = e^2 f + g. \tag{9}$$

Hence, any minimal prime in $T = R[x, y]/(f, g)$ must contain either $\overline{h}_1$ or $\overline{h}_2$. On the other hand, $R[x, y]/(f, g, h_1) \cong R[x]/(f)$, since we can eliminate the variable y using $h_1(x, y) = y - ex - (c - ae)/2$, by sending y into $ex + (c - ae)/2$. In a similar fashion we see that $R[x, y]/(f, g, h_2) \simeq R[x]/(f)$. Since $R[x]/(f(x))$ is a domain, $P_1 = (\overline{h}_1)$ and $P_2 = (\overline{h}_2)$ must be prime ideals of T and since each minimal prime of T must contain either $\overline{h}_1$ or $\overline{h}_2$, then these must be the only minimal primes. □

Now, we are ready to prove Theorem 3.

Proof As we observed in Sect. 2.1 we may assume S is minimally generated as an R-algebra by certain elements s_1 and s_2 whose corresponding monic polynomials $f(x)$ and $g(y)$ over R have degree greater than one. As we observed in that same section, this implies that $f(x)$ is irreducible.

We know that S can be represented as a quotient of the form T/J, where $T = R[x, y]/(f(x), g(y))$ and $J \subset T$ is an ideal of T of height zero. By Lemma 3, $J \subset P_1 = (\overline{h}_1)$ or $J \subset P_2 = (\overline{h}_2)$, the minimal prime ideals of T defined in Lemma 3, with $h_1(x, y) = y - ex - (c - ae)/2$ and $h_2(x, y) = y - ex - (c + ae)/2$.

We can obtain the desired retraction $\rho : S \to R$ as the composition of the following natural chain of R-homomorphisms:

$$S = T/J \to T/P_j \overset{\varphi}{\to} R[x]/(f(x)) \to R \oplus R\overline{x} \overset{\pi_1}{\to} R,$$

where φ is the R-homomorphism defined by defining x into x and y into $h_j - y$, and π_1 is canonical projection on R. □

4 An asymptotic form of Koh's conjecture

In this second part of this article we give an *asymptotic* formulation of Koh's Conjecture as well as its proof. Let us recall that this conjecture states that if R is a Noetherian ring and that if $R \subset S$ a module-finite extension of rings such that the projective dimension of S as an R-module is finite, then there exists a retraction $\rho : S \to R$.

The asymptotic form of Koh's conjecture is the following: given any bound $b > 0$ for the "complexity" of the extension (see Definition 1) the set S_b of prime numbers for which there are counterexamples whose characteristic lie in S_b must be *finite*. We will prove this asymptotic form for rings that are localization at prime ideals of affine k-algebras, where k is a an algebraically closed field. We refer to such rings by the shorter name of *local k-algebras.*

4.1 Lefschetz's principle

Let us begin by recalling Lefschetz's principle:

Theorem 4 (Lefschetz's principle) *Let ϕ be a sentence in the language of rings. The following statements are equivalent.*

1. *The sentence ϕ is true in an algebraic closed field of characteristic zero.*
2. *There exists a natural number m such that for every $p > m$, ϕ is true for every algebraically closed field of characteristic p.*

Proof See Marker (2002, Corollary 2.2.10, page 42). □

We state without proof *the Compactness Theorem*. This theorem guarantees the existence of a model for a $\mathcal{L}$-theory T (we say in this case that T is *satisfiable*) if and only if there exists a model for each finite subset of T.

Theorem 5 (Compactness Theorem) *Suppose T is a $\mathcal{L}$-theory. Then, T is satisfiable if and only if every finite subset of T is satisfiable.*

As a consequence of the Compactness Theorem one can readily deduce the following proposition (Marker 2002, page 42).

Proposition 1 *Let ϕ be a first order sentence which is true in every field k of characteristic zero. Then, there exists a prime number p_0 such that ϕ is true in each field F of characteristic q, for $q > p_0$.*

5 Codes for polynomial rings and modules

Throughout this discussion we will fix a field k and a monomial order in the polynomial ring $A = k[x_1, \dots, x_n]$.

Definition 1 Let R be a finitely generated k-algebra, and let I be an ideal of $k[x_1, \dots, x_n]$. We will say that:

1. The ideal I has **complexity at most** d, if $n \leq d$ and it is possible to choose generators for I, $f_1, \dots, f_s$, with $\deg f_i \leq d$, for $i = 1, \dots, s$.
2. We say R has **complexity at most** d if there is a presentation of R as $k[x_1, \dots, x_n]/I$, with I of complexity at most d.
3. If $J \subset R$ is an ideal, we will say that J **has complexity at most** d, if R has complexity less than or equal to d, and there exists a lifting of J in $k[x_1, \dots, x_n]$, let us say J', with complexity at most d.
4. If R is a local k-algebra, we say it has **complexity at most** d if R can be written as $R = (k[x_1, \dots, x_n]/I)_p$, for some prime ideal $p \subset k[x_1, \dots, x_n]/I$ such that the complexity of R and p is at most d.
5. If M is any finitely generated R-module, we will say that M has **complexity at most** d if R is a k-algebra of complexity at most d, and there exists an exact sequence $R^t \xrightarrow{\Gamma} R^s \to M \to 0$, with $s, t \leq d$, where all the entries of the matrix Γ are polynomials (or quotients of polynomials, in the local case) with degree at most d.
6. Let $M \subset R^d$ be R-submodule. We will say that M has **degree type at most** d (written as $gt(M) \leq d$) if the complexity of R is at most d, and M is generated by d-tuples with all its entries of degree at most d. If M is a finitely generated R-module, we will say that M has **complexity degree at most** d if there exist submodules $N_2 \subset N_1 \subset R^d$, both of degree type at most d, such that $M \cong N_1/N_2$.

Now, for any polynomial $f \in A$ we will denote by a_f the tuple of all the coefficients of f listed according to the fixed order. When the complexity of an ideal I is at most d, and $I = (f_1, \dots, f_s)$, then I can be encoded by the tuple a_I that consists of all the coefficients of the polynomials f_i. It is not difficult to see that the length of this tuple only depends on d (Schoutens 2000). On the other hand, given one of those tuples a we can always reconstruct the ideal where it comes from, an ideal we shall denote by $\mathcal{I}(a)$. Similarly, if R is a k-algebra with complexity at most d then R can be written as $k[x_1, \dots, x_n]/\mathcal{I}(a)$. We will write this fact as $R = \mathcal{R}(a)$.

Let M be an R-module. If the complexity degree of M is at most d then the minimal number of generator for M is bounded in function of d. Hence M can be encoded by a tuple $v = (n_1, n_2)$, where n_1 is a code for N_1 and n_2 is a code for N_2. We will write this as $M \cong \mathcal{M}(v)$ (see Schoutens 2000).

Finally, if $\phi(\xi)$ is a formula with free variable ξ and parameters from a ring R, then by $a \in |\phi|_R$ we will mean $R \models \phi(a)$ (Marker 2002, Definition 1.1.6).

The proof of the following theorem may be found in Schoutens (2000, Remark 2.3), and Schoutens (2010, Theorem 4.4.1, page 59).

Theorem 6 *Given $d > 0$, there exists a formula $IdMem_d$ (Ideal Membership) such that for any field k, any ideal $I \subset k[x_1, \dots, x_n]$, and any k-algebra R, both of* *complexity at most d over k, it holds that $f \in IR$ if an only if $k \models IdMem_d(a_f, a_I)$. Here a_f, and a_I denote codes for f and I, respectively.*

Remark 1 1. Using Corollary 6, it is easy to get for each d formulas Inc_d (Inclusion) and Equal_d (Equality) such that if R is a finitely generated k-algebra with complexity at most d, and if J and I are ideals of R with complexity less than d, then $(a_{I,}a_J) \in |\mathrm{Inc}_d|_K$ (resp. $(a_{I,}a_J) \in \left|\mathrm{Equal}_d\right|_K$) if and only if I is included in J, $I \subset J$ (resp. $I = J$).

2. Given $d, n > 0$ there exists a formula $\mathrm{MaxIdeal}_{d,n}$ such that for any algebraic closed field k and any ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ of complexity at most d we have: $\mathfrak{m}$ is a maximal ideal if and only if $k \models \mathrm{MaxIdeal}_{d,n}(a_m)$, where a_m is a code for $\mathfrak{m}$. In fact, by the Nullstellensatz $\mathfrak{m}$ is maximal if and only if there exist $b_1, \ldots, b_n \in k$ such that $\mathfrak{m} = (x_1 - b_1, \ldots, x_n - b_n)$. Let us call $J = (x_1 - b_1, \ldots, x_n - b_n)$. Then, the required formula is:

$$\mathrm{MaxIdeal}(\xi) = (\exists b_1, \ldots, b_n)(\mathrm{Equal}_d(\xi, a_J)),$$

where ξ and a_J must be replaced by the codes a_m of $\mathfrak{m}$, and a_J of J, respectively.

We will also need the following lemma:

Lemma 4 (Schoutens 2000, Lemma 3.2) *For each $d > 0$ there is a bound $D = D(d)$ with the following property: let T be a local k-algebra of complexity at most d. Let M and M' be submodules of T^d of degree type at most d. Then the degree type of $(M :_T M') = \{t \in T : tM' \subset M\}$ is bounded by D.*

In particular, if T is a local k-algebra of complexity at most d and $J \subset T$ is an ideal of complexity at most d then $\mathrm{Ann}_T J$ must have complexity at most $D = D(d)$, a bound that only depends on d.

5.1 Proof of the asymptotic form of Koh's conjecture

In this section we give an *non standard* proof of the asymptotic version of Koh's conjecture. We start by defining the *complexity of a ring extension* (d will denote a positive integer).

Definition 2 Let $R \subset S$ be a module-finite extension of local k-algebras. We say this extension has complexity $\leq d$ if:

1. The *complexity* over k of the local k-algebras R and S is at most d.
2. The minimal number of generators of S as an R-module is at most d.
3. The projective dimension of S as an R-module is less than or equal to d.

We intend to prove the following: given $d > 0$, there exists a prime p_d such that for any algebraically closed field k of characteristic $p > p_d$, and any modulo-finite extension $R \subset S$ of local k-algebras of complexity less than d, there is a retraction $\rho : S \to R$ and consequently $R \subset S$ splits.

We start by recalling the following definition.

Definition 3 A local ring $(R, \mathfrak{m})$ is called Gorenstein if R is Cohen Macaulay (CM), and if for any system of parameters $\{x_1, \ldots, x_d\}$ in R the socle of $\overline{R} = R/(x_1, \ldots, x_d)$, defined as $\mathrm{Ann}_{\overline{R}}(\mathfrak{m})$, is a 1 -dimensional $R/\mathfrak{m}$-vector space (Eisenbud 1994, page 526).

The following result will be of fundamental importance (Velez and Florez 2000).

Proposition 2 *Let $(R, \mathfrak{m}) \subset (T, \mathfrak{n})$ be a module-finite extension of local rings with T a free R-module. If $T/\mathfrak{m}T$ is Gorenstein, and if $J \subset T$ is any ideal, then there exists a retraction $\rho : T/J \to R$ if and only if $\mathrm{Ann}_T(J) \nsubseteq \mathfrak{m}T$.*

We also need the following (Vélez 1995).

Theorem 7 (Koh in characteristic zero) *Let R be a ring containing a field of characteristic zero, and let $R \subset S$ be a module-finite extension of rings such that the projective dimension of S as an R-module is finite. Then, there exists a retraction $\rho : S \to R$.*

Remark 2 Let $(R, \mathfrak{m})$ be a local ring and let $R \subset S$ be a module-finite extension. Let us take $s_1, \ldots, s_n \in S$ generators of S as an R-algebra. For each $s_i \in S$ choose an arbitrary monic polynomial with coefficients in R, $f_i(t) = x_i^{d_i} + r_{i1}x_i^{d_i-1} + \cdots + r_{id_i}$, that each s_i satisfies. Let T denote the quotient ring $R[x_1, \ldots, x_n]/(f_1(x_1), \ldots, f_n(x_n))$. As in Sect. 2.1, we may represent S as a quotient of T by defining a surjective R-homomorphism $\phi : T \to S$, sending the class of x_i into s_i. If J denotes its kernel then $\mathrm{ht}(J) = 0$, as showed in that same section.

The representation of S as the quotient T/J makes it possible to give a very useful criterion for the existence of a retraction. Let $(R, \mathfrak{m})$ be a local ring and let $R \subset S$ be a module-finite extension. Then, the inclusion map $R \subset T/J$ splits if and only if $\mathrm{Ann}_T(J)$ is not contained in $\mathfrak{m}T$. This follows immediately from Proposition 2.

The following theorem states that given $i \geq 0$ and $d > 0$ there exists a formula $(\mathrm{Tor}_i)_d$ such that for any k-algebra R and R-modules M, N, V, all of complexity at most d, then $\mathrm{Tor}_i^R(M, N) \cong V$ if and only if $(\mathrm{Tor}_i)_d$ evaluated in codes of R, M, N and V is true over k (analogously for Ext).

Theorem 8 *Given $i \geq 0, d > 0$, there exist formulas $(\mathrm{Tor}_i)_d$ and $(\mathrm{Ext}^i)_d$ with the following properties: let k be any field; then, if a tuple (a, m, n, v) is in $|(\mathrm{Tor}_i)_d|_k$ (respectively, in $|(\mathrm{Ext}^i)_d|_k$), then $\mathcal{M}(v)$ is isomorphic to*

$$\mathrm{Tor}_i^{\mathcal{A}(a)}(\mathcal{M}(m), \mathcal{M}(n))$$

(respectively to $\mathrm{Ext}^i_{\mathcal{A}(a)}(\mathcal{M}(m), \mathcal{M}(n))$). Moreover, for each tuple (a, m, n) we can find at least one v such that (a, m, n, v) belongs to $|(\mathrm{Tor}_i)_d|_k$ (respectively, to $|(\mathrm{Ext}^i)_d|_k$).

Proof See Schoutens (2000, Corollary 4.4, page 150). □

We recall the following standard result (Eisenbud 1994, page 167, Theorem 6.8).

Theorem 9 *Let* $(R, \mathfrak{m})$ *be a local ring, and denote by* k *the residue field* $R/\mathfrak{m}$. *If* M *is a finitely generated* R*-module, then* $pd_R(M) \leq n$ *if and only if* $\mathrm{Tor}^R_{n+1}(M, k) = 0$.

Remark 3 It is clear from the previous theorems that there exists a formula $(\mathrm{pd}_{<n})_d$ such that, if $M = \mathcal{M}(v)$ is an $R = \mathcal{R}(a)$-module with complexity less than d, where $(R, \mathfrak{m})$ is a local k-algebra with complexity less than d, then $k \models (\mathrm{pd}_{<n})_d(a, v)$, if and only if, $\mathrm{pd}_R(M) \leq n$.

From this preliminaries we obtain the following main result:

Theorem 10 *For each* $d > 0$ *there exists a first order formula* Koh_d *such that if* $R \subset S$ *is a module-finite extension of local* k*-algebras such that the complexity of this extension is at most* d*, then there exists a retraction* $\rho : S \to R$ *if and only if* $k \models \mathrm{Koh}_d(a, b)$, *where* $R \cong \mathcal{R}(a)$ *and* $S \cong \mathcal{S}(b)$.

Proof As shown in Remark 2 we can represent S as $S \cong T/J$, where the complexity of J and T is less than d. We know there is a retraction $\rho : S \to R$ if and only if $\mathrm{Ann}_T(J) \nsubseteq mT$.
As proved in Remark 3, there exists a formula $(\mathrm{pd}_{<n})_d$ such that if $R = \mathcal{R}(a)$ and $S = \mathcal{R}(b)$ then $k \models \mathrm{pd}_{<n}(a, b)$ if and only if $\mathrm{pd}_R(S) < n$.
Let Koh_d be the formula which establishes the following: if $\mathrm{pd}_R(S) < d$, then $\mathrm{Ann}_T(J) \nsubseteq mT$. Explicitly:

$$\mathrm{Koh}_d(\xi, \xi', \nu, \nu') : \bigvee_{i=0}^{d-1} Pd_R(\xi, \nu) = i \Longrightarrow \neg Inc_d(\nu', \xi').$$

Here ν' and ξ' are reserved for a code of $\mathrm{Ann}_T(J)$ and $\mathfrak{m}T$, respectively. Then, it is clear that $k \models \mathrm{Koh}_d(a, a', b, b')$ if and only if there exist a retraction $\rho : S \to R$. □

Theorem 11 *Let* $R \subset S$ *be module-finite extensions of local* k*-algebras. Fix* $d > 0$, *an arbitrary positive integer. The set of prime numbers* p *for which there are counterexamples to Koh's Conjecture of complexity less than d is finite.*

Proof From Theorem 7 we see that $K \models \mathrm{Koh}_d(a, b)$ for any field K of characteristic zero. Then, by Proposition 1, we deduce that $k \models \mathrm{Koh}_d(a, b)$, for every field k of prime characteristic p sufficiently large. More precisely: given $d > 0$, there exists a prime number p_d such that for any field k of characteristic $p > p_d$, and any modulo-finite extension $R \subset S$ of local k-algebras with complexity at most d there exists a retraction $\rho : S \to R$. □

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References

Dutta, S.P., Hochster, M., McLaughlin: Modules of Finite Projective Dimension with Negative Intersection Multiplicities. Inventiones Mathematicae. Springer, Berlin (1985)

Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Mathematics, vol. 150. Springer, New York (1994)
Evans, E.G., Griffith, P.: The Syzygy problem. Ann. Math. **114**(2), 323–333 (1981)
Heitmann, R.C.: A Counterexample to the rigidity conjecture for rings. Bull. Am. Math. Soc. **29**(1), 94–97 (1993)
Heitmann, R.C.: The direct summand conjecture in dimension three. Ann. Math. **156**(2), 695–712 (2002)
Hochster, M.: Contracted ideals from integral extensions of regular rings. Nagoya Math. J. **51**, 25–43 (1973)
Hochster, M.: Topics in the Homological Theory of Modules over Commutative Rings. CBMS Regional Conference Series in Mathematics, vol. 24. American Mathematical Society, Providence (1975)
Hochster, M.: Canonical elements in local cohomology modules and the direct summand conjecture. J. Algebra **84**(2), 503–553 (1983)
Hochster, M.: Homological conjectures, old and new. Ill. J. Math. **51**(1), 151–169 (2007)
Hochster, M., Huneke, C.: Infinite integral extensions and big Cohen–Macaulay algebras. Ann. Math. **135**(1), 53–89 (1992)
Koh, J.H.: The direct summand conjecture and behavior of codimension in graded extensions. Doctoral dissertation, Ph.D. Thesis, University of Michigan (1983)
Kunz, E.: Introduction to Commutative Algebra and Algebraic Geometry. Springer Science & Business Media, Berlin (2012)
Marker, D.: Model Theory: An Introduction. Graduate texts in Mathematics, vol. 217. Springer, New York (2002)
McCullough, J.: On the strong direct summand conjecture. Ph.D. Thesis, University of Illinois, Urban-Champaigne 2009 (2011). http://math.ucr.edu/~jmccullo/talks/09Urbana
Ohi, T.: Direct summand conjecture and descent for flatness. Proc. Am. Math. Soc. **124**(7), 1967–1968 (1996)
Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie locale. Publications Mathématiques de l'IHÉS **42**(1), 47–119 (1973)
Roberts, P.: Heitmann's proof of the direct summand conjecture in dimension 3 (2002) (preprint). arXiv:math/0212073
Schoutens, H.: Bounds in cohomology. Isr. J. Math. **116**, 125–169 (2000)
Schoutens, H.: The Use of Ultraproducts in Commutative Algebra. Lecture Notes in Mathematics, vol. 1999. Springer, Berlin (2010)
Serre, J.P., Gabriel, P.: Algèbre locale, multiplicités: cours au Collège de France, 1957–1958 (No. 11). Springer Science & Business Media, Berlin (1975)
Vélez, J.D.: Splitting results in module-finite extension rings and Koh's conjecture. J. Algebra **172**(2), 454–469 (1995)
Velez, J.D., Florez, R.: Failure of splitting from module-finite extension rings. Beiträge zur Algebra und Geometrie **41**(2), 345–357 (2000)