Light Pathways in Inhomogeneous and Anisotropic Media

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In this study, we first revisit Fermat's principle and reformulate it as a variational calculus problem for inhomogeneous media, establishing a bridge with Lagrangian mechanics. We then apply this framework to explore light trajectories in materials with spatially dependent refractive indices. Finally, we extend the analysis to anisotropic media by modeling the refractive index as a tensor field, deriving the corresponding geodesic and eikonal equations to capture the directional dependence of light propagation. This approach offers a rigorous basis for analyzing light paths in complex media, with potential applications in advanced optical design and material science.

Fermat's Principle | Variational Calculus | Inhomogeneous Media | Anisotropic Media |

1. Introduction

The fundamental question regarding the path light follows during its propagation has intrigued scientists and philosophers throughout history. From the era of ancient Greeks to contemporary scientific advancements, this question has driven research and discoveries in all of physics [4].

In the context of classical optics, Fermat's theorem states that light follows the optical path that minimizes the propagation time between two points [8]. This principle allows us to understand phenomena such as reflection and refraction, where light selects specific routes to minimize its travel time depending on the properties of the medium.

Recent contributions to this field include the work of Leonhardt (2006) on optical conformal mapping, which enabled the design of materials guiding light through specified paths in inhomogeneous and anisotropic media [9]. Additionally, Pendry, Schurig, and Smith (2006) expanded this concept to transformation optics, paving the way for cloaking devices that manipulate electromagnetic fields to render objects invisible by directing light around them [14]. These studies laid the groundwork for advanced ray-tracing methods in complex media, such as gradient-index (GRIN) materials [7].

The aim of this paper is to formulate Fermat's principle as a variational calculus (or functional optimization) problem, drawing a direct analogy with Lagrangian mechanics, and to explore the resulting light trajectories in inhomogeneous media.

The document is organized as follows: In Section 2, we develop the mathematical framework needed to express Fermat's principle as a variational calculus problem for inhomogeneous media. Section 3 demonstrates applications of this theory by setting a medium with a specific, spatially dependent refractive index and plotting light trajectories within it. In Sections 4 and 5, we extend our discussion to anisotropic materials and, using concepts from special relativity, derive the well-known Eikonal Equation [18]. Finally, in Section 6, we conclude our work.

2. Mathematical Formulation for Inhomogeneous Media

In this section, we present inhomogeneous media, Fermat's principle for such media, and its analogy with classical mechanics.

A. Inhomogeneous Media. Inhomogeneous media are materials in which the optical properties, particularly the refractive index, vary across different points within the medium. Informally, this variation means that light encounters different "speeds" or "resistances" as it traverses regions with distinct refractive indices, resulting in

Significance Statement

Understanding light trajectories in complex media is fundamental to fields such as optical communication, optical device design, and medical imaging. In applications where light travels through anisotropic, inhomogeneous, or nonlinear media, variational calculus, particularly Fermat's principle, offers an intuitive and effective approach to optimizing these trajectories. This method is crucial for enhancing data transmission through optical fibers under varying conditions and for ensuring optimal performance in optical devices with nonlinear properties [1]. Additionally, in medical applications, precise modeling of light in biological tissues aids advancements in diagnostic and treatment techniques [6]. The connection between variational calculus and Lagrangian mechanics further strengthens our analytical framework, allowing us to address complex optical challenges by leveraging principles from both optics and classical mechanics.

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curved trajectories as light seeks the path of least time.

Formally, in an inhomogeneous medium, the refractive index n is defined as a scalar field $n : \mathbb{R}^3 \to \mathbb{R}$, which depends on position $\mathbf{r}$, so that $n = n(\mathbf{r}) = n(x, y, z)$. The refractive index is given by:

$$n = \frac{c}{v}, \qquad [1]$$

where c is the speed of light in a vacuum and v is the speed of light within the medium. Consequently, v must also vary with position, making it a scalar field dependent on the spatial coordinates $\mathbf{r}$.

In daily life, we encounter several examples of inhomogeneous media, such as atmospheric layers [17], water bodies with temperature gradients [10], and the human eye lens [19]. These variations in refractive index influence light paths, creating phenomena like atmospheric refraction, underwater visibility changes, and the eye's focusing ability.

B. Fermat's Principle. Consider a scenario where a ray of light is emitted from point A and detected at point B, as illustrated in Fig. 1. The central question we aim to address is: what trajectory does the light ray follow as it travels from A to B?

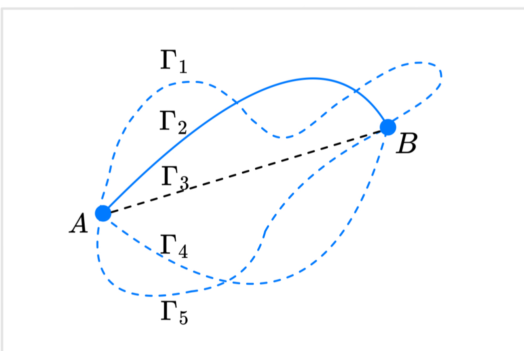

Fig. 1. Light trajectories in an arbitrary medium.

While both optics and electromagnetism offer various approaches to solve this problem [8], our focus will be on **Fermat's Principle**, which, as we will demonstrate, can be elegantly formulated using variational calculus. A first statement of Fermat's principle is as follows:

Theorem 1 (Fermat's Principle for T) *The path taken by a light beam between two points is the one that requires the least time to traverse.*

Let us formalize this. Let Γ be the trajectory that minimizes the time taken by light, and let A, B be points in 2D space with coordinates (x_0, y_0) and (x_1, y_1), respectively (see Fig. 2).

The path Γ has a total arc length S measured from A to B, and T is defined as the time it takes to traverse the path Γ. Clearly, this travel time can be calculated as:

$$T = \int_A^B dt. \qquad [2]$$

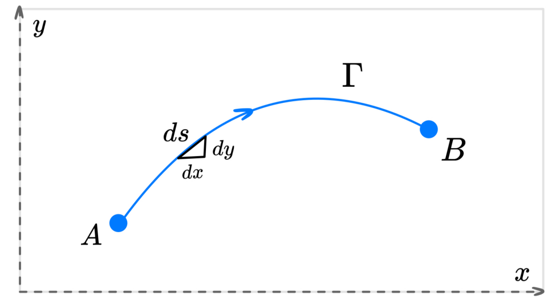

Fig. 2. Path that minimizes the travel time of the ray between A and B.

Furthermore, if we consider the speed of the light ray in the medium v (which is a scalar field when the medium is inhomogeneous and varies in two or more directions in anisotropic materials), the time T can be reformulated as:

$$T = \int_A^B dt = \int_A^B \frac{ds}{v}. \qquad [3]$$

As will be seen in the next subsection, the first-order condition that Γ must satisfy to be the path with the least travel time is that the variation of this time is zero, that is,

$$\delta T = \delta \int_A^B \frac{ds}{v} = 0. \qquad [4]$$

Now, let us consider a stronger version of Fermat's theorem in terms of the *optical path length* (OPL). The OPL corresponds to the distance in a vacuum equivalent to the distance S traversed in the medium with refractive index n. That is, $OPL/\lambda_0 = S/\lambda$ with λ_0 the wavelength of light in a vacuum, preserving the phase as light advances [8].

Then $t = OPL/c$, with c the speed of light in a vacuum, and Fermat's principle can be reformulated as follows:

Theorem 2 (Fermat's Principle for the OPL) *To travel from point A to point B, light follows the path with the shortest optical path length.*

Now, with the definitions made earlier, $S = OPL$ and we have $ds = c\,dt$, so that $OPL = S = cT$. Multiplying Eq. 3 by c:

$$cT = S = \int_A^B c\,dt, \qquad [5]$$

and with the definitions of the speed in the medium $v = ds/dt$, and the refractive index $n = c/v$, we get:

$$S = \int_A^B \frac{c}{v} ds = \int_A^B n\,ds. \qquad [6]$$

Using this last expression, the first-order condition translates to:

$$\delta S = \delta \int_A^B n\,ds = 0. \qquad [7]$$

Therefore, to find the trajectories of light in a medium, one must minimize the functional S, the optical path length. Assuming an isotropic but inhomogeneous medium, the refractive index n is a scalar field $n : \mathbb{R}^2 \to \mathbb{R}$ (if the light moves in a plane). Moreover, using Pythagoras' theorem for the differential of the arc (see Fig. 2), taking x as the independent variable and considering the Lagrange notation for the derivatives $y'(x) = dy/dx$, the arc differential can be written as:

$$ds = \sqrt{1+y'^2}\,dx. \qquad [8]$$

Thus, the optimization problem (or variational calculus problem) to solve is:

$$\left\{ \min_{y} S[y] = \int_{x_0}^{x_1} n(x,y)\sqrt{1+y'^2}\,dx, \right. \qquad [9]$$

with $y(x)$ an **admissible** function, that is, of class C^{2*}, such that $y(x_0) = y_0$ and $y(x_1) = y_1$. The following subsection will delve deeper into the technicalities of variational calculus.

C. Problem Formulation of Variational Calculus. Does a minimum exist for the problem we just posed? What does it mean to minimize a functional? What about the admissible function? Let us start by analyzing the variational calculus problem to be solved. Let x_0, x_1 be real numbers with $x_0 < x_1$. We define the set of functions Ω as:

$$\Omega = \{f : [x_0, x_1] \to \mathbb{R} \mid f \in C^2([x_0, x_1])\}. \qquad [10]$$

In this set of functions, the usual operations of addition and scalar multiplication are considered. Let $f_1, f_2 \in \Omega$, $x \in [x_0, x_1]$, and $\lambda \in \mathbb{R}$; we define $(f_1+f_2)(x) := f_1(x)+f_2(x)$ and $(\lambda f)(x) := \lambda f(x)$. Clearly, the triplet $(\Omega, +, \cdot_{\mathbb{R}})$ is a vector space [3].

In Ω, the following norm is now defined:

$$\begin{aligned} \|\cdot\| &: \Omega \to \mathbb{R} \\ f &\mapsto \|f\| = \max_{x\in[x_0,x_1]} |f(x)|, \end{aligned} \qquad [11]$$

so that $(\Omega, \|\cdot\|)$ is a normed space. Similarly, the notion of *distance* is induced by:

$$d(f_1, f_2) = \|f_1 - f_2\| = \max_{x\in[x_0,x_1]} |f_1(x) - f_2(x)|, \qquad [12]$$

so that $(\Omega, d(\cdot,\cdot))$ is a *metric space*. Finally, we define a **functional** J as any mapping:

$$\begin{aligned} J &: \Omega \to \mathbb{R} \\ f &\mapsto J[f] = J[f(x)]. \end{aligned} \qquad [13]$$

Having defined the space to which the functions of interest belong and what a functional is, let us see what it means to minimize a functional J. Since the functionals of interest to solve are integrals between the initial and final positions,

$$J[f] = \int_{x_0}^{x_1} F[f(x), \dot{f}(x), x]\,dx, \qquad [14]$$

we define the set of **admissible** functions as:

$$\Psi = \{f \in \Omega \mid f(x_0) = y_0, f(x_1) = y_1\}. \qquad [15]$$

*In particular $y \in C^2((x_0, x_1)) \cup C^0([x_0, x_1])$

It is then said that f^* is a global minimum for J if $f \in \Psi$ and $J[f^*] \le J[f], \forall f \in \Psi$. Similarly, a local minimum f^* is defined if $\exists \delta > 0$ such that $\forall f \in B_\delta(f^*)$, $J[f^*] \le J[f]$, where B is a ball of radius δ centered at f^* defined by the distance $d(\cdot,\cdot)$.

Then, we are capable of discerning local minima for the functional J and of posing the optimization problem (**P**) to be solved:

$$(\mathbf{P}) \quad \left\{ \min_{f\in\Psi} J[f] = \int_{x_0}^{x_1} F[f(x), \dot{f}(x), x]\,dx. \right. \qquad [16]$$

Let f^* be the function that minimizes the functional of problem (**P**). The first-order condition it must satisfy is expressed in the following theorem.

Theorem 3 (Euler Condition) *If $f \in \Psi$, and f is a local minimum or maximum of problem* (**P**)*, then f verifies the following condition:*

$$\delta J[f] = \frac{d}{d\varepsilon} J[f + \varepsilon\eta]_{\varepsilon=0} = 0, \qquad [17]$$

with $\varepsilon \in \mathbb{R}$ and $\eta = \eta(x)$ such that $\eta(x_0) = \eta(x_1) = 0$ arbitrary. This variation set to zero implies that f satisfies the Euler-Lagrange equation†:

$$\frac{\partial F}{\partial f} - \frac{d}{dt}\left(\frac{\partial F}{\partial f'}\right) = 0, \qquad [18]$$

for all $x \in [x_0, x_1]$ [3].

D. Analogy with Lagrangian Mechanics. Analyzing what was mentioned in the previous two subsections, it is possible to draw an analogy between Lagrangian mechanics and the problem of light paths. In this context, the refractive index acts as a mechanical potential, and the problem can be solved using the Euler-Lagrange equations.

Returning to the optimization problem, and having already defined the set of admissible functions, in an inhomogeneous and isotropic material one must solve:

$$\left\{ \min_{y\in\Psi} S[y] = \int_{x_0}^{x_1} n(x,y)\sqrt{1+y'^2}\,dx. \right. \qquad [19]$$

Meanwhile, the problem of Lagrangian mechanics is based on the minimization of the action S', formulated in one dimension as $x' = x'(t)$:

$$\left\{ \min_{y} S'[x] = \int_{t_0}^{t_1} L(x, x', t)\,dt. \right. \qquad [20]$$

Then, making a correspondence between the optics (left) problem and the elements of classical mechanics (right), we have:

$$\begin{array}{ccc} x & \longleftrightarrow & t \\ y & \longleftrightarrow & x' \\ n(x,y)\sqrt{1+y'^2} & \longleftrightarrow & L \\ S & \longleftrightarrow & S'. \end{array} \qquad [21]$$

Thus, although the dynamics of the systems are vastly different, Lagrangian mechanics theory models the problem

†This work does not focus on the sufficient first-order or any second-order optimality conditions.

of light paths.

This represents a supremely powerful tool in solving the problems posed here, as it gives us access to concepts from Lagrangian mechanics, such as conserved quantities, symmetries, degrees of freedom and constraints.

E. Curves Not Parameterizable Solely by y. Recall that in formulating the variational calculus problem as in Eq. (16), we implicitly assumed (in Eq. (8)) that the curve Γ could be parameterized by x, i.e., $\mathbf{r} = (x, y) = (x, y(x))$. Naturally, this is not always the case, and we now establish a generalization of the previous formulation for curves parameterizable by a parameter $t \in \mathbb{R}$ in $\mathbb{R}^3$.

Returning to Eq. Eq. (7), recall that for an inhomogeneous medium with $n : \mathbb{R}^3 \to \mathbb{R}$, Fermat's principle could be written as:

$$\delta S = \delta \int_A^B n \, ds = 0. \tag{22}$$

If we denote Γ as a curve parameterizable by $t \in [a, b] \subseteq \mathbb{R}$, then since ds is generally expressed as

$$ds = \sqrt{dx^2 + dy^2 + dz^2}, \tag{23}$$

in terms of the parameter t, we obtain:

$$\begin{aligned} \delta S &= \delta \int_A^B n(x, y, z) \sqrt{dx^2 + dy^2 + dz^2} \\ &= \delta \int_a^b n(x, y, z) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, dt \\ &= \delta \int_a^b n(t) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, dt = 0, \end{aligned} \tag{24}$$

where the dot above a spatial variable represents differentiation with respect to the parameter t. In this case, it is clear that the Lagrangian (referred to as the *Optical Lagrangian*) is $L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = L(\mathbf{r}, \dot{\mathbf{r}}) = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$, and that the analogy with the classical mechanics problem (Eq. Eq. (20)) is:

$$\begin{array}{ccc} t & \longleftrightarrow & t^{\ddagger} \\ x, y, z & \longleftrightarrow & x' \\ n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} & \longleftrightarrow & L \\ S & \longleftrightarrow & S'. \end{array} \tag{25}$$

Similarly, one can use the Euler-Lagrange equation Eq. (18) to solve the problem of minimizing the functional S as given by Hamilton's principle in Eq. (24).

3. Light Pathways in Inhomogeneous Media

In this section, we discuss the pathways of light in materials with scalar fields as indices of refraction.

A. Medium with Speed Proportional to Height: $v \propto y$. Before discussing applications, we revisit a useful theorem, the proof of which can be found in Appendix A.

Theorem 4 *Let $f(y)$ be a function of y. Then the function $y(x)$ optimizing the functional:*

$$J = \int_{x_0}^{x_1} f(y)\sqrt{1 + y'^2} \, dx, \tag{26}$$

satisfies the differential equation

$$1 + y'^2 = B f(y)^2, \tag{27}$$

where $B \in \mathbb{R}$ is an integration constant.

As an initial application, consider a medium where the light propagation speed is proportional to height, $v = v_r \propto y$. Then, by the definition of the refractive index:

$$n(x, y) = \frac{k}{y}. \tag{28}$$

Having defined $n(x, y)$, the functional to minimize becomes:

$$S[y] = \int_{x_0}^{x_1} \frac{k}{y} \sqrt{1 + y'^2} \, dx. \tag{29}$$

Continuing with the development, the Lagrangian is defined as follows:

$$L = f(y)\sqrt{1 + y'^2} = \frac{k\sqrt{1 + y'^2}}{y}. \tag{30}$$

The necessary first-order condition for minimizing the functional $S = S[y]$ is the Euler-Lagrange equation of the system. Applying Theorem 4, the differential equation describing the light's trajectory, if it minimizes the optical path length, is:

$$1 + y'^2 = B f(y)^2. \tag{31}$$

Where $f(y) = k/y$. Substituting $f(y)$ into the above equation yields the first-order differential equation:

$$1 + y'^2 = \frac{k^2 B}{y^2}. \tag{32}$$

Solving for y'^2, we have:

$$y'^2 = \frac{k^2 B}{y^2} - 1 = \frac{k^2 B - y^2}{y^2}. \tag{33}$$

Taking the square root of both sides gives:

$$y' = \frac{dy}{dx} = \pm \sqrt{\frac{k^2 B - y^2}{y^2}}. \tag{34}$$

Separating variables and integrating both sides, we obtain:

$$\int dx = \pm \int \frac{y \, dy}{\sqrt{k^2 B - y^2}}. \tag{35}$$

Using the substitution $u = k^2 B - y^2$, the right-hand integral can be easily computed. The result of these operations is:

$$x + A = \mp \sqrt{k^2 B - y^2}, \tag{36}$$

where A is another integration constant. Therefore, the trajectory followed by light in a material with refractive index $n(x, y) = k/y$ is that of a circular arc given by $(x + A)^2 + y^2 = k^2 B$. It is apparent that the circle is centered at the edge of the half-plane H^+, and to express $y = y(x)$, the positive root is conventionally taken in the positive semiplane.

This refractive index model has applications in the design of lenses and optical systems with graded refractive indices, where the shape and properties of optical surfaces are meticulously designed to achieve specific effects, such as in lenses with corrected aberrations or waveguides with special light propagation characteristics [8].

Fig. 3 displays the scalar field $n(x, y)$ of the refractive index (note that this is a dimensionless quantity). All units on the x and y axes in this document are meters.

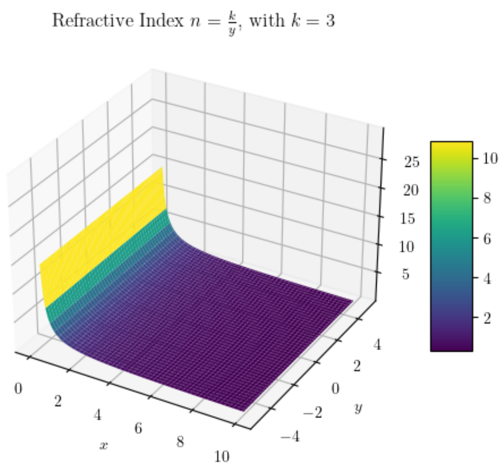

Fig. 3. Scalar field of the refractive index $n(x, y) = k/y$ with $k = 3$.

Furthermore, Fig. 4 illustrates the calculated trajectories for different initial conditions in the previously analyzed medium with $k = 3$.

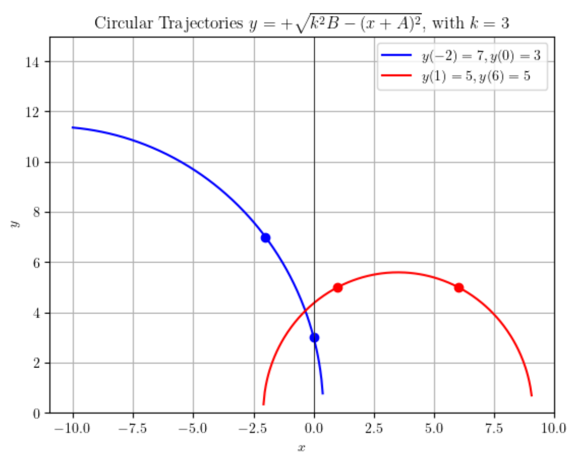

Fig. 4. Circular light trajectories in the medium $n(x, y) = k/y$, with $k = 3$.

A.1. Curious Fact. The set of all light trajectories in the previously described material is known as the **hyperbolic plane**, where what we call a line can be represented by either vertical lines or semicircles centered on the x-axis.

B. Radial Refractive Index: $n = \sqrt{1+y}$. In this application of Fermat's principle, we explore a widely used refractive index in computational optics: the scalar field $n(x, y) = \sqrt{1+y}$. To continue with our established strategy, we will streamline the explanation of the development, assuming familiarity with the detailed steps in the first example.

After defining the refractive index, the optical system's Lagrangian is:

$$L = n(x, y)\sqrt{1+y'^2} = \sqrt{1+y}\sqrt{1+y'^2} = f(y)\sqrt{1+y'^2}. \quad [37]$$

Thus, the action to minimize is:

$$S[y] = \int_{x_0}^{x_1} \sqrt{1+y}\sqrt{1+y'^2}\, dx. \quad [38]$$

The necessary first-order condition for the trajectory $y(x)$ to be a *path of light* (minimizing the optical path length) is formulated using Theorem 4. The resulting equation from the theorem is:

$$1 + y'^2 = Bf(y)^2. \quad [39]$$

Substituting $f(y) = n(x, y)$, we obtain:

$$1 + y'^2 = B(1+y). \quad [40]$$

Solving for y', we derive:

$$y' = \pm\sqrt{B(1+y) - 1}. \quad [41]$$

This is a separable first-order differential equation. Expressing $y' = dy/dx$ and separating the ODE, we get:

$$\int dx = \pm \int \frac{dy}{\sqrt{B(1+y) - 1}}. \quad [42]$$

Using the substitution $u = B(1+y) - 1$, the integral on the right is solved, resulting in:

$$x + A = \pm\frac{2}{B}\sqrt{B(1+y) - 1}. \quad [43]$$

Finally, squaring the equation and solving for y, we obtain the parametrization $y = y(x)$ of the trajectory followed by light in this medium:

$$y(x) = \frac{B}{4}(x+A)^2 + \left(\frac{1}{B} - 1\right). \quad [44]$$

This corresponds to a downward-opening parabola with its vertex on the y-axis:

$$y(x) = a(x+h)^2 + b. \quad [45]$$

We observe that for the inhomogeneous material with the radial refractive index $n = \sqrt{1+y}$, the trajectory between any two points in its domain is described by a parabola focused along the y-axis.

Figure 5 presents the scalar field $n(x, y)$ for which the optical path length optimization problem was just solved.

Furthermore, with appropriate boundary conditions (x_0, y_0) and (x_1, y_1), it is possible to graphically reconstruct trajectories for these paths traversed by light. Examples of these boundary conditions and the resulting trajectories are shown in Figure 6.

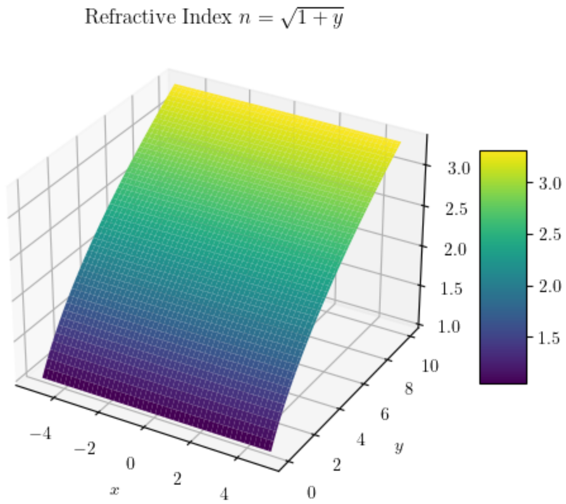

Fig. 5. Scalar field of the refractive index $n(x, y) = \sqrt{1 + y}$.

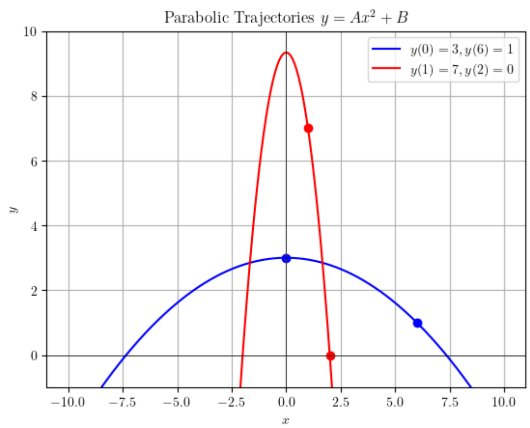

Fig. 6. Parabolic light trajectories in the medium $n(x, y) = \sqrt{1 + y}$.

C. Exponential Trajectory: $n = \sqrt{1 + y^2}$. We now replicate the procedures previously outlined for a new inhomogeneous material. Initially, we consider a refractive index given by the expression:

$$n(x, y) = \sqrt{1 + y^2}. \quad [46]$$

This allows us to formulate the associated Optical Path Length (OPL) functional, S, as follows:

$$S[y] = \int_{x_0}^{x_1} \sqrt{1 + y^2}\sqrt{1 + y'^2}\, dx. \quad [47]$$

Additionally, the Lagrangian is defined as:

$$L = \sqrt{1 + y^2}\sqrt{1 + y'^2} = f(y)\sqrt{1 + y'^2}. \quad [48]$$

Applying Theorem 4, we derive the Euler-Lagrange equation for the system:

$$1 + y'^2 = Bf(y)^2. \quad [49]$$

Substituting $f(y)$, we obtain:

$$1 + y'^2 = B(1 + y^2). \quad [50]$$

Clearing y'^2, the following algebraic steps yield:

$$y'^2 = B(1 + y^2) - 1 \quad [51]$$

$$y' = \frac{dy}{dx} = \pm\sqrt{B(1 + y^2) - 1}. \quad [52]$$

Solving the separable differential equation, we find:

$$\int dx = \pm \int \frac{dy}{\sqrt{B(1 + y^2) - 1}}. \quad [53]$$

Using the hyperbolic trigonometric identity, this integral becomes:

$$x + A = \frac{\operatorname{arsinh}\left(\frac{\sqrt{B}\, y}{\sqrt{B-1}}\right)}{\sqrt{B}}. \quad [54]$$

Clearing the variable y, we derive:

$$y = \sqrt{\frac{B - 1}{B}} \sinh(\sqrt{B}(x + A)). \quad [55]$$

Figure 7 illustrates the scalar field $n(x, y)$ of the refractive index $n(x, y) = \sqrt{1 + y^2}$, where it is important to note that this value is dimensionless. All units on the x and y axes in this document are arbitrary.

This material may be relevant in the study of optical phenomena requiring a medium with a gradient refractive index, such as in the formation of cylindrical lenses or in the design of waveguides, where the variation of the refractive index in a specific direction can influence light propagation along that direction.

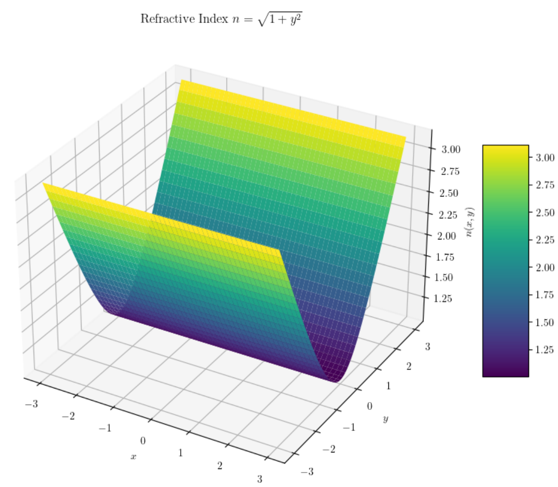

Fig. 7. Scalar field $n(x, y) = \sqrt{1 + y^2}$ of the refractive index.

Subsequently, by graphing the exponential trajectories under different boundary conditions, we obtain the trajectories shown in Figure 8.

Note that trajectories in media with this refractive index are termed exponential because the solution to the Euler-Lagrange equation yields a hyperbolic sine, which can be expressed as a linear combination of increasing and decreasing exponential functions:

$$\sinh(z) = \frac{e^z - e^{-z}}{2}. \quad [56]$$

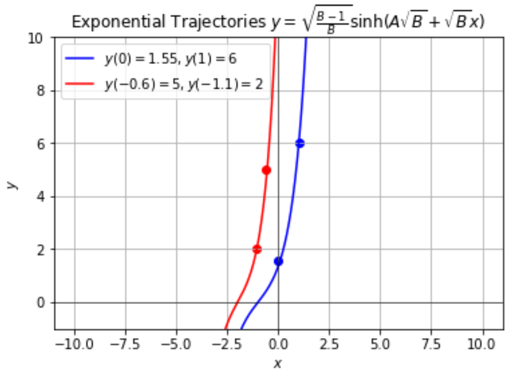

Fig. 8. Exponential light trajectories in the medium $n(x, y) = \sqrt{1 + y^2}$.

4. Fermat's Principle for Anisotropic Media

In this section, we extend the modeling of light propagation from inhomogeneous to anisotropic media. While inhomogeneous media have spatially varying refractive indices, anisotropic media add directional dependence, meaning that the refractive index varies not only by location but also by the direction of light propagation. Anisotropic media, such as birefringent crystals and optical fibers, are critical in applications requiring directional light control.

To accommodate these complexities, the formulation of Fermat's principle must generalize the refractive index from a scalar to a tensor quantity. Following Shen et al. [16] and using special relativity's formalism (with Einstein summation convention), we extend Fermat's principle to anisotropic conditions.

A. Anisotropic Media. In anisotropic media, the refractive index depends on both the spatial position and the direction of light propagation, distinguishing these materials from isotropic ones where the refractive index is uniform in all directions. In anisotropic materials, such as certain crystals (e.g., calcite and quartz) or optical components like polarizers, the refractive index varies along different principal axes, denoted as n_1, n_2, and n_3[§], where each n_i is still a scalar field, i.e. $n_i = n_i(x, y, z)$.

Mathematically, this directional dependence is represented by the dielectric tensor ε, which is a symmetric 3×3 matrix that characterizes the medium's response to the electric field. In its principal coordinate system, the dielectric tensor is diagonal and can be expressed as:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{bmatrix}, \qquad [57]$$

where $\varepsilon_i = n_i^2$ are the principal dielectric constants or permittivities, associated with the principal refractive indices n_1, n_2, and n_3. Thus, the refractive index is no longer a scalar but a tensor, allowing it to vary with both spatial coordinates and propagation direction.

To develop Fermat's principle in an anisotropic medium, we start by generalizing the classical formulation used in isotropic media. In a static isotropic medium, Fermat's principle is traditionally expressed as:

$$\delta \int ds = \delta \int n \, dl = 0, \qquad [58]$$

where n is the scalar refractive index and $dl = \sqrt{dx^2 + dy^2 + dz^2}$ represents the infinitesimal path length. The geometrical structure of a three-dimensional manifold associated with an isotropic medium is thus characterized by the invariant element ds, which is defined as:

$$ds^2 = n^2 dl^2 = g_{ij}(x) dx^i dx^j. \qquad [59]$$

Here, $x = (x, y, z) = (x^1, x^2, x^3)$ represents spatial coordinates, and $g_{ij}(x)$ denotes the metric tensor of the three-dimensional manifold, which in isotropic media reduces to:

$$g_{ij} = n^2 \delta_{ij} = \begin{cases} n^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (i, j = 1, 2, 3), \qquad [60]$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \qquad [61]$$

For inhomogeneous anisotropic media, $\varepsilon_{ij} = \varepsilon_{ij}(x)$ varies with position, affecting both the values of the principal dielectric constants and the orientations of the principal axes. Therefore, we rewrite Fermat's principle to account for anisotropic conditions:

$$\delta \int ds = \delta \int \sqrt{n_1^2 dx^2 + n_2^2 dy^2 + n_3^2 dz^2} = 0. \qquad [62]$$

The invariant element ds now describes a three-dimensional manifold structured by the anisotropic medium, with ds^2 expressed as:

$$ds^2 = n_1^2 dx^2 + n_2^2 dy^2 + n_3^2 dz^2 = g_{ij}(x) dx^i dx^j. \qquad [63]$$

Here, the metric tensor $g_{ij}(x)$ for the manifold is defined by:

$$g_{ij} = n_i^2 \delta_{ij} = \begin{cases} n_i^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (i, j = 1, 2, 3), \qquad [64]$$

or equivalently:

$$g_{ij} = \begin{bmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{bmatrix}. \qquad [65]$$

The determinant of the metric tensor g_{ij} is given by

$$g = \det(g_{ij}) = n_1^2 n_2^2 n_3^2, \qquad [66]$$

and the inverse metric tensor takes the form

$$g^{ij} = \begin{cases} n_i^{-2} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \qquad [67]$$

or explicitly:

$$g^{ij} = \begin{bmatrix} n_1^{-2} & 0 & 0 \\ 0 & n_2^{-2} & 0 \\ 0 & 0 & n_3^{-2} \end{bmatrix}. \qquad [68]$$

It is straightforward to observe that in an isotropic medium (where $n_1 = n_2 = n_3 = n$), Eqs. (59) reduce to the familiar case of isotropic optics.

§ In cartesian coordinates $n_1 = n_x$, $n_2 = n_y$, and $n_3 = n_z$

5. Geodesic Equation and General Eikonal Equation

According to Shen et al. [16], Fermat's principle can be reformulated as follows:

$$\delta \int ds = \delta \int (g_{ij} dx^i dx^j)^{1/2} = 0. \quad [69]$$

This leads to the geodesic equation governing the trajectory of light in a static anisotropic medium:

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad [70]$$

where Γ^i_{jk} represents the affine connection of the three-dimensional manifold associated with the anisotropic medium:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \quad [71]$$

Substituting the metric tensor components, we obtain:

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \begin{cases} n_i^{-1} n_{i,i} & (i = j) \\ -n_i^{-2} n_j n_{j,i} & (i \neq j \neq k \neq i) \\ 0 & (i \neq j \neq k \neq i), \end{cases} \quad [72]$$

where $n_{i,i}$ denotes the partial derivative of n_i with respect to x^i, i.e., $n_{i,i} = \partial n_i / \partial x^i$.

In terms of unit vectors (i, j, k) along the coordinate axes, we define the gradient operator as:

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}. \quad [73]$$

Letting $\mathbf{e} = (i, j, k)$, we define the directional refractive index matrix $\mathcal{N}^2$ as follows:

$$\mathcal{N}^2 = \begin{bmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{bmatrix} = \mathcal{G}, \quad [74]$$

and we can thus propose the following expression:

$$\frac{d}{ds} \left(\mathbf{e} \cdot \mathcal{N}^2 \frac{d\mathbf{x}}{ds} \right) = \frac{1}{2} \mathbf{e} \cdot \mathcal{N}^2 \nabla \mathcal{N}^2 \cdot \frac{d\mathbf{x}}{ds}. \quad [75]$$

In isotropic media, where $n_i = n$ for $i = 1, 2, 3$ and $ds = n\, dl$, this expression reduces to the well-known eikonal equation:

$$\frac{d}{dl} \left(n \frac{dx}{dl} \right) = \nabla n. \quad [76]$$

6. Conclusions

In this study, we developed a systematic application of Fermat's principle for modeling light paths in inhomogeneous and anisotropic media. Beginning with the formulation of light trajectories in inhomogeneous media using variational calculus, we derived the necessary conditions for minimizing optical path length and demonstrated this framework through specific examples with spatially varying refractive indices. Extending this approach to anisotropic media, we represented the refractive index as a tensor field, derived the corresponding geodesic equations, and generalized the eikonal equation to account for directional dependencies. This work offers a rigorous theoretical basis for analyzing light behavior in complex media, with potential applications in advanced optical design and materials science.

A. Future Work. Future research could extend this framework to nonlinear and dynamic media, where refractive indices depend on both light intensity and time. Investigating the influence of nonlinearity and temporal variations on light pathways would deepen the applicability of Fermat's principle in fields like high-power laser optics, adaptive optics, and real-time imaging technologies.

Appendix A. Proof of Theorem 4

We follow the proof outlined in [**god**]. The objective is to find the function $y(x)$ that extremizes the integral:

$$\int_{x_0}^{x_1} f(y) \sqrt{1 + y'^2}\, dx. \quad [77]$$

We define the "Lagrangian" as $L = f(y)\sqrt{1 + y'^2}$, and apply the Euler-Lagrange equation:

$$\frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = \frac{\partial L}{\partial y}$$
$$\Downarrow \quad [78]$$
$$\frac{d}{dx} \left(f \cdot y' \cdot (1 + y'^2)^{-1/2} \right) = f' \sqrt{1 + y'^2}.$$

Using the product rule on the left-hand side, we obtain:

$$\frac{f' y'^2}{\sqrt{1 + y'^2}} + \frac{f y''}{\sqrt{1 + y'^2}} + \frac{f y'^2 y''}{(1 + y'^2)^{3/2}} = f' \sqrt{1 + y'^2}. \quad [79]$$

Multiplying both sides by $(1 + y'^2)^{3/2}$ and simplifying, we reach:

$$f y'' = f'(1 + y'^2). \quad [80]$$

Now, multiplying both sides by y' and rearranging, we obtain the ordinary differential equation:

$$\frac{y' y''}{1 + y'^2} = \frac{f' y'}{f}. \quad [81]$$

Since $f = f(y(x))$, we can rewrite this using the chain rule as:

$$\frac{1}{2} \frac{d}{dx} \left(\ln(1 + y'^2) \right) = \frac{d}{dx} \left(\ln(f) \right). \quad [82]$$

Integrating both sides with respect to x gives:

$$\frac{1}{2} \ln(1 + y'^2) = \ln(f) + C. \quad [83]$$

Taking the exponential of both sides, we obtain:

$$1 + y'^2 = B f(y)^2, \quad [84]$$

where $B = e^{2C}$ is an integration constant, yielding the desired result. Note that here we used $f' = \frac{df}{dy}$.

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