Restricted normal modal logics and levelled possible worlds semantics

JUAN C. AGUDELO-AGUDELO, *Instituto de Matemáticas, Universidad de Antioquia, A.A. 1226, Medellín, Colombia.*
E-mail: jagudelo@matematicas.udea.edu.co

MANUEL H. SIERRA-ARISTIZÁBAL, *Departamento de Ciéncias Básicas, Universidad EAFIT, A.A. 3030, Medellín, Colombia.*
E-mail: msierra@eafit.edu.co

Abstract

Restricted normal modal logics are here defined by imposing conditions on the modal axioms and rules of normal modal systems. The conditions are defined in terms of a *depth* (associated with the modal connective) and a *complexity function*. It is proven that the logics obtained are characterized by a subtle adaptation of the possible worlds semantics in which *levels* are associated with the worlds. Restricted normal modal logics constitute a general framework allowing the definition of a huge variety of modal systems, which can have different applications. For instance, they are useful to define epistemic logics where the logical omniscience problem is partially controlled.

Keywords: Modal logic, epistemic logic, logical omniscience problem.

1 Introduction

The logic K, the *basic normal modal logic*, is usually defined by extending the *classical propositional logic (CPL)*, introducing a unary *necessitation* connective $\Box$ and adding the axiom (K) and the *necessitation rule* (Nec):

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta), \tag{K}$$

$$\vdash_K \alpha \text{ implies } \vdash_K \Box\alpha. \tag{Nec}$$

The modal systems extending K are called *normal modal logics*, and they are semantically characterized by the so-called *relational semantics* (or *possible worlds semantics*).

Since about the half of the twentieth century, many logical systems have been proposed in order to formalize the concepts of knowledge and belief, these formal systems are called *epistemic logics*. Now, the most influential epistemic logics are extensions of the normal modal system K, usually including multiple modal operators (each operator representing the knowledge of an agent) and modal axioms intended to capture the 'essential' properties of the concepts that they formalize.[1]

The logic K^m, considered the *minimal logic of knowledge* (see [1, Section 7.3]), is a multi-modal logic including a finite number of modal connectives $K_1,\dots,K_m$, each of them ruled by the axiom (K) and the necessitation rule. Each modal connective in K^m represents the knowledge of the respective agent, then the formula $K_i\alpha$ is interpreted as 'agent i knows α'. Under this interpretation, the axiom

[1]For an introduction to epistemic logic see [1, Chapter 7].

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(K) in conjunction with the rule (Nec) leads to the conclusion that any agent i knows all the logical consequences of his knowledge, which goes in contradiction with the ordinary sense of 'known'. This is an important puzzle of epistemic logics based on K and is called the 'logical omniscience problem'.[2]

Several approaches have been proposed in order to deal with the logical omniscience problem; most of them are described in [2] and [6]. A new alternative approach is proposed by Manuel Sierra in [4] and [5]. In [4], Sierra introduces the family of systems SMM-n, where n is a positive integer number. The system SMM-n is called the *multi-modal systems of depth n*. The family is inductively defined in the following way: SMM-1 is just CPL and SMM-$(n+1)$ is obtained by adding to SMM-n the results of applying (Nec) only once to each formula of SMM-n. In this family, each system has a different language, which is associated with the deductive capacity of the reasoners whose knowledge is represented by the modal connectives. The union of the family is denoted by SMM and is called the *restricted multi-modal system*. All these systems are semantically characterized by a relational semantics in which the length of the chains of worlds is limited. These systems are extended in [5], obtaining the families of systems LER-n and LDR-n. The system LER-n is called the *epistemic logic with restrictions of depth n*, and the union of the family is denoted by LER; these systems can be viewed as restricted versions of the well-known modal logic $S5$. In an similar way, the system LDR-n is called the *doxastic logic with restrictions of depth n*, and the union of the family is denoted by LDR; these systems can be viewed as a restricted version of the modal logic $KD45$. Both families are semantically characterized by relational semantics with 'embedded worlds'.

The main motivation of Sierra's systems is to allow the representation of reasoners of different types, where the type is associated with the 'deductive capacity' of the reasoner. A reasoner of type 1 can deduce any CPL-tautology, but cannot deduce the necessity of any formula, which under a suitable interpretation means that a reasoner of type 1 is totally unconscious of his knowledge (or of his beliefs). In contrast, a reasoner of type $n+1$ can deduce the necessity of any formula deducible by a reasoner of type $\leq n$, and following the same interpretation, it means that a reasoner of type $n+1$ is conscious of the knowledge (or beliefs) of reasoners of type $\leq n$. Considering that reasoners of type n are also reasoners of type m, for any $m \leq n$, a reasoner of type n is conscious of his knowledge until 'depth' $n-1$. By capturing these ideas, the logical systems introduced by Sierra allow a partial control of the problem of logical omniscience. For instance, in LER-4, a reasoner of type 2 cannot do inferences involving formulas of type 3, while a reasoner of type 3 or 4 can. However, in LER-4 none of the reasoner can do inferences of formulas with type >4 (such formulas are not even included in the language of LER-4).

In this article, a wide generalization of Sierra's proposal is presented, by introducing the family of 'restricted normal modal logics' and the 'levelled possible worlds semantics'. In contrast with Sierra's systems, the systems here defined have no restrictions on the language (the restrictions are established only on the axioms and rules, but not on the language, which is the same language of all the normal modal logics), this feature avoids many technical difficulties. Moreover, the family of restricted normal modal logics covers all the systems that can be defined by an adaptation of the so-called Lemmon–Scott axiomatic schema, which represents an important generalization of the systems proposed by Sierra. On the semantical counterpart, the assignation of 'levels' to worlds allows a natural adaptation of the relational semantics in order to be apt to characterize all the restricted normal modal systems defined here, and this without restricting the language associated with worlds (in the semantics defined by Sierra in [4] and [5], any world has associated a language, and the levels of worlds are defined in terms of the 'depth' of their associated language).

[2]For an interesting discussion about the logical omniscience problem see [7].

It is important to emphasize that the intention of this article is not to define and defend a particular system of epistemic logic, but to provide a general framework allowing the definition of a huge variety of modal systems that could have different purported applications. However, we briefly describe how these systems can be used in order to define epistemic logics where the logical omniscience problem is partially controlled.

This article is structured as follows: in section 2, the logic $K_{n,C}$ is defined as a restriction of K where (K) and (Nec) are conditioned in terms of a depth n and a complexity function C. The system $K_{n,C}$ is first defined in a syntactical way and some properties are proven, then, the possible worlds semantics is modified obtaining a 'levelled' version of this semantics, which characterizes $K_{n,C}$ (i.e. $K_{n,C}$ is sound and complete with respect to the levelled possible worlds semantics). In Section 3, the Lemmon–Scott axiomatic schema is adapted and is used to extend $K_{n,C}$ in such a general approach. Then, it is proven that any extension obtained in this way is characterizable by a levelled possible worlds semantics where the existence of some 'embedded worlds' is assumed. In Section 4, it is described how epistemic logics based on restricted normal modal logics can be defined, and some properties concerning the logical omniscience problem are pointed out. Some possible future works, in connection with the ideas presented here, are described in the final remarks in Section 5.

2 The basic restricted normal modal logic $K_{n,C}$

Let *For* be the set of formulas of K (which is the same language of any monomodal normal logic, and will be the language of all monomodal restricted modal logics defined below). We will consider connectives $\neg, \rightarrow$ and $\Box$ as being the primitive connectives of *For*. More precisely, fixing a set of propositional variables $\mathcal{V} = \{p_i : i \in \mathbb{N}\}$, *For* is the smaller set X satisfying the following conditions: (i) $\mathcal{V} \subseteq X$; (ii) if $\alpha \in X$ then $\neg\alpha \in X$; (iii) if $\alpha, \beta \in X$ then $(\alpha \rightarrow \beta) \in X$; and (iv) if $\alpha \in X$ then $\Box\alpha \in X$. Connectives for conjunction, disjunction, equivalence and possibility are defined in terms of the primitive connectives in the following way: $\alpha \wedge \beta \overset{\text{def}}{=} \neg(\alpha \rightarrow \neg\beta)$; $\alpha \vee \beta \overset{\text{def}}{=} (\neg\alpha \rightarrow \beta)$; $\alpha \leftrightarrow \beta \overset{\text{def}}{=} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$; and $\Diamond\alpha \overset{\text{def}}{=} \neg\Box\neg\alpha$. As it is usual, in some places letters like p, q, r, etc., will be used instead of letters with subscript (p_i) to denote propositional variables, and greek letters will denote arbitrary formulas in *For*. Moreover, parenthesis will be omitted where there is no place for confusion.

A function $C: For \rightarrow \mathbb{N}$ will be called a *complexity function* if it satisfies the following conditions:[3]

$$C(p) = 0 \text{ for at least one propositional variable } p, \tag{C1}$$

$$C(\neg\alpha) = C(\alpha), \tag{C2}$$

$$C(\alpha \rightarrow \beta) = \max\{C(\alpha), C(\beta)\}, \tag{C3}$$

$$C(\Box\alpha) = C(\alpha) + 1. \tag{C4}$$

By the above definitions, it is clear that $C(\alpha \wedge \beta) = C(\alpha \vee \beta) = C(\alpha \leftrightarrow \beta) = C(\alpha \rightarrow \beta)$ and that $C(\Diamond\alpha) = C(\Box\alpha)$. Note also that complexity functions can assign any value to propositional variables, whenever it assigns 0 to at least one of them. This property will be useful to restrict the deduction of the necessitation for some *CPL*-tautologies, which will be illustrated below.

The logic $K_{n,C}$, the *basic restricted normal modal logic*, is obtained from K by associating a natural number n with the necessitation connective (such number will be called the *depth of the*

[3]At first glance, these conditions may seem odd. However, they are enough (and seems to be minimal) to prove at once the completeness of all the restricted normal modal systems with respect to the levelled world semantics. After reading Section 3, the relevance of these conditions can be better understood.

modal connective) and restricting the axiom (K) and the necessitation rule, in accordance with a given complexity function C, in the following way:

$$\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta) \text{ if } C(\alpha\to\beta)<n, \tag{$\mathrm{K}_{n,C}$}$$

$$\vdash_{K_{n,C}}\alpha \text{ and } C(\alpha)<n \text{ implies } \vdash_{K_{n,C}}\Box\alpha. \tag{$\mathrm{Nec}_{n,C}$}$$

Note that the definition of $K_{n,C}$ is parameterized in terms of the depth of the modal connective n and the complexity function C, then different logics can be obtained by only changing the values of these parameters.

In $K_{n,C}$, uniform substitution is not valid in general.[4] For instance, if $n=2$, $C(p)=1$ and $C(q)=2$, then $\vdash_{K_{2,C}}\Box(p\to p)$ while $\nvdash_{K_{2,C}}\Box(q\to q)$. Substitution of equivalents is also not valid in $K_{n,C}$. For instance, $\vdash_{K_{2,C}}(p\to p)\leftrightarrow(q\to q)$ and $\vdash_{K_{2,C}}\Box(p\to p)$ while $\nvdash_{K_{2,C}}\Box(q\to q)$. However, restricted versions of these properties are established in Propositions 2.1, 2.2 and 2.4. In such propositions, the following three notational conventions are used: (i) $\alpha[p/\beta]$ denotes the result of substituting β for every occurrence of the variable p in α; (ii) $\alpha[p_1/\beta_1,\dots,p_m/\beta_m]$ designates the simultaneous substitution of m variables; and (iii) $\alpha(\delta/\beta)$ denotes the result of substituting β for some occurrences of δ in α.

Proposition 2.1 (Uniform substitution on *CPL*-theorem instances)
If α is an instance of a *CPL*-theorem (i.e. if $\alpha=\beta[p_1/\gamma_1,\dots,p_m/\gamma_m]$, where β is a *CPL*-theorem and all γ_i, for $1\le i\le m$, are arbitrary formulas in *For*), then $\vdash_{K_{n,C}}\alpha$.

Proof. Let the sequence of *CPL*-formulas $\beta_1,\beta_2,\dots,\beta_l$ be a proof of β in *CPL*. In such a proof there are no restrictions on axioms and rules. Thus, the sequence $\beta_1[p_1/\gamma_1,\dots,p_m/\gamma_m],\beta_2[p_1/\gamma_1,\dots,p_m/\gamma_m],\dots,\beta_l[p_1/\gamma_1,\dots,p_m/\gamma_m]$ is a proof of $\alpha=\beta[p_1/\gamma_1,\dots,p_m/\gamma_m]$ in $K_{n,C}$. ■

Proposition 2.2 (Uniform substitution restricted by the complexity function)
If $\vdash_{K_{n,C}}\alpha$ and $C(\beta_i)\le C(p_i)$, for $1\le i\le m$, then $\vdash_{K_{n,C}}\alpha[p_1/\beta_1,\dots,p_m/\beta_m]$.

Proof. Let the sequence of formulas $\alpha_1,\alpha_2,\dots,\alpha_l$ be a proof of α in $K_{n,C}$, then $C(\alpha_j[p_1/\beta_1,\dots,p_m/\beta_m])\le C(\alpha_j)$ for any $1\le j\le l$ (due to the fact that $C(\beta_i)\le C(p_i)$, for $1\le i\le m$). Consequently, all restrictions on applications of axiom ($\mathrm{K}_{n,C}$) and rule ($\mathrm{Nec}_{n,C}$) satisfied in the sequence $\alpha_1,\alpha_2,\dots,\alpha_l$ are also satisfied in the sequence $\alpha_1[p_1/\beta_1,\dots,p_m/\beta_m],\alpha_2[p_1/\beta_1,\dots,p_m/\beta_m],\dots,\alpha_l[p_1/\beta_1,\dots,p_m/\beta_m]$, and such a sequence constitutes a proof of $\alpha[p_1/\beta_1,\dots,p_m/\beta_m]$ in $K_{n,C}$. ■

Lemma 2.3
If $\vdash_{K_{n,C}}\delta\leftrightarrow\beta$ and $C(\delta\to\beta)<n$ then $\vdash_{K_{n,C}}\Box\delta\leftrightarrow\Box\beta$.

Proof. From $\vdash_{K_{n,C}}\delta\leftrightarrow\beta$, by *CPL*, it follows that $\vdash_{K_{n,C}}\delta\to\beta$ and $\vdash_{K_{n,C}}\beta\to\delta$. Taking into account that $C(\beta\to\delta)=C(\delta\to\beta)<n$, by ($\mathrm{Nec}_{n,C}$) and ($\mathrm{K}_{n,C}$) we have that $\vdash_{K_{n,C}}\Box(\beta\to\delta)$, $\vdash_{K_{n,C}}\Box(\delta\to\beta)$, $\vdash_{K_{n,C}}\Box(\beta\to\delta)\to(\Box\beta\to\Box\delta)$ and $\vdash_{K_{n,C}}\Box(\delta\to\beta)\to(\Box\delta\to\Box\beta)$. Thus, by *CPL*, it follows that $\vdash_{K_{n,C}}\Box\delta\leftrightarrow\Box\beta$. ■

Proposition 2.4 (Substitution of equivalents restricted by the complexity function)
If $\vdash_{K_{n,C}}\alpha$, $\vdash_{K_{n,C}}\delta\leftrightarrow\beta$ and $C(\alpha\to\alpha(\delta/\beta))\le n$ then $\vdash_{K_{n,C}}\alpha(\delta/\beta)$.

[4] In order to C validate the condition (C1) in all the examples, it will be considered hereafter that $C(r)=0$, for a propositional variable r that would not explicitly appear in the examples.

PROOF. By induction on the structure of α: if α is a propositional variable p, since $K_{n,C}$ is a restriction of K and $\nvdash_K p$, then $\nvdash_{K_{n,C}} p$. Consequently, the hypothesis of the proposition is not validated and the implication is fulfilled. In the inductive step, if $\alpha = \Box\delta$ then $C(\delta \to \beta) < C(\alpha \to \alpha(\delta/\beta)) \leq n$, and by Lemma (2.3) and *CPL* it follows that $\vdash_{K_{n,C}} \Box\beta$. The other cases are just like in *CPL*. ■

Restricted versions of many theorems of K can be proven in $K_{n,C}$. Some instances are shown in the following:

PROPOSITION 2.5
If $C(\alpha) < n$ and $C(\beta) < n$ then:

$$\vdash_{K_{n,C}} \alpha \to \beta \text{ implies } \vdash_{K_{n,C}} \Box\alpha \to \Box\beta, \tag{1}$$

$$\vdash_{K_{n,C}} \alpha \to \beta \text{ implies } \vdash_{K_{n,C}} \Diamond\alpha \to \Diamond\beta, \tag{2}$$

$$\vdash_{K_{n,C}} \Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta), \tag{3}$$

$$\vdash_{K_{n,C}} \Diamond(\alpha \vee \beta) \leftrightarrow (\Diamond\alpha \vee \Diamond\beta), \tag{4}$$

$$\vdash_{K_{n,C}} (\Box\alpha \vee \Box\beta) \to \Box(\alpha \vee \beta), \tag{5}$$

$$\vdash_{K_{n,C}} \Diamond(\alpha \wedge \beta) \to (\Diamond\alpha \wedge \Diamond\beta). \tag{6}$$

PROOF. Similar to the demonstrations in K, restrictions of $(\mathrm{K}_{n,C})$ and $(\mathrm{Nec}_{n,C})$ are validated due to the assumptions that $C(\alpha) < n$ and $C(\beta) < n$. ■

It is important to point out that all items in Proposition 2.5 are valid only under the condition that $C(\alpha) < n$ and $C(\beta) < n$, if $C(\alpha) \geq n$ or $C(\beta) \geq n$, then none of the items are demonstrable (this is a consequence of Theorem 2.13).

Up to here, we have defined $K_{n,C}$ in a syntactic fashion. In the following, the well-known 'relational semantics' (or 'possible worlds semantics') will be adapted in order to obtain an adequate semantics for $K_{n,C}$, the new semantics will be called *levelled possible worlds semantics*.

In the relational semantics, a *frame* is a structure $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set (whose elements are called *worlds*), and R is a binary relation on W (called the *accessibility relation*). A *model* is a structure $\mathcal{M} = \langle W, R, V \rangle$, where $\langle W, R \rangle$ is a frame and V is a valuation function assigning truth values ('false' represented by 0 and 'true' represented by 1) to all formulas of K in each world (i.e. $V: For \times W \to \{0,1\}$), validating the following conditions: (i) $V(\neg\alpha, w_i) = 1$ if and only if $V(\alpha, w_i) = 0$; (ii) $V(\alpha \to \beta, w_i) = 1$ if and only if $V(\alpha, w_i) = 0$ or $V(\beta, w_i) = 1$; and (iii) $V(\Box\alpha, w_i) = 1$ if and only if $V(\alpha, w_j) = 1$ for all w_j such that $w_i R w_j$. A formula α is *valid in a model* $\mathcal{M} = \langle W, R, V \rangle$ if $V(\alpha, w_i) = 1$ in all worlds $w_i \in W$, and is *valid* (in general) if it is valid in all models. These notions are adapted in the following definitions:

DEFINITION 2.6 (L-frame)
An *L-frame* is a structure $\mathcal{F} = \langle W, L, R \rangle$, where $\langle W, R \rangle$ is a frame (as defined above), L is a function $L: W \to \mathbb{N}$ assigning *levels* (represented by natural numbers) to worlds, and R is restricted by the following condition: if $w_i R w_j$ then $L(w_i) = L(w_j) + 1$, for any $w_i, w_j \in W$.

DEFINITION 2.7 (L_C-model)
An *L_C-model* is a structure $\mathcal{M} = \langle W, L, R, V \rangle$, where $\langle W, L, R \rangle$ is an L-frame and V is an L_C-*valuation function* $V: For \times W \to \{0,1\}$ which assigns truth values to all formulas of $K_{n,C}$ in each world validating the following conditions:

(1) $V(\neg\alpha, w_i) = 1$ if and only if $V(\alpha, w_i) = 0$;
(2) $V(\alpha \to \beta, w_i) = 1$ if and only if $V(\alpha, w_i) = 0$ or $V(\beta, w_i) = 1$; and

(3) if $C(\alpha) < L(w_i)$, then $V(\Box\alpha, w_i) = 1$ if and only if $V(\alpha, w_j) = 1$ for all w_j such that $w_i R w_j$.

An L_C*-model* $\mathcal{M} = \langle W, L, R, V\rangle$ *is based* on an L-frame $\mathcal{F}$ if and only if $\mathcal{F}$ is just $\langle W, L, R\rangle$.

Note that, by the third condition of the previous definition, an L_C-valuation function is not totally determined by the values assigned to the propositional variables; i.e. given a function $f : \mathcal{V} \times W \to \{0,1\}$ (where $\mathcal{V}$ is the fixed set of propositional variables), the extension of f to an L_C-valuation function is not unique (since values assigned by L_C-valuation functions to formulas $\Box\alpha$ and worlds w_i such that $C(\alpha) \geq L(w_i)$ are not ruled by any condition).

Definition 2.8 (n_C-validity)
A formula α is n_C*-valid in an* L_C*-model* $\mathcal{M} = \langle W, L, R, V\rangle$ (which will be denoted by $\mathcal{M} \models_{n,C} \alpha$) if $V(\alpha, w_i) = 1$ in all worlds $w_i \in W$ such that $L(w_i) = n$. Accordingly, α is n_C*-valid* (in general) if α is n_C-valid in all L_C-models (i.e. if $\mathcal{M} \models_{n,C} \alpha$ for every L_C-model $\mathcal{M}$). Moreover, α is n_C*-valid in an* L*-frame* $\mathcal{F}$ if it is n_C-valid in all models based on $\mathcal{F}$.

Soundness and completeness of $K_{n,C}$ with respect to the levelled worlds semantics (Theorems 2.13 and 2.14) will be proven in a constructive way. The proofs are obtained by first adapting the method of relational tableaux (described in [1]).

We will call $K_{n,C}$*-tableau* for a formula α (called the *input of the tableau*) a diagram composed of rectangles, with formulas and truth values inside, and directed arrows among some rectangles. The diagram is build by following an algorithmic procedure described below. In a $K_{n,C}$-tableau, rectangles represent 'fragments' of worlds, arrows represent the accessibility relation and values under formulas represent the truth values associated with formulas by the L_C-valuation function. A label w_i^n on the right side of a rectangle indicates that such a rectangle represents the world w_i and that $L(w_i) = n$. A rectangle is called *contradictory* if any formula inside it receives both truth values. The aim of depicting a $K_{n,C}$-tableau for a formula α is to obtain a graphical representation of an attempt to build an L_C-model that falsifies the n_C-validity of α, the algorithm to build such a representation is described in the following steps (considering, without loss of generality, that α only contains primitive connectives):

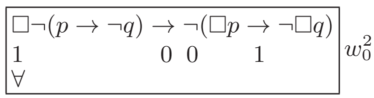
(1) Depict a first rectangle with label w_0^n and write the formula α inside with value 0 under its main connective.
(2) While there are subformulas to which truth values can be assigned and there are no contradictory rectangles, do the following:
  (a) derive truth values of subformulas applying, as far as possible, conditions 1 and 2 for L_C-valuation functions and write them under the main connective of the respective subformulas.
  (b) if new truth values were written in the previous step then:
    (i) if value 0 (respectively, value 1) is written under a connective $\Box$ in a rectangle labelled with w_i^m, the subformula under the scope of $\Box$ is β and $C(\beta) < m$, then write $\exists$ (respectively, $\forall$) under the truth value.
    (ii) for each symbol $\exists$ written in the previous step in a rectangle labelled with w_i^m, depict a new rectangle labelled with w_j^{m-1} (where j is a new consecutive number), write inside the subformula β (in the scope of the respective $\Box$) and 0 under its main connective, and draw a directed arrow from w_i^m to w_j^{m-1}.
    (iii) for each new rectangle and for each symbol $\forall$ in the rectangles accessing them, write in the new world the subformula β in the scope of the respective $\Box$ (marked with $\forall$) and write 1 under its main connective.

(c) otherwise (if no new truth values were written in step (a)) then take *alternatives* if possible; i.e. if there is a subformula $\beta \rightarrow \delta$ with assigned truth value 1, create three copies of the $K_{n,C}$-tableau and in each copy consider a different possible assignment of truth values to β and δ. Continue the loop for each alternative.

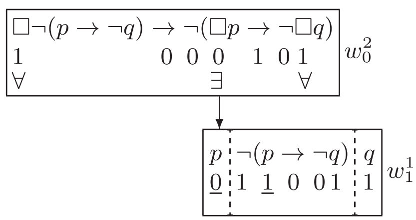
(3) if all alternatives finish with contradictory rectangles, then the $K_{n,C}$-tableau for α is *closed* and α is n_C-valid (because any attempt to build an L_C-model that falsifies the n_C-validity of α leads to contradictions). Otherwise, the $K_{n,C}$-tableau for α is *open* and alternatives without contradictory rectangles represent L_C-models that falsify the n_C-validity of α.

EXAMPLE 2.9
The $K_{2,C}$-tableau for $\delta = \Box\neg(p \rightarrow \neg q) \rightarrow \neg(\Box p \rightarrow \neg\Box q)$ (which by the definition of $\wedge$ is equivalent to $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$), in two different instants of its construction and considering that $C(p) = C(q) = 1$, is depicted below:

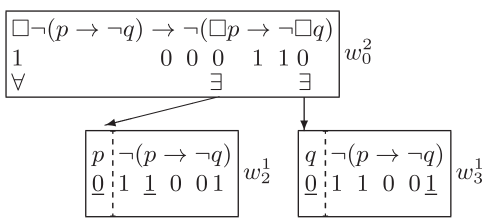
- before taking alternatives the diagram obtained is:



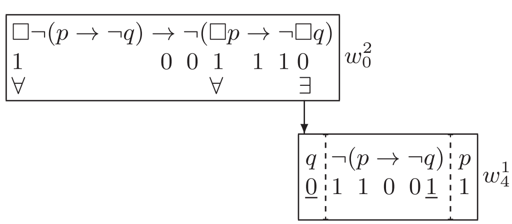
- after taking alternatives the following diagrams are obtained:
Alternative 1:



Alternative 2:



Alternative 3:



Note that in all alternatives there are contradictions (marked by underlines), thus the $K_{2,C}$-tableau for δ is closed and δ is 2_C-valid.

If we consider $C(p)=C(q)=2$ (instead of $C(p)=C(q)=1$), then none of the symbols $\exists$ and $\forall$ can be written in the $K_{2,C}$-tableau for δ (since in this case, $C(\neg(p\to\neg q))=C(p)=C(q)=2\not< L(w_0)=2$), and no new rectangles can be depicted. In this case, all the three alternatives consist of a single rectangle (the rectangle depicted before taking alternatives), each of them assigning different possible values to $\Box p$ and $\neg\Box q$. Each of these alternatives represents an L_C-model falsifying the 2_C-validity of δ.

Definition 2.10 (Chain)
In a $K_{n,C}$-tableau, a *chain of length m* is a sequence of rectangles $w_0^n, w_1^{n-1}, \ldots, w_m^{n-m}$ such that w_i is connected to w_{i+1} $(0\le i<m)$ by a directed arrow.

Due to the fact that w_iRw_j only if $L(w_i)=L(w_j)+1$, it is clear that the longest chain in a $K_{n,C}$-tableau for any formula α is of length n. Depending on the complexity of α we also have that:

Proposition 2.11
If $C(\alpha)<n$ the longest chain in the $K_{n,C}$-tableau for α is of length $n-1$.

Proof. If $C(\alpha)<n$ the greatest number of connectives $\Box$ in α is $n-1$ (due to condition (C4) in the definition of complexity function), and these are the only connectives that can generate new rectangles in the $K_{n,C}$-tableau for α. ■

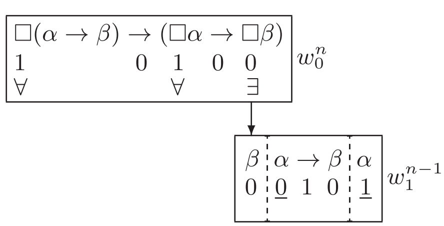
Lemma 2.12
If α is n_C-valid and $C(\alpha)<n$ then α is $(n-1)_C$-valid.

Proof. Let α be an n_C-valid formula, then the $K_{n,C}$-tableau $\mathcal{T}$ for α is closed. By Proposition 2.11, the longest chain in $\mathcal{T}$ is of length $n-1$, then the least level of a world in the diagram is 1. By decreasing in 1 the level of each world in $\mathcal{T}$ it is obtained a closed $K_{n-1,C}$-tableau $\mathcal{T}'$ for α, therefore α is $(n-1)_C$-valid. Symbols $\exists$ and $\forall$ in $\mathcal{T}'$ are justified due to the fact that $C(\alpha)<n$ implies that $C(\beta)<n-1$ for any subformula β of α in the scope of a $\Box$. ■

Theorem 2.13 (Soundness of $K_{n,C}$)
If $\vdash_{K_{n,C}}\alpha$ then α is n_C-valid.

Proof. The n_C-validity of *CPL*-axioms and the preservation of the n_C-validity for the *modus ponens* rule are direct consequences of conditions 1 and 2 for L_C-valuation functions (Definition 2.7). Then, it is enough to prove that $(\mathrm{K}_{n,C})$ is n_C-valid and that $(\mathrm{Nec}_{n,C})$ preserves n_C-validity:

- $(\mathrm{K}_{n,C})$ is n_C-valid: the $K_{n,C}$-tableau for any valid instance of $(\mathrm{K}_{n,C})$ is closed, which is shown in the following diagram (condition $C(\alpha\to\beta)<n$ implies that $n\ge 1$):



- $(\mathrm{Nec}_{n,C})$ preserves n_C-validity: let α be n_C-valid and $C(\alpha)<n$. Suppose, by way of contradiction, that $\Box\alpha$ is not n_C-valid. Thus, there exists an L_C-model $\mathcal{M}=\langle W,L,R,V\rangle$ with a world $w_i\in W$ such that $L(w_i)=n$ and $V(\Box\alpha,w_i)=0$. Due to condition 3 for L_C-valuation

functions (Definition 2.7), there exists a world $w_j \in W$ such that $L(w_j) = n-1$, $w_i R w_j$ and $V(\alpha, w_j) = 0$. Consequently, α is not $(n-1)_C$-valid, which contradicts Lemma 2.12.

■

THEOREM 2.14 (Completeness of $K_{n,C}$)
If α is n_C-valid then $\vdash_{K_{n,C}} \alpha$.

PROOF. Let α be an n_C-valid formula, thus the $K_{n,C}$-tableau $\mathcal{T}$ for α is closed. A proof for α in $K_{n,C}$, based on $\mathcal{T}$, is constructed by performing the following steps:

(1) taking into account that any rectangle w_i^m in a $K_{n,C}$-tableau has the following general format:

$$\begin{array}{|c:c:c:c|} \beta & \delta_1 & \dots & \delta_l \\ 0 & 1 & & 1 \end{array}\; w_i^m$$

associate with w_i^m a *characteristic formula* $\chi_i = \beta \vee \neg\delta_1 \vee \ldots \vee \neg\delta_l$ (some rectangles w_i^m have inside only the formula β, in such cases $\chi_i = \beta$; a particular case is χ_0 which is just α). By conditions of classical valuations, for any subformula γ of χ_i receiving an initial[5] or derived truth value (but not an alternative truth value) in w_i^m we have the following cases:

(a) if γ has associated truth value 0 in w_i^m, then $\gamma \to \chi_i$ is an instance of a *CPL*-tautology.
(b) if γ has associated truth value 1 in w_i^m, then $\neg\gamma \to \chi_i$ is an instance of a *CPL*-tautology.

(2) Since $\mathcal{T}$ is closed, for any alternative of $\mathcal{T}$ a contradictory rectangle w_i^m exists. Let γ be a formula receiving both truth values in w_i^m, by the previous item, $\gamma \to \chi_i$ and $\neg\gamma \to \chi_i$ are instances of *CPL*-tautologies, thus χ_i is a *CPL*-tautology. By the completeness of *CPL*, χ_i is an instance of a *CPL*-theorem, therefore $\vdash_{K_{n,C}} \chi_i$.

(3) If in $\mathcal{T}$ we have that $w_i^m R w_j^{m-1}$, no alternatives were taken in w_i^m and $\vdash_{K_{n,C}} \chi_j$, then $\vdash_{K_{n,C}} \chi_i$: the formula $\chi_j = \beta \vee \neg\delta_1 \vee \ldots \vee \neg\delta_l$ is equivalent (by *CPL*) to $(\delta_1 \wedge \ldots \wedge \delta_l) \to \beta$. By condition 3 of Definition 2.7, each of the formulas $\beta, \gamma_1, \ldots, \gamma_l$ has to have complexity less than $L(w_i)$, therefore less than n, thus by (Nec$_{n,C}$) we have that $\vdash_{K_{n,C}} \Box((\delta_1 \wedge \ldots \wedge \delta_l) \to \beta)$. By applying (K$_{n,C}$) and *modus ponens* it follows that $\vdash_{K_{n,C}} \Box(\delta_1 \wedge \ldots \wedge \delta_l) \to \Box\beta$, which is equivalent by Proposition 2.5 (item 3) and *CPL* to (i) $\vdash_{K_{n,C}} \Box\beta \vee \neg\Box\delta_1 \vee \ldots \vee \neg\Box\delta_l$. Taking into account that $\Box\beta$ has to be a subformula of χ_i receiving value 0 in w_i^m, by item 1, we have that $\Box\beta \to \chi_i$ is an instance of a *CPL*-tautology, thus (ii) $\vdash_{K_{n,C}} \Box\beta \to \chi_i$. In a similar way, taking into account that formulas $\Box\gamma_1, \ldots, \Box\gamma_l$ have to be subformulas of χ_i receiving value 1 in w_i^m, we have that (iii) $\vdash_{K_{n,C}} \neg\Box\gamma_1 \to \chi_i, \ldots, \vdash_{K_{n,C}} \neg\Box\gamma_l \to \chi_i$. From (i), (ii) and (iii) it follows that $\vdash_{K_{n,C}} \chi_i$.
When alternatives for a formula $\varphi \to \psi$ in a rectangle w_i^m are taken, values assigned to φ and ψ are considered initial values, thus the following formulas are defined: $\chi_i(\mathrm{i}) = \chi_i \vee \varphi \vee \psi$ (for the alternative where φ and ψ receive value 0); $\chi_i(\mathrm{ii}) = \chi_i \vee \varphi \vee \neg\psi$ (for the alternative where φ receives value 0 and ψ receives value 1); and $\chi_i(\mathrm{iii}) = \chi_i \vee \neg\varphi \vee \neg\psi$ (for the alternative where φ and ψ receive value 1). By departing from the formulas proven in the previous item, and iterating the procedure described in the previous paragraph, it follows that $\vdash_{K_{n,C}} \chi_i(\mathrm{i})$, $\vdash_{K_{n,C}} \chi_i(\mathrm{ii})$ and $\vdash_{K_{n,C}} \chi_i(\mathrm{iii})$. Which, in conjunction with $\vdash_{K_{n,C}} \neg(\varphi \to \psi) \to \chi_i$, leads to $\vdash_{K_{n,C}} \chi_i$ (due to the fact that $(\neg(\varphi \to \psi) \to \chi_i) \to (((\chi_i \vee \varphi \vee \psi) \wedge (\chi_i \vee \varphi \vee \neg\psi) \wedge (\chi_i \vee \neg\varphi \vee \neg\psi)) \to \chi_i)$ is an instance of a *CPL*-tautology).

[5] Formulas receiving initial truth values in w_i^m are just $\beta, \delta_1, \ldots, \delta_l$.

The proof of $\vdash_{K_{n,C}} \alpha$ is then constructed by iterating the procedures described above until reaching $\vdash_{K_{n,C}} \chi_0$.

■

3 Extending $K_{n,C}$ in a monomodal fashion

It is well known that the modal system K can be extended with an arbitrary finite number of axioms, yielding the so-called family of 'normal modal logics'. Moreover, there are correspondences among modal axioms extending K and properties of the accessibility relation of the frames where such axioms are valid. Some of these correspondences are:

Axiom	**Property of the accessibility relation**
(D): $\Box\alpha \to \Diamond\alpha$,	R is serial: $\forall w \exists w'(wRw')$;
(T): $\Box\alpha \to \alpha$,	R is reflexive: $\forall w(wRw)$;
(4): $\Box\alpha \to \Box\Box\alpha$,	R is transitive: $\forall w, w', w''((wRw' \wedge w'Rw'') \to wRw'')$;
(B): $\alpha \to \Box\Diamond\alpha$,	R is symmetric: $\forall w, w'(wRw' \to w'Rw)$;
(5): $\Diamond\alpha \to \Box\Diamond\alpha$,	R is Euclidean: $\forall w, w', w''((wRw' \wedge wRw'') \to w'Rw'')$.

In a similar way, $K_{n,C}$ can be extended by adding restricted versions of the axioms above:

$$\Box\alpha \to \Diamond\alpha \text{ if } C(\alpha)+1 \leq n, \tag{$\mathrm{D}_{n,C}$}$$

$$\Box\alpha \to \alpha \text{ if } C(\alpha)+1 \leq n, \tag{$\mathrm{T}_{n,C}$}$$

$$\Box\alpha \to \Box\Box\alpha \text{ if } C(\alpha)+2 \leq n, \tag{$4_{n,C}$}$$

$$\alpha \to \Box\Diamond\alpha \text{ if } C(\alpha)+2 \leq n, \tag{$\mathrm{B}_{n,C}$}$$

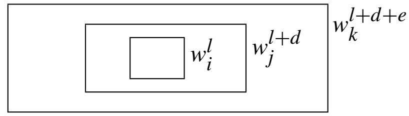
$$\Diamond\alpha \to \Box\Diamond\alpha \text{ if } C(\alpha)+2 \leq n. \tag{$5_{n,C}$}$$

Following the conventional naming of normal modal systems, if $(X_1),\ldots,(X_m)$ are labels of restricted versions of modal axioms, all based in the same depth n and complexity function C, then $K_{n,C}X_1 \ldots X_m$ will be the name of the restricted normal modal system obtained by adding to $K_{n,C}$ the axioms $(X_1),\ldots,(X_m)$ (subscript n, C is avoided from labels X_i in the name of the system in order to simplify notation). For instance, $K_{n,C}T4$ will denote the extension of $K_{n,C}$ with axioms ($\mathrm{T}_{n,C}$) and ($4_{n,C}$).

In order to establish correspondences among these axioms and properties of the accessibility relation of L-frames, we will first present some definitions (where $\mathcal{M} = \langle W, L, R, V \rangle$ is an arbitrary L_C-model):

DEFINITION 3.1 (Embedded world)
A world w_i is *embedded in depth d* into a world w_j in $\mathcal{M}$ ($w_i, w_j \in W$ and $d \in \mathbb{N}$), which will be denoted by $w_i \subseteq_d w_j$, if and only if $L(w_j) - L(w_i) = d$ and $V(\alpha, w_i) = V(\alpha, w_j)$ for any formula α such that $C(\alpha) \leq L(w_i)$.

In the tableaux for restricted normal modal logics, $w_i \subseteq_d w_j$ will be graphically represented by drawing w_i inside of w_j. The depth of the embedded can be easily deduced from its graphical representation, due to the fact that the labels of the rectangles include the levels of the worlds that they represent. For instance, the following diagram illustrates that $w_i \subseteq_d w_j \subseteq_e w_k$, considering that $L(w_i) = l$:

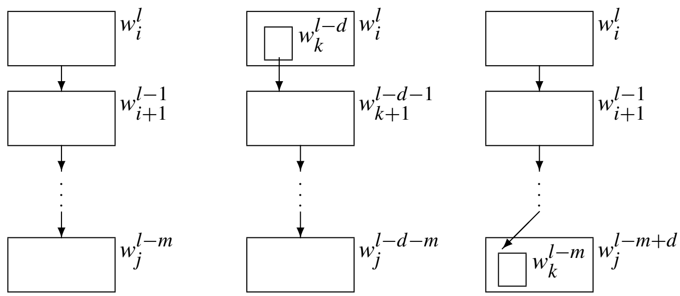


DEFINITION 3.2 (Accessibility in m steps)
A world w_i *accesses* a world w_j *in m steps*, which will be denoted by $w_iR^mw_j$, if there exist worlds $w_{i+1}, w_{i+2}, \ldots, w_{i+m}$ in W such that w_kRw_{k+1} for $i \leq k < i+m$ and $w_{i+m} = w_j$. When $m=0$, the expression $w_iR^0w_j$ means just that $w_i = w_j$.

DEFINITION 3.3 (L-accessibility in m steps)
A world w_i *L-accesses* a world w_j *in m steps*, which will be denoted by $w_iR_L^mw_j$, if and only if:

- $L(w_i) - L(w_j) = m$ and $w_iR^mw_j$; or
- $L(w_i) - L(w_j) > m$ and there exist $w_k \in W$ such that $w_k \subseteq_d w_i$, where $d = (L(w_i) - L(w_j)) - m$, and $w_kR^mw_j$; or
- $L(w_i) - L(w_j) < m$ and there exist $w_k \in W$ such that $w_k \subseteq_d w_j$, where $d = m - (L(w_i) - L(w_j))$, and $w_iR^mw_k$.

In the tableaux, $w_iR_L^mw_j$ will be represented in the following way (figures (a), (b) and (c) correspond, respectively, to the cases where $L(w_i) - L(w_j) = m$, $L(w_i) - L(w_j) > m$ and $L(w_i) - L(w_j) < m$):



(a) (b) (c)

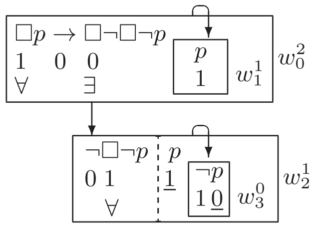
Now we can define the following properties on the accessibility relation of L-frames:

- R is *n-serial* if $\forall w(1 \leq L(w) \leq n \rightarrow \exists w'(wRw'))$,
- R is *n-reflexive* if $\forall w(1 \leq L(w) \leq n \rightarrow wR_L^1w)$,
- R is *n-transitive* if $\forall w, w', w''((2 \leq L(w) \leq n \wedge wRw' \wedge w'Rw'') \rightarrow wR_L^1w'')$,
- R is *n-symmetric* if $\forall w, w'((2 \leq L(w) \leq n \wedge wRw') \rightarrow w'R_L^1w)$,
- R is *n-Euclidean* if $\forall w, w', w''((2 \leq L(w) \leq n \wedge wRw' \wedge wRw'') \rightarrow w'R_L^1w'')$.

Correspondences among axioms ($\mathrm{D}_{n,C}$)–($5_{n,C}$) and properties of the accessibility relation of L-frames, similar to the described above for axioms (D)–(5), can be established (we will prove such correspondences in a more general framework below).

A $K_{n,C}X_1 \ldots X_m$-tableau (where $(X_1)\ldots(X_m)$ are axioms in (D)–(5)) for a formula α is built by following the algorithm described above to construct $K_{n,C}$-tableaux, and depicting obligatory arrows and embedded worlds in each iteration (in accordance with the correspondences among axioms and conditions on the accessibility relation). In order to illustrate this, the $K_{n,C}T$-tableau for $\Box p \rightarrow$

$\Box\neg\Box\neg p$ is depicted in the next diagram (assuming that $C(p)=0$):



This $K_{n,C}T$-tableau shows that $\Box p\to\Box\neg\Box\neg p$ is 2_C-valid in all n-reflexive frames.

Soundness and completeness of extensions of $K_{n,C}$, with different combinations of axioms, can be proven in a constructive way; similarly as for normal modal logics. However, in the procedure to obtain a proof of a formula α based on a closed $K_{n,C}X_1\ldots X_m$-tableau for it, in order to prove completeness, it is necessary to take into account axioms $(X_1),\ldots,(X_m)$, and the construction of the proof becomes more complicated depending on the number of axioms added. Due to this fact, we will prove soundness and completeness of an infinite family of extensions of $K_{n,C}$ at once, by adapting the (non-constructive) method of proof used by Lemmon and Scott in [3] (this method is also described in [1, Chapter 4], and our adaptation follows this description very closely).

The following is a *restricted* version of the so-called 'Lemmon–Scott axiomatic schema' (superscripts i, j, k and l represents natural numbers and stand for the number of iterated operators of the same kind):

$$G^{i,j,k,l}_{n,C}:\Diamond^i\Box^j\alpha\to\Box^k\Diamond^l\alpha \text{ if } C(\alpha)+\max\{i+j,k+l\}\le n.$$

Note that $(D_{n,C})$–$(5_{n,C})$ are all instances of $G^{i,j,k,l}_{n,C}$: $D_{n,C}=G^{0,1,0,1}_{n,C}$; $T_{n,C}=G^{0,1,0,0}_{n,C}$; $4_{n,C}=G^{0,1,2,0}_{n,C}$; $B_{n,C}=G^{0,0,1,1}_{n,C}$ and $5_{n,C}=G^{1,0,1,1}_{n,C}$. The schema of conditions on the accessibility relation R corresponding to $G^{i,j,k,l}_{n,C}$ is:

$$C^{i,j,k,l}_{n,L}:\forall w_0,w_1,w_2((\max\{i+j,k+l\}\le L(w_0)\le n\wedge w_0R^iw_1\wedge w_0R^kw_2)\to\exists w_3(w_1R^j_Lw_3\wedge w_2R^l_Lw_3)).$$

It can be easily proven that the properties defined above on the accessibility relation of L-frames are equivalent, respectively, to $C^{0,1,0,1}_{n,L}$, $C^{0,1,0,0}_{n,L}$, $C^{0,1,2,0}_{n,L}$, $C^{0,0,1,1}_{n,L}$ and $C^{1,0,1,1}_{n,L}$.

Considering that $L(w_0)=m$, the schema $C^{i,j,k,l}_{n,L}$ is graphically represented by the following diagram:

$$\begin{tikzcd} & w_0^m \arrow[dl, "R^i"'] \arrow[dr, "R^k"] & \\ w_1^{m-i} \arrow[dr, "R^j_L"'] & & w_2^{m-k} \arrow[dl, "R^l_L"] \\ & w_3^{m-\max\{i+j,k+l\}} & \end{tikzcd}$$

Depending on the values of j and l, the existence of embedded worlds is necessary. For instance, the property of L-transitivity, which corresponds to $C^{0,1,2,0}_{n,L}$, is valid if $2\le L(w_0)\le n$, $w_0=w_1$, $w_2=w_3$, $w_0R^2w_2$, there exists a world w_4 such that $w_4\subseteq_1 w_1$ and $w_4R^1w_3$ (i.e. $w_1R^1_Lw_3$).

Now, we will proceed to prove soundness and completeness for any extension of $K_{n,C}$ obtained by adding a finite number of instances of $G^{i,j,k,l}_{n,C}$.

LEMMA 3.4
If β is an instance of $G^{i,j,k,l}_{n,C}$, then β is n_C-valid in all L-frames in which the accessibility relation satisfies the condition $C^{i,j,k,l}_{n,L}$.

PROOF. Let β be an instance of $G^{i,j,k,l}_{n,C}$; i.e. $\beta = \Diamond^i \Box^j \alpha \to \Box^k \Diamond^l \alpha$ (for some fixed i,j, k and l) and $C(\beta) = C(\alpha) + \max\{i+j, k+l\} \leq n$. Suppose that β is not n_C-valid in some L-frame $\mathcal{F} = \langle W, L, R\rangle$ which satisfies condition $C^{i,j,k,l}_{n,L}$. Then, there exists an L_C-model $\mathcal{M} = \langle W, L, R, V\rangle$, based on $\mathcal{F}$, in which β is not n_C-valid. Consequently, there exists a world $w_0 \in W$, with $L(w_0) = n$, such that $V(\Diamond^i \Box^j \alpha, w_0) = 1$ and $V(\Box^k \Diamond^l \alpha, w_0) = 0$. By conditions of L_C-valuation functions, there exist worlds $w_1, w_2 \in W$, such that $w_0 R^i w_1$, $w_0 R^k w_2$, $V(\Box^j \alpha, w_1) = 1$ and $V(\Diamond^l \alpha, w_2) = 0$. Since $\mathcal{F}$ satisfies condition $C^{i,j,k,l}_{n,L}$, there exists a world $w_3 \in W$ such that $w_1 R^j_L w_3$ and $w_2 R^l_L w_3$. Taking into account that $C(\Box^j \alpha) \leq C(\beta) - i \leq n - i$, it follows from $V(\Box^j \alpha, w_1) = 1$, $w_1 R^j_L w_3$ and the conditions of L_C-valuation functions that $V(\alpha, w_3) = 1$. In a similar way, it follows from $V(\Diamond^l \alpha, w_2) = 0$ and $w_2 R^l_L w_3$ that $V(\alpha, w_3) = 0$, leading to a contradiction. Consequently, β has to be n_C-valid in any L-frame satisfying condition $C^{i,j,k,l}_{n,L}$. ■

In order to simplify notation, hereafter we will write G^* to denote some specific instance of $G^{i,j,k,l}_{n,C}$ and C^* to denote the corresponding condition on the accessibility relation. Different instances of G^* and C^* will be distinguished by subscripts.

Soundness of $K_{n,C}G^*_1 \ldots G^*_m$ is then a direct consequence of Lemma 3.4:

THEOREM 3.5 (Soundness of $K_{n,C}G^*_1 \ldots G^*_m$)
If α is a theorem of $K_{n,C}G^*_1 \ldots G^*_m$ then α is n_C-valid in all L-frames in which the accessibility relation jointly satisfies the corresponding conditions $C^*_1 \ldots C^*_m$.

Before proving completeness some definitions and lemmas are in order.

DEFINITION 3.6 (m_C-maximal consistent extension)
A set X of formulas in *For* is an *m_C-maximal consistent extension* of a restricted normal modal system S if:

(1) $S' \subseteq X$ (where S' is the set of theorems of S; i.e. $S' = \{\alpha : \vdash_S \alpha\}$);
(2) X is S-consistent (i.e. there is no formula α such that $X \vdash_S \alpha$ and $X \vdash_S \neg\alpha$); and
(3) for every formula α in *For*, if $C(\alpha) \leq m$ then $\alpha \in X$ or $\neg\alpha \in X$.

Moreover, an m_C-maximal consistent extension X is *strict* if for any formula $\beta \in For$ such that $C(\beta) > m$ we have that $\beta \in X$ if and only if $\vdash_S \beta$.

LEMMA 3.7 (Lindenbaum lemma)
Let S be a restricted normal modal system. For each S-consistent set of formulas X there exists at least one m_C-maximal consistent extension of S containing X, for any $m \in \mathbb{N}$.

PROOF. A maximal consistent extension of S containing X can be obtained by the standard Lindenbaum's construction, such an extension is m_C-maximal for any $m \in \mathbb{N}$. ■

LEMMA 3.8
Let S be a restricted normal modal system. If X is an S-consistent set of formulas in *For* such that $C(\alpha) \leq m$ for any $\alpha \in X$, then there exists at least one strict m_C-maximal consistent extension of S containing X.

PROOF. By Lemma 3.7, there exists at least one m_C-maximal consistent extension X' of S containing X. Thus, $X'' = X' \setminus \{\beta \mid C(\beta) > m \wedge \nvdash_S \beta\}$ is a strict m_C-maximal consistent extension of S containing X. ■

LEMMA 3.9
Let S be an extension of $K_{n,C}$ with a finite number of instances of $G^{i,j,k,l}_{n,C}$ (i.e., $S = K_{n,C}G_1^* \ldots G_u^*$), then there exists at least one strict m_C-maximal consistent extension of S for any $m \in \mathbb{N}$.

PROOF. Any extension of K with a finite number of instances of the Lemmon–Scott axiomatic schema is consistent (see [1]), then any extension S of $K_{n,C}$ with a finite number of instances of $G^{i,j,k,l}_{n,C}$ is also consistent (since these systems prove less axioms that their non-restricted versions). Consequently, by Lemma 3.8 (taking $X = \emptyset$), S has at least one strict m_C-maximal consistent extension for any $m \in \mathbb{N}$. ■

LEMMA 3.10
If $S = K_{n,C}G_1^* \ldots G_u^*$ and X is a strict m_C-maximal consistent extension of S, then X is not a strict l_C-maximal consistent extension of S for any $l \neq m$.

PROOF. Let X' be an arbitrary strict l_C-maximal consistent extension of S. By condition (C1) in the definition of complexity function, there exists a propositional variable p such that $C(p) = 0$. Then, for any $k \in \mathbb{N}$, we have that $\Box^k p$ and $\neg\Box^k p$ are not n_C-valid and $C(\Box^k p) = C(\neg\Box^k p) = k$. By Theorem 3.5, it follows that $\nvdash_S \Box^m p$, $\nvdash_S \neg\Box^m p$, $\nvdash_S \Box^{m+1} p$ and $\nvdash_S \neg\Box^{m+1} p$. If it is supposed that $l \neq m$, there are two options: (i) if $l < m$, then $\Box^m p \notin X'$ and $\neg\Box^m p \notin X'$, while $\Box^m p \in X$ or $\neg\Box^m p \in X$; consequently $X \neq X'$; and (ii) if $m < l$, then $\Box^{m+1} p \notin X$ and $\neg\Box^{m+1} p \notin X$, while $\Box^{m+1} p \in X'$ or $\neg\Box^{m+1} p \in X'$; therefore $X \neq X'$. In both cases $X \neq X'$. ■

PROPOSITION 3.11
If $S = K_{n,C}G_1^* \ldots G_u^*$, X is a strict m_C-maximal consistent extension of S and α, β are formulas in *For* such that $C(\alpha) \leq m$ and $C(\beta) \leq m$, then:

(1) if $\vdash_S \gamma$ then $\gamma \in X$ (independently of the value of $C(\gamma)$);
(2) $\alpha \wedge \beta \in X$ if and only if $\alpha \in X$ and $\beta \in X$;
(3) $\alpha \vee \beta \in X$ if and only if $\alpha \in X$ or $\beta \in X$;
(4) if $\alpha \in X$ and $\alpha \rightarrow \beta \in X$ then $\beta \in X$.

PROOF. Similar to the classical case. ■

DEFINITION 3.12 (Denecessitation)
Let X be a set of formulas in *For*, the *denecessitation* with complexity m of X is the set $\mathrm{Den}_{m,C}(X) = \{\alpha \mid \Box\alpha \in X \wedge C(\alpha) < m\}$.

By applying denecessitation i times to a set X it is obtained the set $\mathrm{Den}^i_{m,C}(X) = \{\alpha \mid \Box^i\alpha \in X \wedge C(\alpha) \leq m - i\}$.

DEFINITION 3.13 (Canonical L_C-model)
Given a restricted normal modal system S, the *canonical L_C-model* of S is the L_C-model $\mathcal{M} = \langle W, L, R, V \rangle$ where:

(1) $W = \{w \mid w$ is a strict m_C-maximal consistent extension of S, for some $m \in \mathbb{N}\}$;
(2) $L(w) = m$ if and only if w is a strict m_C-maximal consistent extension of S (by Lemma 3.10, L is well defined);
(3) $w_i R w_j$ if and only if $L(w_i) = m$, $m = L(w_j)+1$ and $\mathrm{Den}_{m,C}(w_i) \subseteq w_j$; and
(4) for each formula $\alpha \in For$ and each world w, $V(\alpha, w) = 1$ if and only if $\alpha \in w$ (if $\alpha \notin w$ then $V(\alpha, w) = 0$, independently if $\neg\alpha$ belongs or not to w).

The L-frame $\mathcal{F} = \langle W, L, R\rangle$ will be called the *canonical L-frame* of S.

By the definitions above, in the canonical L_C-model of a restricted normal modal system S we have that $wR^i w'$ if and only if $L(w) = L(w')+i$ and $\mathrm{Den}^i_{m,C}(w) \subseteq w'$.

LEMMA 3.14
Let α be a formula in For and S be an extension of $K_{n,C}$ with a finite number of instances of $G^{i,j,k,l}_{n,C}$, then $\vdash_S \alpha$ if and only if α is n_C-valid in the canonical L_C-model of S.

PROOF. Let $\mathcal{M} = \langle W, L, R, V\rangle$ be the canonical L_C-model of S. Suppose that $\vdash_S \alpha$, then $\alpha \in X$ for any strict n_C-maximal consistent extension of S (by Proposition 3.11, item 1). Consequently, for any world $w \in W$ such that $L(w) = n$ we have that $\alpha \in w$, then $V(\alpha, w) = 1$; i.e. α is n_C-valid in $\mathcal{M}$.

In the other direction, suppose that $\nvdash_S \alpha$, then $\{\neg\alpha\}$ is S-consistent. If $C(\alpha) > n$, then $\alpha \notin w$ for any $w \in W$ such that $L(w) = n$, and by Lemma 3.9 there exists at least one $w' \in W$ such that $L(w') = n$, consequently $V(\alpha, w') = 0$ and α is not n_C-valid in $\mathcal{M}$. If $C(\alpha) \leq n$, by Lemma 3.8, there exists a strict n_C-maximal consistent extension of S containing $\{\neg\alpha\}$. Such an extension is a world w in the canonical L_C-model of S, $L(w) = n$ and $V(\alpha, w) = 0$. Hence, α is not n_C-valid in the canonical L_C-model of S. ■

DEFINITION 3.15
A restricted normal modal system S is canonical with respect to a class of L-frames $\mathcal{F}$ if and only if the canonical L-frame of S belongs to $\mathcal{F}$.

LEMMA 3.16
Let S be any consistent restricted normal modal system containing $K_{n,C}G^*$, for some G^*. Then, the accessibility relation of the canonical L-frame of S satisfies condition C^*.

PROOF. Let $\mathcal{F} = \langle W, L, R\rangle$ be the canonical L-frame of S. Suppose that $w_0, w_1, w_2 \in W$, $L(w_0) = m$, $\max\{i+j, k+l\} \leq m \leq n$, $w_0 R^i w_1$ and $w_0 R^k w_2$; then $L(w_1) = m-i$ and $L(w_2) = m-k$. Consider the following set:

$$X = \{\alpha \mid \Box^j \alpha \in w_1 \wedge C(\alpha) \leq m - \max\{i+j, k+l\}\} \cup \\ \{\beta \mid \Box^l \beta \in w_2 \wedge C(\beta) \leq m - \max\{i+j, k+l\}\}.$$

We will first prove that X is S-consistent: suppose that X is S-inconsistent, then we have that there are formulas $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_h \in X$ such that $\vdash_S \neg(\alpha_1 \wedge \ldots \wedge \alpha_g \wedge \beta_1 \wedge \ldots \wedge \beta_h)$. By defining $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_g$ and $\beta = \beta_1 \wedge \ldots \wedge \beta_h$ we have that $\vdash_S \neg(\alpha \wedge \beta)$, which is equivalent to $\vdash_S \alpha \to \neg\beta$. Moreover, by the definition of X, $C(\alpha_u) \leq m-(i+j)$ (for $1 \leq u \leq g$) and $C(\beta_v) \leq m-(k+l)$ (for $1 \leq v \leq h$), then $C(\alpha) \leq m-(i+j)$ and $C(\beta) \leq m-(k+l)$. Taking into account that $\Box^j \alpha_u \in w_1$, $C(\Box^j \alpha_u) \leq m-i$ and w_1 is an $(m-i)_C$-maximal consistent extension of S, by Proposition 3.11 (item 2) $\Box^j \alpha_1 \wedge \ldots \wedge \Box^j \alpha_g \in w_1$; then by Proposition 2.5 (item 3) $\Box^j \alpha \in w_1$. By a similar argument $\Box^l \beta \in w_2$.

Now, since $\Box^j \alpha \in w_1$, $w_0 R^i w_1$ and $C(\Box^j \alpha) \leq m-i$ then $\Diamond^i \Box^j \alpha \in w_0$. Taking into account that $C(\alpha) \leq m - \max\{i+j, k+l\}$, then $C(\Diamond^i \Box^j \alpha \to \Box^k \Diamond^l \alpha) \leq m \leq n$ and $\Diamond^i \Box^j \alpha \to \Box^k \Diamond^l \alpha$ is an instance of G^*. By Proposition 3.11 (item 4) it follows that $\Box^k\Diamond^l\alpha\in w_0$. Thus, due to the presupposition that $w_0R^kw_2$, we have that $\Diamond^l\alpha\in w_2$. Moreover, since $\vdash_S\alpha\to\neg\beta$, by applying Proposition 2.5 (item 2) l times it follows that $\vdash_S\Diamond^l\alpha\to\Diamond^l\neg\beta$. Then, by Proposition 3.11 (item 4), $\Diamond^l\neg\beta\in w_2$. Since $C(\Diamond^l\neg\beta)=C(\neg\Box^l\beta)\leq m-k$ and $\vdash_S\Diamond^l\neg\beta\to\neg\Box^l\beta$, by Proposition 3.11 (item 4) it is obtained that $\neg\Box^l\beta\in w_2$, leading to a contradiction (since w_2 is S-consistent, and in the previous paragraph it was obtained that $\Box^l\beta\in w_2$). Consequently, X has to be S-consistent.

Provided that X is S-consistent, by Lemma 3.8 there exists a strict z_C-maximal consistent extension of X, where $z=m-\max\{i+j,k+l\}$. Let w_* be such an extension. Now, we will proceed by cases:

(1) If $i+j=k+l$: let $w_3=w_*$, by definition of X and the accessibility relation of $\mathcal{F}$, $w_1R^jw_3$ and $w_2R^lw_3$, then $w_1R^j_Lw_3$ and $w_2R^l_Lw_3$ (if $j=0$ then $w_*=w_1$, and if $l=0$ then $w_*=w_2$). Therefore, $\mathcal{F}$ satisfies condition C^*.

(2) If $i+j>k+l$: in this case we have the following options:

 (a) $l>0$: let $w_3=w_*$ and $w_4=w_2\backslash\{\gamma\mid C(\gamma)>m-i-j+l\wedge\nvdash_S\gamma\}$. By definition, w_4 is embedded in depth $(i+j)-(k+l)$ into w_2. Then, by definition of X and the accessibility relation of $\mathcal{F}$, $w_1R^jw_3$ (if $j=0$ then $w_*=w_1$) and $w_4R^lw_3$. Consequently, $w_1R^j_Lw_3$ and $w_2R^l_Lw_3$. Then, $\mathcal{F}$ satisfies condition C^*.

 (b) $l=0$: let $w_3=w_2$ (if $j=0$ then w_3 also equals w_1) and $w_4=w_*$. By definition of X, w_* is embedded in depth $(i+j)-k$ into w_2. Then, by definition of the accessibility relation of $\mathcal{F}$, $w_1R^jw_4$. Consequently, $w_1R^j_Lw_3$ and $w_2R^0_Lw_3$. Then $\mathcal{F}$ satisfies condition C^*.

(3) If $k+l>i+j$: similar to case 2.

■

Now, we have all the elements to prove the completeness of $K_{n,C}G_1^*\ldots G_m^*$:

THEOREM 3.17 (Completeness of $K_{n,C}G_1^*\ldots G_m^*$)
If α is n_C-valid in all L-frames in which the accessibility relation jointly satisfies conditions $C_1^*\ldots C_m^*$ then α is a theorem of $K_{n,C}G_1^*\ldots G_m^*$.

PROOF. Let S be the system $K_{n,C}G_1^*\ldots G_m^*$. Suppose that $\nvdash_S\alpha$, then α is not n_C-valid in the canonical L_C-model of S (by Lemma 3.14) and the canonical L-frame of S satisfies conditions $C_1^*\ldots C_m^*$ (by Lemma 3.16). Therefore, α is not n_C-valid in all L-frames in which the accessibility relation jointly satisfies conditions $C_1^*\ldots C_m^*$. ■

4 Defining epistemic logics based on restricted normal modal logics

Epistemic logics are defined as multi-modal logics based on normal modal logics, each modal connective representing the knowledge of an agent. In this context, as it is mentioned in the Introduction in Section 1, the system K^m is considered the *minimal logic of knowledge*. This system includes a finite number of modal connectives $K_1,\ldots,K_m$, each of them ruled by the axiom (K) and the necessitation rule. K^m is semantically characterized by *multi-agent models*, which are structures $\mathcal{M}=\langle W,P_1,\ldots,P_m,V\rangle$, where W and V are defined like in models for mono-modal logics and $P_1,\ldots,P_m$ are binary relations used to interpret the respective modal connectives. In epistemic terms, the relation P_i represents the ‘plausibility’ with respect to the agent i; i.e. if wP_iw' this means that for agent i in ‘state’ w the state w' is ‘plausible’ (for details see [1, Chapter 7]).

In an similar way, the system $K^m_{n,C}$ could be defined as the restricted multi-modal system containing a finite number of modal connectives $K_1,\dots,K_m$, each of them ruled by the axiom ($\mathrm{K}_{n,C}$) and the rule ($\mathrm{Nec}_{n,C}$). Then, *levelled multi-agent models* could be defined as structures $\mathcal{M}=\langle W,L,P_1,\dots,P_m,V\rangle$, where W and L are defined as in Definition 2.7; $P_1,\dots,P_m$ are binary relations on W, subject to the condition that if $w_iP_iw_j$ then $L(w_i)=L(w_j)+1$; and condition 3 in Definition 2.7 have to be adapted in the obvious way (if $C(\alpha)<L(w_i)$, then $V(K_i\alpha,w_i)=1$ if and only if $V(\alpha,w_j)=1$ for all w_j such that $w_iP_iw_j$). Similarly to the case of K^m, the soundness and completeness of $K^m_{n,C}$ with respect to levelled multi-agent models are corollaries of soundness and completeness of $K_{n,C}$ with respect to L_C-models (Theorems 2.13 and 2.14).

Some interesting properties of $K^m_{n,C}$ are the following:

- Depending on n and C, it is possible to have that $\Gamma\vdash_{K^m_{n,C}}\alpha$ and $Nec_i(\Gamma)\nvdash_{K^m_{n,C}}K_i\alpha$, where $Nec_i(\Gamma)=\{K_i\gamma:\gamma\in\Gamma\}$ (for instance, if $\Gamma=\{p\vee q,\neg q\}$, $\alpha=p$, $n=1$ and $C(p)=C(q)=1$). This property shows how the logical omniscience problem can be partially controlled in $K^m_{n,C}$. Note that this control depends on the parameters n and C. For instance, if $n>0$ and C is such that $C(p)=0$ for any propositional variable p, then $\nvdash_{K^m_{n,C}}K_i\alpha$ for any CPL-tautology α. However, if $n=1$ and C is such that $C(p)=C(q)=1$, and $C(r)=0$ for any propositional variable r different of p and q, then $\nvdash_{K^m_{n,C}}K_i(((p\vee q)\wedge\neg q)\to p)$ and $((p\vee q)\wedge\neg q)\to p$ is a CPL-tautology (the same situation happens with any CPL-tautology involving p or q).
- Condition (C1), in the definition of complexity function, allows to assign different complexities to different propositional variables, this property can be used to represent the fact that to 'know' some atomic propositions can be more difficult than to 'know' other atomic propositions. For instance, to know that '7,919 is an odd number' is easier than to know that '7,919 is a prime number', and both propositions can be represented by atomic formulas.
- It is possible that, for some formulas α and β, we have that $\vdash_{K^m_{n,C}}\alpha\leftrightarrow\beta$, $\vdash_{K^m_{n,C}}K_i\alpha$ and $\nvdash_{K^m_{n,C}}K_i\beta$ (for instance, if $\alpha=p\to p$, $\beta=q\to q$, $n=1$, $C(p)=0$ and $C(q)=1$). This property could be useful to deal with the problem of accessibility of knowledge described in [7].

In $K^m_{n,C}$, C can be viewed as a function that assigns 'knowledge complexity' to propositions; i.e. for each formula α, the function C assigns a complexity or difficulty of 'knowledge' the fact represented by α, independently of what 'knowledge' means. The parameter n can consequently be interpreted as the 'knowledge depth' of the agents, in the sense that agents only can know propositions with complexity $<n$. Under these interpretations, in $K^m_{n,C}$ it is assumed that all agents have the same knowledge depth. More interesting epistemic logics could be defined if parameters n and C are independently associated with each modal connective, in this way groups of agents with different knowledge capacities could be more adequately formalized.

Moreover, an infinity of restricted normal multi-modal logics could be defined by adding to $K^m_{n,C}$ different instances of $G^{i,j,k,l}_{n,C}$, and some of the systems obtained could capture the notion of knowledge in a more realistic way, in accordance with different philosophical or technical assumptions. The theorems proved in Section 3 assure the soundness and completeness, with respect to the corresponding classes of levelled multi-agent models, of any multi-modal logic defined in this way.

5 Final remarks

The logical framework here introduced is general enough to allow the construction of a huge variety of modal logics, each of them with different purported applications. In Section 4, we sketch a way in which restricted normal multi-modal logics can be used to define epistemic logics, but the full definition of particular systems of epistemic logic based on these ideas is a task to be addressed in future works.

Moreover, by analysing the proofs of soundness and completeness for $K_{n,C}$ (Theorems 2.13 and 2.14), it seems to be possible to relax the conditions in the definition of complexity function for this specific system, but some technical details have to be addressed. By proving soundness and completeness for specific extension of $K_{n,C}$, in the same constructive way followed in the proofs of Theorems 2.13 and 2.14, it could be possible to extend the class of complexity functions for these systems, in this way opening the possibility of defining even more restricted normal modal systems and extending its possible applications. In each application, the interpretation of the parameters n and C can be totally different.

As a final remark, in view of the multiple applications that modal logic has in computer science and other fields, it is also important to investigate if this new approach could be advantageous for some of these applications. In this context, a detailed study of the algorithmic decision procedures for these logics (based on tableaux, like those described above, or perhaps other methods) and their respective computational complexities are of great importance.

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References

[1] W. Carnielli and C. Pizzi. *Modalities and Multimodalities*. Springer, 2008.
[2] R. Fagin, Y. Moses, J. Y. Halpern and M. Y. Vardi. *Reasoning About Knowledge*. The MIT Press, 1995.
[3] E. J. Lemmon and D. S. Scott. *An Introduction to Modal Logic*. Blackwell, 1977.
[4] M. Sierra A. Sistemas multi-modales de profundidad restringida (multimodal systems with restricted depth). *Ingeniería y Ciencia*, **4**, 175–202, 2008. Available at http://publicaciones.eafit.edu.co/index.php/ingciencia/article/view/222.
[5] M. Sierra A. Lógicas epistémica y doxástica con restricciones (epistemic and doxastic logic with restrictions). *Ingeniería y Ciencia*, **6**, 81–115, 2010. Available at http://publicaciones.eafit.edu.co/index.php/ingciencia/article/view/335.
[6] K. M. Sim. Epistemic logic and logical omniscience: a survey. *International Journal of Intelligent Systems*, **12**, 57–81, 1997.
[7] R. Stalnaker. The problem of logical omniscience, I. *Synthese*, **89**, 425–440, 1991.

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