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(Q, r) MODEL WITH $CVaR_\alpha$ OF COSTS MINIMIZATION

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Abstract. In the classical stochastic continuous review, (Q, r) model [18, 19], the inventory cost $c(Q, r)$ has an averaging term which is given as an integral of the expected costs over the different inventory positions during the lead time on any given cycle. The main objective of the article is to study risk averse optimization in the classical (Q, r) model using $CVaR_\alpha$ as a coherent risk measure with respect to the probability distribution of the demand D on inventory position costs (the sum of the inventory holding and backorder penality cost), for any given (generic) confidence level $\alpha \in [0, 1)$.

We show that the objective function is jointly convex in (Q, r). We also compare the risk averse solution and some other solutions in both analytical and computational ways. Additionally, some general and useful results are obtained.

1. **Introduction.**

The model studied in this paper is a classical single-item continuous review inventory system (Q, r). The demand during the lead time is assumed to be Poisson distributed and a fixed quantity order Q is placed as soon as the inventory position falls to the reorder point r. Replenishment orders are delivered after a positive fixed lead time L. The cost factors are: fixed cost K to place an order, cost holding h per unit in inventory for one unit of time, and penality cost p

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for one unit backorder per one unit of time. This model has received great attention in cases of low inventory stocks, slow-moving and critical items.

The following literature provides a good basis for the study of the (Q, r) model.

In their book Analysis of Inventory Systems, Hadley and Whittin [7], formally derived in 1963 the exact cost function for the case where the demand is Poisson and the lead time is fixed or has a normal distribution. Some restrictions are then added such as: the demand during lead time is less than the order point. The inventory-carrying costs are defined as the sum of the ordering, holding and backorder costs.

In 1979 Platt et al. [12] analytically developed a solution of the model for a single product with a Poisson demand, random lead time and uncompleted orders being considered as backorders. They provided two heuristic solutions based on extension to the limit when an order size tends to infinity or to zero; in both cases, they consider the situation of pending orders when a new order arrives. The article derives a heuristic for the cost with service level constraints.

Federgruen and Zheng [5] provided in 1992 a simple algorithm to determine the optimal values of r and Q. In 2000 Zipkin [19] developed the first algorithm to compute the optimal values of r and Q with Poisson demand and constant lead time. He also proved the joint convexity of the exact cost function.

Hopp and Spearman in 2001 [8] considered a multi-item model with Poisson demand, fixed lead time and handling of backorders as deficits. They minimized restricted inventory investments for a given frequency of orders and a certain level of satisfaction. They also presented an algorithm based on the bisection technique to estimate Lagrange Multipliers and the decision variables of the model.

Zheng's paper [18] is closely related to the present work. In the paper the author obtains optimal conditions for Q and r by sequentially minimizing the cost function over both variables, obtaining two simultaneous equations that the optimal decision variables must satisfy. He makes a comparison with the deterministic EOQ model (with backorders allowed) and a sensitivity analysis is also carried out. The analysis is performed using the expected cost as risk measurement.

In Zipkin 2000 [19] two relevant criteria for the performance system are mentioned: we do not want to order too frequently, because of scale economies, nor do we want to carry too much inventory, as these guidelines are too broad to guide action.

According to Zipkin it seems plausible that the "natural" approach focuses on long-run averages cost over the time interval $[0, \infty)$. However, does the average cost reflect the economic impact of alternative policies? Does the average cost really capture the significant differences in this cost pattern? In the (Q, r) model studied, the Conditional Value at Risk presents one form of considering large losses with no control as opposite to profits modeling where control is placed on the prevention of suffering great loss. Managers with risk-averse in the real world may be more concerned with the large losses brought by uncertain demands.

The literature on risk-averse optimization of the (Q, r) inventory model is rather limited. Some works in that area are the following:

Vinod et al. in 1994 [16] analyzing the effect of risk on decision variables, derived optimal conditions for the lot size and the reorder point. The model is then studied under the objective of minimizing the present value of total cost, using CAPM (Capital Asset Pricing Model) to value the uncertain cash flow; the relevant measure of risk being the covariance of demand with the market return.

Moosa et al. in 2009 [9] evaluated the risk-averse approach to stock-out and obsolescence. They also derived the quasi-stationary distribution for the time to stock-out for a continuous review (Q, r) inventory system as a coherent measure of risk which behaved well, unlike the usual service level measures used in inventory control.

In this paper, we adopt the risk measure called Conditional Value at Risk $(CVaR)$, in the (Q, r) model. The $CVaR$ in known as a coherent risk measure [2]. The preferable properties of this risk measure are included from some axiomatization of rational investors' behavior under uncertainty and, thus they are meaningful also to a manager who faces uncertain loss situations as in the (Q, r) model. In particular, the lower partiality of the $CVaR$ plays an important role in preserving the concavity of the convexity of the cost. Some papers studying the $CVaR$ measure that applied the newsvendor model are Chen et al. 2007 [3], Gotoh and Takano 2007 [6], Ahmed et al. 2007 [1] and Cheng et al. 2009 [4].

We concentrate on characterizing the optimal solution in the classical (Q, r) model using $CVaR_\alpha$ as a risk measure with respect to the probability distribution of the demand D and for any given confidence level $\alpha \in [0, 1)$ during the lead time on any given cycle, according to the model in Zheng's paper [18], where the structure of the cost function has an averaging term which is given as an integral of the expected costs over the different inventory positions. This is assumed as a random variable with uniform distribution, so that the risk-averse might be considered in two levels: one on the inventory position and another on the demand distribution. We focus on the last one, which led to maintaining the convexity properties of the objective function, the optimal conditions of the variables Q and r, additionally this structure led to comparisons made with the EOQ model and adapts the algorithm made in [5] for computing the best solution for Q and r. We consider a fixed lead time. In future works it may be interesting to consider risk-averse with inventory positions and variability on lead time.

This paper is organized as follows. The next section describes the proposed optimization model, the classical model and the deterministic EOQ model. In section 3 some basic computations on the $CVaR_\alpha$ of costs associated to any given inventory position are performed. In section 4 the joint convexity on Q and r of the overall cost function $c(Q, r)$, for the proposed model, is proven. The optimal conditions for the model with $CVaR_\alpha$ are studied and finally the section focuses on the characterization of the optimal inventory policy. In section 5 a comparison between the $CVaR_\alpha$ proposed model and the deterministic EOQ model and a sensitivity analysis is performed. Section 6 presents several numerical examples and finally section 7 provides some concluding remarks.

2. **Classical and proposed models.** Let us introduce some basic concepts and notations. Given a random variable (r.v.) X in a probability space, we denote its distribution function (d.f.) by F_X, unless otherwise stated.

For any given confidence level $\alpha \in [0, 1)$ the risk measure Value at Risk is defined as

$$VaR_\alpha[X] := \inf\{x | F_X(x) \geq \alpha\},$$

which is the α-quantile of X. Although the measure so defined is able to reflect risk aversion, it lacks some desirable properties such as subadditivity, which is the mathematical statement of the response of risk concentration, a basic reality in risk management. Among other objections raised on VaR we can also mention that it is unable to account for the consequences of the established threshold being surpassed and that, in general, it is not continuous on the parameter α.

A measure of risk, closely related to VaR is the Conditional Value at Risk ($CVaR$), defined as the conditional expected value of the $(1-\alpha)-tail$ i.e.

$$CVaR_\alpha[X] := E\{X|X \geq VaR_\alpha[X]\},$$

$CVaR$ has been defined in several ways, depending on the conditions met by the d.f. as to continuity and strict monotonicity. A complete discussion on the matter can be found in [14].

We define the function

$$F_\alpha(x,\varsigma) = \varsigma + \frac{1}{1-\alpha}E[f(x,y)-\varsigma]^+,$$

where $[t]^+ = max\{0,t\}$ and $f(x,y)$ is the loss associated with the decision vector x, to be chosen from a certain subset S of $\mathbb{R}^n$, and random vector y in $\mathbb{R}^m$. Theorem 10 in [14], states that:

$$CVaR_\alpha[X] = F_\alpha(x, VaR_\alpha[X]) \tag{1}$$

due to $VaR_\alpha[X] \in argmin_\varsigma F_\alpha(x,\varsigma)$, see [11].

The model studied in this paper is a classical single-item continuous review inventory system (Q,r) , with fixed ordering cost K for each replenishment order and inventory costs composed of holding and backorder linear costs are incurred. The delivering time, or lead time L is fixed, and demand D, is assumed to be Poisson distributed with a rate λ units by time. The inventory position y, is continuously reviewed and considered continuous uniform on the real numbers in the interval $[r, r+Q]$.

We will use Zheng's [18] approach to combine holding and backorder costs under the so called inventory costs. The cost function at inventory position y at time t corresponding to demand D during lead time L is given by

$$\Phi(y,D) = h(y-D)^+ + p(D-y)^+,$$

where h is the holding cost, in dollars per unit per time period, and p is the stockout cost, in dollars per backorder per time period. Let F be the distribution function of D; we will assume that it is continuous and strictly increasing.

Let us define $F_{\Phi(y,D)}$ as the distribution function of the cost function at inventory position y. Notice that

$$\begin{align}
F_{\Phi(y,D)}(x) &= Pr(\Phi(y,D) \leq x) \tag{2}\\
&= Pr(h(y-D)^+ + p(D-y)^+ \leq x) \tag{3}\\
&= Pr[h(y-D) \leq x, y \geq D] + Pr[p(D-y) \leq x, D \geq y] \tag{4}\\
&= Pr[y - \frac{x}{h} \leq D, y \geq D] + Pr[y + \frac{x}{p} \geq D, D \geq y] \tag{5}\\
&= F\left(y + \frac{x}{p}\right) - F\left(y - \frac{x}{h}\right) \tag{6}
\end{align}$$

for any given real y.

In the classical model, the long-run average total costs per time unit are given by the expression

$$c(Q, r) = \frac{1}{Q}\left[\lambda K + \int_r^{r+Q} G(y)dy\right], \tag{7}$$

where Q is the fixed order quantity, r is the reorder point and

$$G(y) := E[h(y - D)^+ + p(D - y)^+] = E[\Phi(y, D)], \tag{8}$$

is the rate at which the expected inventory costs accumulate at time $t + L$ when the inventory position at time t equals y. For (7) to hold, the inventory position in steady state it must be uniformly distributed on $(r, r + Q]$ and independent of D.

As can be seen in [13], the inventory policy is find the solution of $\min_{(Q,r)} c(Q, r)$. This model tries to obtain the minimum expected cost as much as possible. However, as stated in [6], such a criteria may be insufficient as it may result in an unacceptably large losses. To reduce this loss, the paper proposes to minimize the Conditional Value at Risk of the inventory costs, as well as calculating the average costs of $CVaR$ with $\alpha = 0$ to calculate potential losses and improve reliability of the inventory, for an appropriate level of confidence, so the CVaR decreases the likelihood of Stockout.

We substitute the risk measure used in (8) by

$$\varrho_\alpha(y) := CVaR_\alpha[\Phi(y, D)],$$

and defining

$$\rho_\alpha(y) := \mathrm{VaR}_\alpha(\Phi(y, D)).$$

According to definition of $CVaR_\alpha[X]$ in (1) we have calculeted $CVaR_\alpha[\Phi(y, D)]$

$$\begin{aligned}
\varrho_\alpha(y) &= E[\Phi(y, D)|\Phi(y, D) > \rho_\alpha(y)] \\
&= \rho_\alpha(y) + \frac{1}{1-\alpha}\int_0^\infty [h(y - \xi)^+ + p(\xi - y)^+ - \rho_\alpha(y)]^+ dF(\xi) \\
&= \rho_\alpha(y) + \frac{h}{1-\alpha}\int_0^{\underline{y}_\alpha} (\underline{y}_\alpha - \xi)dF(\xi) + \frac{p}{1-\alpha}\int_{\overline{y}_\alpha}^\infty (\xi - \overline{y}_\alpha)dF(\xi),
\end{aligned} \tag{9}$$

as our chosen way to reflect risk aversion, where $\underline{y}_\alpha := y - \frac{\rho_\alpha(y)}{h}$ and $\overline{y}_\alpha := y + \frac{\rho_\alpha(y)}{p}$. From $F_{\Phi(y,D)}(0) = 0$, it becomes clear that $\rho_\alpha(y) \geq 0,\ \forall y$, hence we can obtain the following limits

$$\lim_{y\to\infty} \overline{y}_\alpha = \infty \quad \text{and} \quad \lim_{y\to-\infty} \underline{y}_\alpha = -\infty. \tag{10}$$

The optimization problem we want to study now is the minimization of the cost function

$$c_\alpha(Q, r) = \frac{1}{Q}\left[\lambda K + \int_r^{r+Q} \varrho_\alpha(y)dy\right]. \tag{11}$$

3. **Derivatives of $\rho_\alpha(y)$ and $\varrho_\alpha(y)$.** In this section the basic results used in sections 4, 5 are obtained.

By definition of ρ_α we get the the basic and useful equation

$$F_{\Phi_{(y,D)}}(\rho_\alpha(y)) = F(\overline{y}_\alpha) - F(\underline{y}_\alpha) = \alpha. \tag{12}$$

Taking limits to $\pm\infty$ in this formula and using (10) we can easily compute the following limits

$$\lim_{y\to-\infty} \overline{y}_\alpha = VaR_\alpha(D) \quad \text{and} \quad \lim_{y\to\infty} \underline{y}_\alpha = VaR_{1-\alpha}(D). \tag{13}$$

Using equation (12) we obtain the following expression for the derivative of ϱ_α:

$$\begin{aligned}\varrho'_\alpha(y) &= \rho'_\alpha(y) + \frac{h}{1-\alpha}\int_0^{\underline{y}_\alpha}\left(1-\frac{\rho'_\alpha(y)}{h}\right)dF(\xi) - \frac{p}{1-\alpha}\int_{\overline{y}_\alpha}^{\infty}\left(1+\frac{\rho'_\alpha(y)}{p}\right)dF(\xi)\\ &= \frac{1}{1-\alpha}\left[hF(\underline{y}_\alpha) - p\overline{F}(\overline{y}_\alpha)\right].\end{aligned}$$

To compute the limits of $\varrho'_\alpha(y)$ when $|y|$ goes to ∞ we use equation (12) to rewrite first $\varrho'_\alpha(y)$ in terms of $F(\overline{y}_\alpha)$ as

$$\varrho'_\alpha(y) = \frac{1}{1-\alpha}\left[(h+p)F(\overline{y}_\alpha) - h\alpha - p\right].$$

Using (10) in the last equation we obtain

$$\lim_{y\to\infty}\varrho'_\alpha(y) = h. \tag{14}$$

In a similar manner we rewrite $\varrho'_\alpha(y)$ as

$$\varrho'_\alpha(y) = \frac{1}{1-\alpha}\left[(h+p)F(\underline{y}_\alpha) + p(\alpha-1)\right],$$

and compute

$$\lim_{y\to-\infty}\varrho'_\alpha(y) = -p. \tag{15}$$

Some useful results on $\rho_\alpha(y)$ can be deducted as an easy consequence of the previous computations. From the definition of $\overline{y}_\alpha$ and $\underline{y}_\alpha$ and the limits stated in (10) and (13) we can obtain

$$\lim_{|y|\to\infty}\rho_\alpha(y) = \infty.$$

We can also differentiate (12) to find $\rho'_\alpha(y)$ and its asymptotic behavior

$$f(\overline{y}_\alpha)\left(1+\frac{\rho'_\alpha(y)}{p}\right) - f(\underline{y}_\alpha)\left(1-\frac{\rho'_\alpha(y)}{h}\right) = 0,$$

therefore

$$\rho'_\alpha(y) = \frac{ph[f(\underline{y}_\alpha) - f(\overline{y}_\alpha)]}{hf(\overline{y}_\alpha) + pf(\underline{y}_\alpha)}.$$

and

$$\lim_{y\to\infty}\rho'_\alpha(y) = h; \qquad \lim_{y\to-\infty}\rho'_\alpha(y) = -p.$$

4. **Optimality conditions for** $c_\alpha(Q,r)$**.**

We want to characterize the optimal solution of $\min\limits_{(Q,r)} c_\alpha(Q,r)$ (see (11)).

Theorem 4.1. *$c_\alpha(Q,r)$ is jointly convex in Q and r.*

Proof. According to Corollary 11 in [14] $\varrho_\alpha(y)$ is convex, therefore we can resort to the Theorem on pg. 3 in [17] to assert that $c_\alpha(Q,r)$ is jointly convex in Q and r. □

Now we know that $c_\alpha(Q,r)$ and $\varrho_\alpha(y)$ meet correspondingly the same conditions as $c(Q,r)$ and the mean $G(y)$ in [18], therefore the arguments used in section 2 in that paper are still valid and the corresponding propositions can be restated in our case:

The minimization problem

$$\min_{(Q,r)} c_\alpha(Q,r) \tag{16}$$

is carried out as $\min_Q[\min_r c_\alpha(Q,r)]$. As $c_\alpha(Q,r)$ is convex in r, let $r(Q)$ be optimal for a given Q and let y^0_α be a minimum point of ϱ_α.

With the same arguments used in [18] we can prove the following proposition summarizing the properties of $r(Q)$.

Proposition 1. *For any $Q > 0$ and any real r,*

1. *$r = r(Q)$ if and only if $\varrho_\alpha(r) = \varrho_\alpha(r+Q)$,*
2. *$r(Q)$ is solely determined by ϱ_α, hence it is independent of K,*
3. *$r(Q) < y^0_\alpha < r(Q) + Q$,*
4. *$-1 < r'(Q) < 0$, $r(Q)$ is decreasing and $r(Q) + Q$ is increasing,*
5. *$\lim_{Q\to\infty} r(Q) = -\infty$ and $\lim_{Q\to\infty} r(Q) + Q = \infty$.*
6. *$\min(\varrho_\alpha(r), \varrho_\alpha(r+Q)) \le \varrho_\alpha(r(Q)) \le \max(\varrho_\alpha(r), \varrho_\alpha(r+Q))$.*

For the characterization of the minimal Q in $\min_Q c_\alpha(Q, r(Q))$, let us define $C_\alpha(Q) := c_\alpha(Q, r(Q))$. It can be shown that

$$C_\alpha(Q) = \frac{1}{Q}\left[\lambda K + \int_0^Q H_\alpha(y)dy\right], \tag{17}$$

where

$$H_\alpha(Q) := \begin{cases} \varrho_\alpha(r(Q)); & Q > 0 \\ \varrho_\alpha(y^0_\alpha); & Q = 0. \end{cases} \tag{18}$$

Notice that $\varrho_\alpha(y^0_\alpha) = \lim_{Q\to 0^+} \varrho_\alpha(r(Q))$.

Using (14) and (15) and the same argument as in [18] we can prove the following

Proposition 2.
1. *$H_\alpha(Q)$ is an increasing convex function with asymptotic slope $hp/(h+p)$ as $Q \to \infty$.*
2. *$C_\alpha(Q)$ is convex.*

Therefore, differentiating (17) we obtain

$$C'_\alpha(Q) = \frac{1}{Q^2}\left[H_\alpha(Q)Q - \lambda K - \int_0^Q H_\alpha(y)dy\right]. \tag{19}$$

Let Q^*_α be the optimal order quantity and let us define $A_\alpha(Q) := H_\alpha(Q)Q - \int_0^Q H_\alpha(y)dy$. Equating (19) to zero we obtain the optimality condition. Using this condition together with (17), (18) and Proposition (1), as in [18], we get three equivalent ways to characterize the optimal solution of (16):

Theorem 4.2. *The following statements are equivalent*

1. *(Q,r) minimizes (16),*
2. *$H_\alpha(Q^*_\alpha) = C_\alpha(Q^*_\alpha)$,*
3. *$c_\alpha(Q,r) = \varrho_\alpha(r) = \varrho_\alpha(r+Q)$,*
4. *$A_\alpha(Q) = \lambda K$.*

5. **Comparison with the EOQ model and sensitivity analysis.** In addition to the two models already mentioned we will also use the deterministic EOQ model for comparison and sensitivity analysis. In that model the leadtime demand is constant and equal to its expected value, i.e. $E[D] = \lambda L$. To perform the comparison we need the following variables for the proposed model.

$\forall Q > 0;\ \alpha \in [0, 1):$

$$H^0_\alpha(Q) := H_\alpha(Q) - \varrho_\alpha(y^0_\alpha)$$

Total average costs:

$$C_\alpha(Q) := \frac{1}{Q}[\lambda K + \int_0^Q H_\alpha(y)dy] = O(Q) + I_\alpha(Q) = \varrho_\alpha(y^0_\alpha) + C^0_\alpha(Q)$$

Total average controllable costs:

$$C^0_\alpha(Q) := \frac{1}{Q}[\lambda K + \int_0^Q H^0_\alpha(y)dy] = O(Q) + I^0_\alpha(Q)$$

Optimal average cost: C^*_α
Optimal average controllable cost: C^{0*}_α

In agreement with the notation used in [18], for the EOQ model, we will use the same variables with the subscript d (for deterministic) instead of α. For the classical model we will omit the subscript.

Theorems 2 through 5 in [18] and the corresponding Lemmas, about comparison of the classical and EOQ models and sensitivity analysis, depend basically on the results on optimality conditions which, for the proposed model, also hold true. In Lemma 7, two results are also used, namely: $G_d(y) \leq G(y)\ \forall y,$ and $H'(Q) \leq H'_d(Q)$.

In the proposed model the equality $G(y) \leq \varrho_\alpha(y);\ \forall y$, is valid by definition of $CVaR_\alpha$, therefore $G_d(y) \leq \varrho_\alpha(y);\ \forall y$, is valid as well. The inequality $H'_\alpha(Q) \leq H'_d(Q)$ can be proved using the same argument as in Lemma 7, in view of proposition (2) numeral 1.

As a consequence, the following results may be summarized in terms of the proposed model:

Theorem 5.1. *The following relations are valid*

1. $Q^*_d \leq Q^*_\alpha$.
2. $C^{0*}_\alpha \leq \dfrac{Q^*_d}{Q^*_\alpha} C^*_d,\quad C^*_d \leq C^*_\alpha \leq \varrho_\alpha(y^0_\alpha) + \dfrac{Q^*_d}{Q^*_\alpha} C_d *.$
3. $\dfrac{C_\alpha(\theta Q^*_\alpha)}{C^*_\alpha} \leq \dfrac{1}{2}\left(\theta + \dfrac{1}{\theta}\right);\quad \forall \theta > 0.$
4. $R := \dfrac{C_\alpha(Q^*_d) - C^*_\alpha}{C^*_\alpha} \leq \dfrac{1}{8}.$

6. **Numerical examples.** In this section we report some results obtained with the algorithm based on [5], with a Poisson demand, $\Delta\varrho(t) = \varrho(t+1) - \varrho(t)$, and fixed leadtime $L = 1$. We performed the computations for 200 cases, combining different parameter values for λ, α, h, p and k. The values given to parameters are λ: 5, 25 and 50; α: 0.9, 0.94, 0.96 and 0.965; h: 1, 10, 25 and 100; p : 5, 10, 25, 100 and k:

1, 5, 25, 100 and 1.000. We only tested combinations with $p \geq h$ because, according to Zhang and Chu [17], that condition is met by most of the real word applications. In tables 1,2 and 3 we list the results for a few set of examples.

The control parameters Q^*, Q^*_α, r^*, r^*_α, C^* and C^*_α are computed by the discrete algorithm in [5], hence they are taken as integer numbers, in particular Q^* and r^* are calculated by the before algorithm with $\alpha = 0$. Finally Q^*_d, r^*_d and C^*_d are computed by the EOQ expression allowing backorders.

The modified algorithm used to compute the optimal (Q, r) policy, which minimizes the CVaR of costs, is the following.

Algorithm

$t = 0$
While $\Delta\varrho(t) < 0, t = t + 1$
End
$s = c^* = \lambda K + \varrho(t), Q = 1, r = t - 1, R = t + 1$
If $\varrho(r) \leq \varrho(R)$
    If $c^* \leq \varrho(r)$
        End
    Else $S = S + \varrho(r), r = r - 1$
Else
    If $c^* \leq \varrho(R)$
        End
    Else $S = S + \varrho(R), R = R + 1$
$Q^* = Q + 1, r^* = r, C^* = \frac{S}{Q^*}$.

For a detailed discussion of the complexity of this algorithm see [5].

The purpose of our numerical study is to verify the actual gap between Q^*_d and Q^*_α; C^*_d and C^*_α, and some other results stated on Theorem 5.1. In tables 1 through 4 it can be verified that the inequality $Q^*_d \leq Q^*_\alpha$ in Theorem 5.1 holds.

The inequality $C^*_d \leq C^*_\alpha$ in numeral 2 of Theorem 5.1 is also verified in said tables. As well as, the parameter values used in calculations, in all reported cases. The comments in the last paragraphs regarding the stability of the first inequality under parameters changes and the theoretical basis of the second inequality are valid for the two inequalities in this paragraph.

As it happens with r^*_d and r^*, there is no order relationship between r^*_d and r^*_α. That is, r^*_α can be either larger or smaller than r^*_d, depending heavily upon the relation between h and p. In tables 1 and 2, in the cases $h = 10$, $p = 25$, $r^*_d \leq r^*_\alpha$. But when $h = 25$ and $p = 25$ the results show in tables 3 and 4 that $r^*_d \geq r^*_\alpha$. The fact that the inequality is not valid for the expected cost ($\alpha = 0$) makes it invalid for the $CVaR_\alpha$ of costs as well.

TABLE 1. $\lambda = 50$, $L = 1$, $h = 10$, $p = 25$, $\alpha = 0.90$

K	λK	Q_d^*	Q^*	Q_α^*	r_d^*	r^*	r_α^*	c_d^*	c^*	c_α^*
1	50	3.7	8	7	48.9	50	54	26.73	95.68	227.90
5	250	8.4	13	12	47.6	48	52	59.76	115.48	252.64
25	1250	18.7	24	20	44.7	44	50	133.63	171.49	318.37
100	5000	37.4	41	39	39.3	38	44	267.26	289.39	448.09
1000	50000	118.3	121	120	16.2	15	21	845.15	852.56	1023.23

TABLE 2. $\lambda = 50$, $L = 1$, $h = 10$, $p = 25$, $\alpha = 0.965$

K	λK	Q_d^*	Q^*	Q_α^*	r_d^*	r^*	r_α^*	c_d^*	c^*	c_α^*
1	50	3.7	8	6	48.9	50	56	26.73	95.68	269.24
5	250	8.4	13	11	47.6	48	54	59.76	115.48	295.01
25	1250	18.7	24	21	44.7	44	51	133.63	171.49	362.73
100	5000	37.4	41	39	39.3	38	46	267.26	289.39	493.90
1000	50000	118.3	121	120	16.2	15	23	845.15	852.56	1068.81

TABLE 3. $\lambda = 50$, $L = 1$, $h = 25$, $p = 25$, $\alpha = 0.90$

K	λK	Q_d^*	Q^*	Q_α^*	r_d^*	r^*	r_α^*	c_d^*	c^*	c_α^*
1	50	2.8	6	4	48.6	46	48	35.36	153.35	381.80
5	250	6.3	11	8	46.9	44	46	79.06	177.25	413.29
25	1250	14.1	19	15	43.0	40	42	176.78	245.58	499.99
100	5000	28.3	31	29	35.9	34	35	353.55	670.96	671.36
1000	50000	89.4	91	90	5.3	4	5	1118.03	1131.89	1431.33

TABLE 4. $\lambda = 50$, $L = 1$, $h = 25$, $p = 25$, $\alpha = 0.965$

K	λK	Q_d^*	Q^*	Q_α^*	r_d^*	r^*	r_α^*	c_d^*	c^*	c_α^*
1	50	2.8	6	7	48.6	46	46	35.36	153.35	466.68
5	250	6.3	11	8	46.9	44	46	79.06	177.25	495.40
25	1250	14.1	19	15	43.0	40	43	176.78	245.58	580.36
100	5000	28.3	31	29	35.9	34	35	353.55	670.96	750.45
1000	50000	89.4	91	90	5.3	4	5	1118.03	1131.89	1510.09

7. **Conclusions.** The distribution function of demand is Poisson, whose standard deviation is relatively small with respect to the mean, which makes it light tailed. This condition is inherited by the costs distribution. In most cases like this, the effect of risk aversion is negligible, because major deviations above the mean have low probability of occurrence.

A clear sign of the situation just described is reflected in the assertion in Theorem 5.1, which shows that the sensibility to uncertainty of optimal costs is low, as it happens with Sheng Zheng's model as well, as shown by the values of R in the last columns of the tables. By the same reason, for values of α close to one, the probabilities of costs at the VaR_α tail being exceeded are too small and values provided by the algorithm become erratic, as mentioned in Section 6.

It would be valuable to explore a model with distribution function of demand with different characteristics . A clear research effort could be provided by the mixture

Poisson and Gamma, that is, taking the Poisson mean as a random variable with Gamma distribution. As it is known that mixture leads to a negative binomial distribution which, with an appropriate selection of parameters, becomes heavy tailed.

Another promissory research front might be to state the model under various coherent measures [10], well suited for risk averse optimization, and compare their theoretical and computational behavior.

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