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A q-exponential statistical Banach manifold

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ABSTRACT

Let μ be a given probability measure and $\mathfrak{M}_\mu$ the set of μ-equivalent strictly positive probability densities. In this paper we construct a Banach manifold on $\mathfrak{M}_\mu$, modeled on the space $L^\infty(p \cdot \mu)$ where p is a reference density, for the non-parametric q-exponential statistical models (Tsallis's deformed exponential), where $0 < q < 1$ is any real number. This family is characterized by the fact that when $q \to 1$, then the non-parametric exponential models are obtained and the manifold constructed by Pistone and Sempi is recovered, up to continuous embeddings on the modeling space. The coordinate mappings of the manifold are given in terms of Csiszár's Φ-divergences; the tangent vectors are identified with the one-dimensional q-exponential models and q-deformations of the score function.

1. Introduction

1.1. On information manifolds

In 1995 Pistone and Sempi [1,2] considered the set $\mathfrak{M}_\mu$ of μ-equivalent strictly positive probability densities and introduced the exponential statistical information manifold modeled on Orlicz function spaces [3]. Consequently each such exponential parametric model was identified with the tangent space of $\mathfrak{M}_\mu$ and the coordinate maps are naturally defined in terms of relative entropies in the context of Shannon's. There have been applications of these geometric tools to the study of information theory, giving new mathematical developments [4–6] in the theory of geometric information. Also, in 2009 Pistone [7] presented a discussion about the use of the Kaniadakis k-exponential in the construction of a statistical manifold modeled on Lebesgue spaces of real random variables. Some algebraic features of the deformed exponential models were also considered.

Recently, Amari and Ohara [8] gave a geometrical structure to the q-exponential family or q-Gibbs distribution for the parametric case (finite dimensional) where the Riemannian geometry induced by divergences depends on the q-potential function given by $\psi_q(\theta) = -\ln_q p_0$, with p_0 being an a priori given distribution; the divergence constructed is one of the Bregman type and differs from the Tsallis one, by a functional factor.

In this paper, we discuss the use of the Tsallis q-exponential in the construction of a non-parametric statistical manifold modeled on essentially bounded function spaces, such that each q-exponential parametric model is identified with the tangent space of $\mathfrak{M}_\mu$ and the coordinate maps are naturally defined in terms of relative entropies in the context of Tsallis. To this effect, we give in this section some preliminaries. In the second section we define the q-exponential model, which permits partitioning of $\mathfrak{M}_\mu$ into equivalence classes, so that later it will possible to define an atlas on $\mathfrak{M}_\mu$ modeled on essentially bounded function spaces. In the third section we study the analyticity of certain maps K_p, which are q-deformations of cumulants maps used by Pistone. This is necessary later for proving that the atlas is actually C^∞. In the fourth section we define the manifold such that for each $p \in \mathfrak{M}_\mu$ the charts for the given atlas are defined in terms of q-exponential connections and the maps K_p. We also prove that the atlas constructed is C^∞. In the fifth section we

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construct the tangent bundle over the manifold by means of regular curves passing through $p \in \mathfrak{M}_\mu$; these regular curves are parametric models associated with every point of the manifold [9]. As usual, the tangent space will be identified with a subspace of the modeling Banach space. Moreover, the tangent vectors are related to the q-score function [10] which is a deformation of the statistical notion of a score function. Finally, the last section is dedicated to characterizing a divergence functional, which is defined as the Tsallis relative entropy functional [10–12] and is a particular case of the Csiszár Φ-divergences [13]; we show that coordinate mappings of the manifold are, in fact, given in terms of Tsallis divergence functionals.

1.2. *q*-deformed exponential and logarithmic functions

Tsallis defined an entropy functional depending of a real parameter q called the entropic index [14,15]. Such a functional is fundamental in the non-extensive statistical mechanics introduced by him in 1988. To obtain useful mathematical developments in the theory, C. Tsallis also considered q-deformed exponential and logarithmic functions. As in the present paper we consider $0 < q < 1$, the above functions are respectively defined by

$$\mathrm{e}_q^x = (1 + (1-q)x)^{1/(1-q)}, \quad \text{if } \frac{-1}{1-q} \le x \quad \text{and} \quad \ln_q(x) = \frac{x^{1-q}-1}{1-q}, \quad \text{if } x > 0.$$

These functions satisfy similar properties to the natural exponential and logarithmic functions. We consider, associated with these, the operations defined for real numbers x and y by

$$x \oplus_q y := x + y + (1-q)xy \quad \text{and} \quad x \ominus_q y := \frac{x-y}{1+(1-q)y},$$

for $y \neq (q-1)^{-1}$. Borges [16] presents a compilation of properties of such functions and non-extensive statistical mechanics introduced by Tsallis.

Remark 1. The following properties will be very useful for us later on.

1. $\mathrm{e}_q^{(x \oplus_q y)} = \mathrm{e}_q^x \mathrm{e}_q^y$ and $\mathrm{e}_q^{(x \ominus_q y)} = \frac{\mathrm{e}_q^{(x)}}{\mathrm{e}_q^{(y)}}$.
2. $\ln_q(xy) = \ln_q(x) \oplus_q \ln_q(y)$ and $\ln_q(\frac{x}{y}) = \ln_q(x) \ominus_q \ln_q(y)$.
3. $d[\ln_q(u)] = \frac{1}{u^q} du$ and $d[\mathrm{e}_q^u] = (\mathrm{e}_q^u)^q du$.

Tsallis theory has many mathematical developments and applications in various disciplines [14,17].

2. *q*-exponential models

Let (Ω, Σ, μ) be a probability space and q a real number such that $0 < q < 1$. Denote by $\mathfrak{M}_\mu$ the set of strictly positive probability densities μ-a.e. For each $p \in \mathfrak{M}_\mu$ consider the probability space $(\Omega, \Sigma, p \cdot \mu)$, where $p \cdot \mu$ is the probability measure given by $(p \cdot \mu)(A) = \int_A p d\mu$. Also, denote by $\|\cdot\|_{p,\infty}$ the essential supremum norm in the Banach space of essentially bounded functions $L^\infty(p \cdot \mu)$. On several occasions, we use the following fact. If $\|u\|_{p,\infty} < 1$, then $u < 1$ almost everywhere on Ω and so

$$\|u\|_{p,\infty} < 1 \quad \text{implies} \quad \|\mathrm{e}_q^u\|_{p,\infty} \le \mathrm{e}_q^{(1)} = (2-q)^{\frac{1}{1-q}}. \tag{1}$$

Given $p \in \mathfrak{M}_\mu$, we say that a function f is a one-dimensional q-exponential model if there exist $u \in L^\infty(p \cdot \mu)$ and a real function of a real variable ψ such that for all t in an open interval I containing zero, $f(t) = \mathrm{e}_q^{tu \ominus_q \psi(t)} p$ is valid. In this case, it is easily verified that $u \in C_{f(t)}$, i.e. $\hat{u}_{f(t)}(\theta) < \infty$ for θ in a neighborhood of zero such that $\theta t \ge 0$.

We say that probability densities $p, z \in \mathfrak{M}_\mu$ are connected by a one-dimensional q-exponential model if there exist $r \in \mathfrak{M}_\mu$, $u \in L^\infty(r \cdot \mu)$, a real function of a real variable ψ and $\delta > 0$ such that for all $t \in (-\delta, \delta)$, the function f defined by

$$f(t) = \mathrm{e}_q^{tu \ominus_q \psi(t)} r$$

satisfies that there are $t_0, t_1 \in (-\delta, \delta)$ for which $p = f(t_0)$ and $z = f(t_1)$.

$\mathfrak{M}_\mu$ is partitioned into equivalence classes using a relation: given $p, z \in \mathfrak{M}_\mu$ we say that $p \sim_q z$ if and only if p and z are connected by a one-dimensional q-exponential model. This equivalence relation and the next proposition are necessary for being able to define an atlas on $\mathfrak{M}_\mu$ modeled on Banach spaces. To establish the proposition, let $p, z \in \mathfrak{M}_\mu$ be probability densities connected by a one-dimensional q-exponential model. There exist $r \in \mathfrak{M}_\mu$, $u \in L^\infty(r \cdot \mu)$, a real function of a real variable ψ, $\delta > 0$ and $t_0, t_1 \in (-\delta, \delta)$ such that $p = \mathrm{e}_q^{t_0 u \ominus_q \psi(t_0)} r$ and $z = \mathrm{e}_q^{t_1 u \ominus_q \psi(t_1)} r$. Using Remark 1, we obtain $p = \mathrm{e}_q^s z$, where $s = (t_0 u \ominus_q \psi(t_0)) \ominus_q (t_1 u \ominus_q \psi(t_1))$ and $\frac{-1}{1-q} < s = \ln_q(\frac{p}{z})$ μ-a.e. Given $g \in L^\infty(p \cdot \mu)$, defining $A_g := \{t \in \Omega : |g(t)| = \infty\}$, we have $0 = (p \cdot \mu)(A_g) = \int_{A_g} p d\mu = \int_{A_g} \mathrm{e}_q^s z d\mu$. Since $\mathrm{e}_q^s = \frac{p}{z} > 0$ μ-a.e., we obtain $\int_{A_g} z d\mu = 0$ (i.e., $g \in L^\infty(z \cdot \mu)$) and clearly $\|g\|_{p,\infty} = \|g\|_{z,\infty}$. Then as the relationship $\sim_q$ is symmetric, we have the following result.

Proposition 2. *Let $p, z \in \mathfrak{M}_\mu$ be probability densities connected by a one-dimensional q-exponential model. Then the measures $p \cdot \mu$ and $z \cdot \mu$ are equivalent and hence the spaces of bounded functions $L^\infty(p \cdot \mu)$ and $L^\infty(z \cdot \mu)$ are just equal. In addition, $\frac{-1}{1-q} < \ln_q(\frac{p}{z})$ holds.*

3. Analyticity

In this section, we define the maps M_p and K_p as q-deformations of the moment-generating functional and the cumulant [1,2]. The maps M_p and K_p are then used to define transition maps for the q-exponential manifold which will be of Class C^∞ under analyticity of such maps. Therefore we study analyticity and some properties of such maps.

We shall define an analytic function between the open unit ball $B_{p,\infty}(0, 1)$ of $L^\infty(p \cdot \mu)$ and the space $L^\infty(p \cdot \mu)$. Let $0 < q < 1$, $p \in \mathfrak{M}_\mu$, $u \in B_{p,\infty}(0, 1) \subset L^\infty(p \cdot \mu)$ and $n \in \mathbb{N}$. We define the map $\Lambda_n(u) : \prod_{i=1}^n L^\infty(p \cdot \mu) \to L^\infty(p \cdot \mu)$ by

$$(v_1 \cdots v_n) \to v_1 \cdots v_n e_q^u. \tag{2}$$

Denoting by $\mathcal{L}_s^n(X, Y)$ the space of all symmetric continuous n-multilinear maps from space X into space Y, we have the following result.

Proposition 3. *For all $u \in L^\infty(p \cdot \mu)$ with $\|u\|_{p,\infty} < 1$, $\Lambda_n(u) \in \mathcal{L}_s^n(L^\infty(p \cdot \mu), L^\infty(p \cdot \mu))$.*

Proof. If $\|u\|_{p,\infty} < 1$, it is clear that $\Lambda_n(u)$ is a multilinear symmetric operator. By (1), $\|e_q^u\|_{p,\infty} \le (2-q)^{\frac{1}{1-q}}$. Since $v_1 \cdots v_n \in L^\infty(p \cdot \mu)$ we obtain $(v_1 \cdots v_n e_q^u) \in L^\infty(p \cdot \mu)$ and

$$\|\Lambda_n(u)(v_1 \cdots v_n)\|_{p,\infty} \le (2-q)^{\frac{1}{1-q}} \|v_1\|_{p,\infty} \cdots \|v_n\|_{p,\infty}. \tag{3}$$

As $(v_1 \cdots v_n)$ is arbitrary in $\prod_{i=1}^n L^\infty(p \cdot \mu)$, then $\Lambda_n(u)$ is well defined. By the criterion for multilinear bounded operators, (3) implies that $\Lambda_n(u)$ is continuous. □

Now for $\|u\|_{p,\infty} < 1$, we write $\hat{\Lambda}_0(u) := e_q^u$. As $\|u\|_{p,\infty} < 1$, by (1) we have $e_q^{(u)} = \hat{\Lambda}_0(u) \in L^\infty(p \cdot \mu)$ and $\|\hat{\Lambda}_0(u)\|_{p,\infty} \le e_q^1 = (2-q)^{\frac{1}{1-q}} < \infty$. For $n \ge 1$, let $\hat{\Lambda}_n(u)$ be the n-homogeneous polynomial determined by the polar form $\Lambda_n(u)$, i.e., for all $v \in L^\infty(p \cdot \mu)$, $\hat{\Lambda}_n(u) \cdot (v) = \Lambda_n(u)(v, v, \dots, v)$, n times, and denoted as $\hat{\Lambda}_n(u) \cdot (v) = v^n e_q^u$.

Let $A : L^\infty(p \cdot \mu) \to L^\infty(p \cdot \mu)$ be such that

$$v \to A(v) := 1 + \sum_{i=1}^\infty \frac{1}{i!} Q_{i-1}(q) \, (v)^i, \tag{4}$$

where for each $q \in (0, 1)$ we define $Q_i(q) = q(2q-1)(3q-2) \cdot \cdot (iq - (i-1)) = \prod_{n=1}^i (nq - (n-1))$ and $Q_0(q) = 1$.

Proposition 4. *If $p \in \mathfrak{M}_\mu$, the function A is a power series from $L^\infty(p \cdot \mu)$ into $L^\infty(p \cdot \mu)$, with radius of convergence $\hat{\rho} > 1$.*

Proof. Note that

$$A(v) = 1 + \left[v + \frac{1}{2!} Q_1(q) v^2 + \frac{1}{3!} Q_2(q) v^3 + \cdots \right] = 1 + \sum_{i=0}^\infty \frac{1}{i!} Q_i(q) \hat{\Lambda}_i(0) \cdot (v).$$

Now, the function $A = \frac{1}{i!} Q_i(q) \hat{\Lambda}_i(0)$ is a power series from $L^\infty(p \cdot \mu)$ into $L^\infty(p \cdot \mu)$ for which we discuss the radius of convergence below. If $v \in L^\infty(p \cdot \mu)$ with $\|v\|_{p,\infty} \le 1$ and we fix $u \in B_{p,\infty}(0, 1)$, we see that

$$\left\| \frac{1}{i!} Q_i(q) \hat{\Lambda}_i(0) \cdot (v) \right\|_{p,\infty} = \left\| \frac{1}{i!} Q_i(q) (v)^i e_q^u \right\|_{p,\infty} \le \frac{1}{i!} Q_i(q) (2-q)^{\frac{1}{1-q}}.$$

Let $\mathcal{P}^n := \mathcal{P}^n(L^\infty(p \cdot \mu), L^\infty(p \cdot \mu))$ be the normed linear space of continuous n-homogeneous polynomials determined by the polar form $\Lambda_n(u)$ as in the definition (2). Then $\|\hat{\Lambda}\|_{\mathcal{P}^n} = \sup_{\|v\|_{p\infty}=1} \|\hat{\Lambda}(v)\|_{p,\infty} \le (2-q)^{\frac{1}{1-q}}$ and

$$\left\| \frac{1}{i!} Q_i(q) \hat{\Lambda}_{a,i}(0) \right\|_{\mathcal{P}^n} \le \frac{1}{i!} Q_i(q) (2-q)^{\frac{1}{1-q}}. \tag{5}$$

By the Cauchy–Hadamard formula [18], if $\hat{\rho}$ is the radius of convergence of the series, it follows that

$$\frac{1}{\hat{\rho}} = \overline{\lim} \left(\|\hat{\Lambda}\|_{\mathcal{P}^n} \right)^{\frac{1}{n}} = \overline{\lim} \left(\|\Lambda\|_{\mathcal{L}_s^n} \right)^{\frac{1}{n}} \le \lim_{n \to \infty} (2-q)^{\frac{1}{n(1-q)}} = 1,$$

and so $\hat{\rho} > 1$. Then the series (4) converges absolutely and uniformly on the closed ball of radius r $\bar{B}_{p,\infty}(0, r)$ for each $r < \hat{\rho}$. □

Before proving the next theorem, we see that the series (4) equals $e_q^{(v)}$. By [16], it is known that $1+\sum_{i=0}^{\infty}\frac{1}{i!}Q_i(q)\,(v)^i = e_q^{(v)}$ pointwise. As $p\cdot\mu$ is a finite measure, $1+\sum_{i=0}^{k}\frac{1}{i!}Q_i(q)\,(v)^i \to e_q^{(v)}$ in $(p\cdot\mu)$-measure. The previous proposition shows the convergence $1+\sum_{i=0}^{k}\frac{1}{i!}Q_i(q)\,(v)^i \to 1+\sum_{i=0}^{\infty}\frac{1}{i!}Q_i(q)\,(v)^i$ in $L^\infty(p\cdot\mu)$-norm and therefore also the convergence in mean. By uniqueness it follows that $1+\sum_{i=0}^{\infty}\frac{1}{i!}Q_i(q)\,(v)^i = e_q^{(v)}$.

Proposition 5. *Let $p\in\mathfrak{M}_\mu$ and $v\in B_{p,\infty}(0,1)$. The series $v\to 1+\sum_{i=0}^{\infty}\frac{1}{i!}Q_i(q)\,(v)^i$ is an analytic function.*

Proof. Since $\Lambda_n(u)\in\mathcal{L}_s^n(L^{p,\infty}(p\cdot\mu),L^{p,\infty}(p\cdot\mu))$, it suffices to prove analyticity in a neighborhood of zero. First, there is the inequality $\|\hat{\lambda}\|_{\mathcal{P}^n}\le\|\lambda\|_{\mathcal{L}_s^n}\le\frac{n^n}{n!}\|\hat{\lambda}\|_{\mathcal{P}^n}$, [18]. By the last inequality and (5), we have

$$\left\|\frac{1}{n!}Q_n(q)\Lambda_n(0)\right\|_{\mathcal{L}_s^n}\le\frac{n^nQ_n(q)(2-q)^{\frac{1}{1-q}}}{(n!)^2}.$$

Applying the Cauchy–Hadamard formula and given that $Q_n(q)\le 1$, we obtain

$$\frac{1}{\hat{\rho}}=\lim_{n\to\infty}\left(\left\|\frac{1}{n!}\frac{Q_n(q)}{(1-q)^n}\Lambda_n(0)\right\|_{\mathcal{L}_s^n}\right)^{\frac{1}{n}}\le\lim_{n\to\infty}\left(\frac{n^nQ_n(q)(2-q)^{\frac{1}{1-q}}}{(n!)^2}\right)^{\frac{1}{n}}\le e.$$

Thus $\hat{\rho}\ge\frac{1}{e}$ and so the restricted radius of convergence of the series is positive and greater than or equal to $\frac{1}{e}$. Applying the above procedure to the n-homogeneous polynomial $\frac{1}{n!}Q_n(q)\hat{\Lambda}_n(v_0)$, where $v_0\in B_{p,\infty}(0,1)$, the restricted radius of convergence satisfies $0<\frac{(1-\|v_0\|_{p,\infty})}{e}\le\hat{\rho}$ and so in a neighborhood of v_0 the map

$$v\to 1+\sum_{n=0}^{\infty}\frac{1}{n!}\frac{Q_n(q)}{(1-q)^n}\hat{\Lambda}_n(v_0)(v)$$

is analytic. □

Definition 6. Defining $D_{M_p}:=\{u\in L^\infty(p\cdot\mu):\frac{-1}{1-q}<u,\,E_p[e_q^{(u)}]<\infty\}$, we define the mapping $M_p:L^\infty(p\cdot\mu)\to[0,\infty]$ by $M_p(u)=E_p[e_q^{(u)}]$.

The domain D_{M_p} of M_p contains the open unit ball $B_{p,\infty}(0,1)\subset L^\infty(p\cdot\mu)$. Also if we restrict its domain to $B_{p,\infty}(0,1)$, we will see that this function is analytic and infinitely Fréchet differentiable.

Theorem 7. *The functional M_p satisfies:*

1. *$M_p(0)=1$ and if $u\ne 0$, $M_p(u)>1$.*
2. *M_p is analytic on the open unit ball $B_{p,\infty}(0,1)$.*
3. *M_p is infinitely Fréchet differentiable and its nth derivative in $u\in B_{p,\infty}(0,1)$ evaluated at $(v_1,v_2,\dots,v_n)\in B_{p,\infty}(0,1)\times\cdots\times B_{p,\infty}(0,1)$ is given by*

$$D^nM_p(u)(v_1,v_2,\dots,v_n)=Q_n(q)E_p[v_1\cdots v_ne_q^{(u)}].$$

Proof. (1) It is obvious from the definition.

(2) For $u\in B_{p,\infty}(0,1)$, $\frac{1}{n!}Q_n(q)E_p[\hat{\Lambda}_n(u)]\in\mathcal{P}^n(L^\infty(p\cdot\mu),\mathbb{R})$ and so $\sum_{n=0}^{\infty}\frac{1}{n!}Q_n(q)E_p[\hat{\Lambda}_n(u)]$ is a power series from $L^\infty(p\cdot\mu)$ into $\mathbb{R}$ with positive radius of convergence. For each u in a neighborhood of $u_0\in B_{p,\infty}(0,1)$, and by Proposition 5 we have

$$e_q^u=1+\sum_{n=0}^{\infty}\frac{1}{n!}\frac{Q_n(q)}{(1-q)^n}\Lambda_n(u_0)(u-u_0)^n,$$

and from this, we obtain

$$M_p(u)=\sum_{n=0}^{\infty}\frac{Q_n(q)}{n!(1-q)^n}E_p[\hat{\Lambda}_n(u_0)(u-u_0)].$$

Consequently, M_p is analytic in $B_{p,\infty}(0,1)$.

(3) It is known that $D^nM_p(u)(v_1,v_2,\dots,v_n)=n!\lambda_n(u)(v_1,v_2,\dots,v_n)$, [18], where $\lambda_n(u)(v_1,v_2,\dots,v_n)=\frac{1}{n!}Q_n(q)E_p[\Lambda_n(u)(v_1,v_2,\dots,v_n)]$ and so

$$\begin{aligned}D^nM_p(u)(v_1,v_2,\dots,v_n)&=Q_n(q)E_p[\Lambda_n(u)(v_1,v_2,\dots,v_n)]\\&=Q_n(q)E_p[v_1\cdots v_n\,e_q^{(u)}].\quad\square\end{aligned}$$

Definition 8. For each $u\in B_{p,\infty}(0,1)$, define $K_p:B_{p,\infty}(0,1)\to[0,\infty]$ by $K_p(u)=\ln_q[M_p(u)]$.

This functional becomes important for defining coordinate maps for the q-exponential information manifold. Some of its properties allow us to obtain explicit expressions for the tangent space of the manifold. Note that $K_p(0) = 1$ and it is clear, by q-deformed logarithmic properties, that if $u \neq 0$, then K_p is strictly positive, continuous, increasing and concave.

Theorem 9. *The functional K_p satisfies:*

1. *For each $u \in B_{p,\infty}(0, 1)$, the function $z = \mathrm{e}_q^{u \ominus_q K_p(u)} p$ is a probability density on $\mathfrak{M}_\mu$.*
2. *K_p is infinitely Fréchet differentiable and its nth derivative at u evaluated in the directions $(v_1 \cdots v_n) \in B_{p,\infty}(0, 1) \times \cdots \times B_{p,\infty}(0, 1)$ (i.e. the continuous n-linear form $D^n K_p(u) \cdot (v_1 \cdots v_n)$) is given by*

$$D^n K_p(u) \cdot (v_1 \cdots v_n) = [M_p(u)]^{1-q} Q_n(q) E_z[v_1 \cdots v_n].$$

3. *K_p is analytic in $B_{p,\infty}(0, 1)$.*

Proof. (1) Since $p \in \mathfrak{M}_\mu$, it follows that $z > 0$. Now, since $z = \frac{\mathrm{e}_q^u}{\mathrm{e}_q^{K_p(u)}} p$ and $\mathrm{e}_q^{K_p(u)} = M_p(u)$, we have $\int z \, p d\mu = \frac{1}{\mathrm{e}_q^{K_p(u)}} \int \mathrm{e}_q^u p d\mu = 1$.

(2) Taking derivatives of K_p (in view of Theorem 7 and Remark 1), it follows that

$$\begin{aligned} D^n K_p(u) \cdot (v_1 \cdots v_n) &= \frac{1}{[M_p(u)]^q} D^n M_p(u) \cdot (v_1 \cdots v_n) \\ &= \frac{1}{[M_p(u)]^q} Q_n(q) E_p[v_1 \cdots v_n \mathrm{e}_q^u]. \end{aligned}$$

Since $z = \mathrm{e}_q^{u \ominus_q K_p(u)} p$, then $\mathrm{e}_q^u \cdot p = z \cdot M_p(u)$ and therefore

$$D^n K_p(u) \cdot (v_1 \cdots v_n) = \frac{1}{[M_p(u)]^q} Q_n(q) M_p(u) E_z[v_1 \cdots v_n].$$

(3) K_p is a composition of analytic functions. □

4. The *q*-exponential statistical manifold

Now we will define a geometrical model similar to the k-exponential model of Pistone [7], where k indicates deformation in the sense of Kaniadakis. The coordinate maps to be defined for the manifold induce a topology on $\mathfrak{M}_\mu$, which is stronger than the topology of $L^1(p \cdot \mu)$.

Let (Ω, Σ, μ) be a probability space and q a real number with $0 < q < 1$. For each $p \in \mathfrak{M}_\mu$, let B_p be the subset of $L^\infty(p \cdot \mu)$ consisting of essentially bounded random variables u such that $E_p[u] = 0$. Clearly B_p is a Banach space and we let $\mathcal{V}_p$ be the open unit ball of B_p, that is, $\mathcal{V}_p := \{u \in B_p : \|u\|_{p,\infty} < 1\}$. We define the maps $\mathrm{e}_{q,p} : \mathcal{V}_p \to \mathfrak{M}_\mu$ by

$$\mathrm{e}_{q,p}(u) := \mathrm{e}_q^{(u \ominus_q K_p(u))} p, \tag{6}$$

which is well defined since $\|u\|_{p,\infty} < 1$ implies $\frac{-1}{1-q} < u$ and hence $\frac{-1}{1-q} < \frac{u - K_p(u)}{1+(1-q)K_p(u)} = u \ominus_q K_p(u)$.

Proposition 10. *Let $p \in \mathfrak{M}_\mu$, then:*

1. *The map $\mathrm{e}_{q,p}$ is injective.*
2. *Let $\mathcal{U}_p$ be the range of $\mathrm{e}_{q,p}$; the map $s_{q,p} : \mathcal{U}_p \to \mathcal{V}_p$ given by*

$$s_{q,p}(z) := \ln_q\left(\frac{z}{p}\right) \ominus_q E_p\left[\ln_q\left(\frac{z}{p}\right)\right] = \frac{\ln_q\left(\frac{z}{p}\right) - E_p\left[\ln_q\left(\frac{z}{p}\right)\right]}{1 + (1-q)E_p\left[\ln_q\left(\frac{z}{p}\right)\right]}, \tag{7}$$

is the inverse function of $\mathrm{e}_{q,p}$.

Proof. (1) Let $p \in \mathfrak{M}_\mu$ and $u_1, u_2 \in \mathcal{V}_p$ be such that $\mathrm{e}_{q,p}(u_1) = \mathrm{e}_{q,p}(u_2)$; that is

$$\frac{u_1 - K_p(u_1)}{1 + (1-q)K_p(u_1)} = \frac{u_2 - K_p(u_2)}{1 + (1-q)K_p(u_2)}. \tag{8}$$

Applying the linear operator $E_p[\cdot]$ on both sides of (8) we obtain

$$\frac{E_p[u_1] - K_p(u_1)}{1 + (1-q)K_p(u_1)} = \frac{E_p[u_2] - K_p(u_2)}{1 + (1-q)K_p(u_2)}.$$

Since $u_1, u_2 \in \mathcal{V}_p \subset B_p$, we have $E_p[u_1] = E_p[u_2] = 0$ and the above expression transforms into

$$-K_p(u_1) - (1-q)K_p(u_1)K_p(u_2) = -K_p(u_2) - (1-q)K_p(u_1)K_p(u_2).$$

So $K_p(u_1) = K_p(u_2)$ and finally, substituting in (8), it follows that $u_1 = u_2$.

(2) Let $p, q \in \mathcal{M}_\mu$. First, we show that $u \in \mathcal{V}_p$ implies $s_{q,p}[\mathrm{e}_{q,p}(u)] = u$. By direct calculation on expressions (6) and (7), we obtain

$$s_{q,p}[\mathrm{e}_{q,p}(u)] = \frac{u - K_p(u) - E_p[u - K_p(u)]}{1 + (1-q)K_p(u) + (1-q)E_p[u - K_p(u)]}. \tag{9}$$

Since $K_p(u)$ is a real constant and $E_p[u] = 0$, applying E_p in (9) we obtain $s_{q,p}[\mathrm{e}_{q,p}(u)] = u$. In an analogous manner, we prove that $z \in \mathcal{U}_p$ implies $\mathrm{e}_{q,p}[s_{q,p}(z)] = z$, and the required result follows. □

Maps $s_{q,p}$ will be coordinate maps for the manifold, these maps assign a unique random variable $u = s_{q,p}(z) \in \mathcal{V}_p$ to a given $z \in \mathcal{U}_p$. We will prove that the family of pairs $(\mathcal{U}_p, s_{q,p})_{p \in \mathfrak{M}_\mu}$ define an atlas on $\mathfrak{M}_\mu$. Consider $p_1, p_2 \in \mathfrak{M}_\mu$ with $\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2} \neq \emptyset$. By direct calculation we have that the maps $s_{p_2} \circ e_{p_1} : s_{q,p_1}(\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2}) \to s_{q,p_2}(\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2})$ are given for each $u \in s_{q,p_1}(\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2})$ by

$$s_{p_2}(e_{p_1}(u)) = \frac{u + [1+(1-q)u]\ln_q\left(\frac{p_1}{p_2}\right) - E_{p_2}\left[u + [1+(1-q)u]\ln_q\left(\frac{p_1}{p_2}\right)\right]}{1 + (1-q)E_{p_2}\left[u + [1+(1-q)u]\ln_q\left(\frac{p_1}{p_2}\right)\right]}. \tag{10}$$

This maps will be the transition ones for the manifold.

Proposition 11. *Let $p_1, p_2 \in \mathfrak{M}_\mu$. The set $s_{q,p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$ is open in the B_{p_1}-topology.*

Proof. Let $u \in s_{q,p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$. We will see that u is an interior point of $s_{q,p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$. Clearly $\|u\|_{p_1,\infty} < 1$ since $s_{q,p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2}) \subset \mathcal{V}_{p_1}$. For $r < 1 - \|u\|_{p_1,\infty}$, consider the ball $A_r = \left\{v \in \mathcal{V}_{p_1} : \|v - u\|_{p_1,\infty} < r\right\}$. Provided that $p_1 \sim_q p_2$, by Proposition 2 we get $\|v-u\|_{p_2,\infty} = \|v-u\|_{p_1,\infty}$, and also for $\|u\|_{p_1,\infty}$; hence $\|v\|_{p_2,\infty} - \|u\|_{p_2,\infty} \le \|v-u\|_{p_2,\infty} = \|v-u\|_{p_1,\infty}$. Thus $\|v\|_{p_2,\infty} - \|u\|_{p_1,\infty} < r$ and $\|v\|_{p_2,\infty} < 1$. This implies that $A_r \subset s_{q,p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$, so the desired result holds. □

This result and Proposition 2 allow us to prove the next result.

Proposition 12. *For each $p_1,\ p_2 \in \mathfrak{M}_\mu$, the function $s_{q,p_2} \circ e_{q,p_1}$ is a topological homeomorphism.*

The maps given in (10) can be written in terms of the q-deformed operations in the following manner. Let $u \in \mathcal{V}_{p_1}$, $\bar{u} \in \mathcal{V}_{p_2}$ and $z \in (\mathcal{U}_{q,p_1} \bigcap \mathcal{U}_{q,p_2})$ with $s_{q,p_2} \circ e_{q,p_1}(z) = \bar{u}$. Then

$$\begin{aligned}
\bar{u} &= \ln_q\left(\mathrm{e}_q^{u \ominus_q K_{p_1}(u)} \frac{p_1}{p_2}\right) \ominus_q E_{p_2}\left[\ln_q\left(\mathrm{e}_q^{u \ominus_q K_{p_1}(u)} \frac{p_1}{p_2}\right)\right] \\
&= \left(u \ominus_q K_{p_1}(u)\right) \oplus_q \ln_q\left(\frac{p_1}{p_2}\right) \ominus_q E_{p_2}\left[\left(u \ominus_q K_{p_1}(u)\right) \oplus_q \ln_q\left(\frac{p_1}{p_2}\right)\right] \\
&= \left[\left(u \ominus_q K_{p_1}(u)\right) \oplus_q \ln_q\left(\frac{p_1}{p_2}\right)\right] \ominus_q E_{p_2}\left[\left(u \ominus_q K_{p_1}(u)\right) \oplus_q \ln_q\left(\frac{p_1}{p_2}\right)\right].
\end{aligned}$$

Hence

$$\bar{u} = f(u) \ominus_q E_{p_2}[f(u)], \quad \text{where} \tag{11}$$

$$f(u) = \left[\left(u \ominus_q K_{p_1}(u)\right) \oplus_q \ln_q\left(\frac{p_1}{p_2}\right)\right]. \tag{12}$$

Expressions (11), (12) permit simplifying calculations to prove the following proposition.

Proposition 13. *Given $u \in s_{q,p_1}(\mathcal{U}_{q,p_1} \bigcap \mathcal{U}_{q,p_2})$, the derivative of map $s_{q,p_2} \circ s_{q,p_1}^{-1}$ evaluated at u in the direction of $v \in L^\infty(p_1 \cdot \mu)$ is of the form*

$$D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = A(u) - B(u)E_{p_2}[A(u)],$$

where $A(u)$, $B(u)$ are constants depending on u.

Proof. We know that $\bar{u} = s_{q,p_2} \circ s_{q,p_1}^{-1}(u) = \frac{f(u)-E_{p_2}[f(u)]}{1+(1-q)E_{p_2}[f(u)]}$ and therefore

$$D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = \frac{[1+(1-q)E_{p_2}[f(u)]]\frac{\partial f(u)}{\partial u} - [1+(1-q)f(u)]\frac{\partial E_{p_2}[f(u)]}{\partial u}}{\big[1+(1-q)E_{p_2}[f(u)]\big]^2},$$

which can be written as

$$D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = \frac{\frac{\partial f(u)}{\partial u}}{[1+(1-q)E_{p_2}[f(u)]]} - \left[\frac{[1+(1-q)f(u)]}{[1+(1-q)E_{p_2}[f(u)]]^2}\right]\frac{\partial E_{p_2}[f(u)]}{\partial u}. \tag{13}$$

Note that

$$\begin{aligned} E_{p_2}[f(u)] &= E_{p_2}\left[\left(u \ominus_q K_{p_1}(u)\right) \oplus_q \ln_q\left(\frac{p_1}{p_2}\right)\right] \\ &= E_{p_2}\left[\left(u \ominus_q K_{p_1}(u)\right) + \ln_q\left(\frac{p_1}{p_2}\right) + (1-q)\left(u \ominus_q K_{p_1}(u)\right)\ln_q\left(\frac{p_1}{p_2}\right)\right] \\ &= E_{p_2}\left[\ln_q\left(\frac{p_1}{p_2}\right)\right] + \left[1+(1-q)\ln_q\left(\frac{p_1}{p_2}\right)\right]E_{p_2}\left[u \ominus_q K_{p_1}(u)\right] \end{aligned}$$

and applying the partial derivative with respect to u, we obtain

$$\frac{\partial E_{p_2}[f(u)]}{\partial u} = E_{p_2}\left[\frac{\left[1+(1-q)K_{p_1}(u) - [1+(1-q)u]DK_{p_1}(u)\cdot v\right]W}{[1+(1-q)K_{p_1}(u)]^2}\right],$$

where $W = 1+(1-q)\ln_q\left(\frac{p_1}{p_2}\right)$. Now, taking the partial derivative with respect to u in (12) we have that the argument in E_{p_2} of this last expression is precisely

$$\frac{\partial f(u)}{\partial u} = \left[\frac{\left[1+(1-q)K_{p_1}(u) - DK_{p_1}(u)v[1+(1-q)u\right]]W}{[1+(1-q)K_{p_1}(u)]^2}\right].$$

From all the above, using (13), it follows that

$$D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = \frac{\frac{\partial f(u)}{\partial u}}{[1+(1-q)E_{p_2}[f(u)]]} - \frac{[1+(1-q)f(u)]}{[1+(1-q)E_{p_2}[f(u)]]^2}E_{p_2}\left[\frac{\partial f(u)}{\partial u}\right]$$

and since $E_{p_2}[f(u)]$ is constant we can write this last expression as $D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = \frac{\frac{\partial f(u)}{\partial u}}{[1+(1-q)E_{p_2}[f(u)]]} - \frac{[1+(1-q)f(u)]}{[1+(1-q)E_{p_2}[f(u)]]}$ $E_{p_2}\left[\frac{\frac{\partial f(u)}{\partial u}}{[1+(1-q)E_{p_2}[f(u)]]}\right]$. Now, taking

$$A(u) = \frac{\frac{\partial f(u)}{\partial u}}{\big[1+(1-q)E_{p_2}[f(u)]\big]} \quad \text{and} \quad B(u) = \frac{[1+(1-q)f(u)]}{[1+(1-q)E_{p_2}[f(u)]]},$$

the derivative can be written finally in the form

$$D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = A(u) - B(u)E_{p_2}[A(u)]. \quad \square \tag{14}$$

We can then establish the main theorem of this section.

Theorem 14. *The collection of pairs $\{(\mathcal{U}_p, s_{q,p})\}_{p\in\mathfrak{M}_\mu}$ is a C^∞-atlas modeled on B_p.*

Proof. Consider the family of pairs $(\mathcal{U}_p, s_{q,p})_{p\in\mathfrak{M}_\mu}$. Obviously the family of $\mathcal{U}_p$ is an open cover of $\mathfrak{M}_\mu$, each $\mathcal{U}_p$ being open in $\mathfrak{M}_\mu$, and $s_{q,p}$ is bijective by Proposition 10. This means that each $(\mathcal{U}_p, s_{q,p})$ is a chart and the corresponding transition map for $p_1, p_2 \in \mathfrak{M}_\mu$ (with $\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2} \neq \varnothing$) is given by $s_{q,p_2} \circ e_{q,p_1} : s_{q,p_1}(\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2}) \to s_{q,p_2}(\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2})$, as defined in (10). By Proposition 11, sets $s_{q,p}(\mathcal{U}_p \bigcap \mathcal{U}_{p_2})$ are open in the topology of B_p, which in turn implies that coordinate maps $s_{q,p}$ are homeomorphisms. Proposition 12 guarantees that spaces $s_{q,p_1}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$ and $s_{q,p_2}(\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2})$ are topologically homeomorphic (and therefore the map defined by (10) is a topological isomorphism). It remains to prove that the transition map $s_{q,p_2} \circ e_{q,p_1}$ is C^∞. To do this, observe that the map defined by (10) can be written in the form $s_{q,p_2} \circ e_{q,p_1}(u) = \frac{1}{g(u)}h(u)$, where

$$h(u) = u + [1+(1-q)u]\ln_q\left(\frac{p_1}{p_2}\right) - E_{p_2}\left[u + [1+(1-q)u]\ln_q\left(\frac{p_1}{p_2}\right)\right]$$

and

$$g(u) = 1 + \left\{ [1 + (1-q)K_{p_2}(u)](1-q)E_{p_2}\left[u + [1+(1-q)u]\ln_q\left(\frac{p_1}{p_2}\right)\right]\right\}.$$

So finally it suffices to prove that h and g are C^∞. There is $z \in (\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2})$ such that $p_2 = e_{q,z}(u)$ and thus $u + [1 + (1-q)u]\ln_q(\frac{p_1}{p_2}) \in \mathcal{V}_z$. Defining $v = u + [1 + (1-q)u]\ln_q(\frac{p_1}{p_2})$ and applying (2) of Theorem 9, we obtain $DK_z(u)v = [M_p(u)]^{1-q}E_{p_2}[v]$. Now, given that K_z is C^∞ in $\mathcal{V}_z$, it follows that $E_{p_2}[u + [1 + (1-q)u]\ln_q(\frac{p_1}{p_2})]$ is C^∞. Since every affine function is C^∞ and h can be expressed as the sum of an affine function and a C^∞-function, h is certainly C^∞. Analogously, since K_{p_2} is C^∞ in $\mathcal{V}_{p_2}$, it follows that g is also C^∞. Proposition 12 also guarantees that the defined atlas is modeled on Banach spaces. □

5. The tangent bundle

Now we construct the tangent bundle for the manifold given in the previous section, showing its natural identification with one-dimensional parametric q-exponential models. Also, we prove the relationship between the tangent vector of the manifold and a q-deformation of the score function.

Let $p \in \mathfrak{M}_\mu$; a curve through p is a one-dimensional parametric q-exponential model: $g : I \subset \mathcal{R} \to \mathfrak{M}_\mu$ such that $t \to g(t) \in \mathfrak{M}_\mu$, where $g(t_0) = p$ for some $t_0 \in I$. Usually it is assumed that $t_0 = 0$; nevertheless this is unnecessary in this framework. Let $(\mathcal{U}_{p_1}, s_{q,p_1})$ and $(\mathcal{U}_{p_2}, s_{q,p_2})$ be charts around $p \in \mathfrak{M}_\mu$; then $g(t) = e_{q,p_1}(u_1) = e_q^{u_1 \ominus_q K_{p_1}(u_1)} p_1$, where for every $t \in I$ it follows that $u_1(t) = s_{q,p_1}(g(t))$. Additionally $g(t) = e_{q,p_2}(u_2)$ and $u_2(t) = s_{q,p_2}(g(t))$. The random variables $u_1(t_0)$ and $u_2(t_0)$ are related by $u_2(t_0) = \left(s_{q,p_2} \circ e_{q,p_1}\right)(u_1(t_0))$. We have that $u_1(t_0) = s_{q,p_1}(p)$, and by the chain rule $u_2'(t_0) = \left(s_{q,p_2} \circ e_{q,p_1}\right)'(u_1(t_0)) \cdot u_1'(t_0)$ which equals $u_2'(t_0) = \left(s_{q,p_2} \circ e_{q,p_1}\right)'(s_{q,p_1}(p)) \cdot u_1'(t_0)$. The above expression is an equivalence relation which fulfills the requirements for defining the tangent space over a manifold [9], and therefore the tangent space for $p \in \mathfrak{M}_\mu$ is a topological vector space with the induced topology of any of spaces $L^\infty(z \cdot \mu)$ such that $p \in \mathcal{U}_z$, and is given by $\mathcal{T}_p(\mathfrak{M}_\mu) = \{[(\mathcal{U}_p, s_{q,p}, u')] : p \in \mathfrak{M}_\mu\}$.

Proposition 15. *Let $g(t)$ be a regular curve for $\mathfrak{M}_\mu$ where $g(t_0) = p$, and $u(t) \in \mathcal{V}_z$ be its coordinate representation over $s_{q,z}$. Then $g(t) = e_q^{[u(t) \ominus_q K_z(u(t))]} z$ and also:*

1. $\frac{d}{dt}\ln_q\left(\frac{g(t)}{p}\right)_{t=t_0} = Tu'(t) - Q[M_p(u(t))]^{1-q}E_p[u'(t)]$ *for some constants T and Q.*
2. *If $z = p$, i.e. the charts are centered at the same point; the tangent vectors are identified with the q-score function in t given by* $\frac{d}{dt}\ln_q\left(\frac{g(t)}{p}\right)_{t=t_0} = Tu'(t_0)$.
3. *Consider a two-dimensional q-exponential model*

$$f(t, q) = e_q^{(tu \ominus_q K_p(tu))} p \tag{15}$$

where if q tends to 1 in (15), one obtains the one-dimensional exponential models $e^{(tu - K_p(tu))}p$. The q-score function $\frac{d}{dt}\ln_q\left(\frac{f(t)}{p}\right)_{t=t_0}$ *for a one-dimensional q-exponential model (15) belongs to span[u] at $t = t_0$, where $u \in \mathcal{V}_p$.*

Proof. 1.

$$\begin{aligned}
\frac{d}{dt}\ln_q\left(\frac{g(t)}{p}\right)_{t=t_0} &= \frac{d}{dt}\ln_q\left[\frac{e_q^{[u(t)\ominus_q K_z(u(t))]}z}{e_q^{[u(t_0)\ominus_q K_z(u(t_0))]}z}\right]_{t=t_0} \\
&= \frac{d}{dt}\left[\frac{[u(t) - K_z(u(t))]\left[\frac{1+(1-q)K_z(u(t_0))}{1+(1-q)K_z(u(t))}\right] - [u(t_0) - K_z(u(t_0))]}{[1+(1-q)K_z(u(t_0))]}\right]_{t=t_0} \\
&= \frac{d}{dt}\left[\frac{u(t) - K_z(u(t))}{1+(1-q)K_z(u(t))}\right]_{t=t_0} \\
&= \frac{1}{[1+(1-q)K_z(u(t_0))]}u'(t_0) - \frac{[1+(1-q)u(t_0)]}{[1+(1-q)K_z(u(t_0))]^2}\left(\frac{d}{dt}K_z(u(t_0))\right) \\
&= Tu'(t_0) - Q\left(\frac{d}{dt}K_z(u(t_0))\right),
\end{aligned} \tag{16}$$

where $T = \frac{1}{[1+(1-q)K_z(u(t_0))]}$ and $Q = \frac{[1+(1-q)u(t_0)]}{[1+(1-q)K_z(u(t_0))]^2}$. From Theorem 9 it follows that $\frac{d}{dt}K_z(u(t)) = [M_p(u(t))]^{1-q}E_{g(t)}[u'(t)]$. Hence $\frac{d}{dt}\ln_q\left(\frac{g(t)}{p}\right)_{t=t_0} = Tu'(t_0) - Q[M_p(u(t))]^{1-q}E_{g(t_0)}[u'(t_0)]$.

2. If $z = p$ then $E_p[u'(t_0)] = E_z[u'(t_0)]$; since $u(t)$ is centered then $E_p[u'(t_0)] = E_z[u'(t_0)] = 0$ and the desired result follows.
3. Taking $u(t) = tu$ in (1), a simple calculation leads to

$$\frac{d}{dt}\ln_q\left(\frac{f(t)}{p}\right)_{t=t_0} = Tu. \quad \square$$

So the tangent space $T_p(\mathfrak{M}_\mu)$ is identified with the collection of q-score functions or the one-dimensional q-exponential models (15).

The model $\mathfrak{M}_\mu$ glued together with the collection of tangent spaces is the tangent bundle, denoted as $\mathcal{T}(\mathfrak{M}_\mu)$. So,

$$\mathcal{T}(\mathfrak{M}_\mu) := \left\{(f, u) : f \in \mathcal{U}_p \subset \mathfrak{M}_\mu \text{ and } u \text{ is the class of tangent vectors to } f\right\}.$$

It is known that this is a manifold, where the charts (trivializing mappings) are given by $(g, u) \in \mathcal{T}(\mathcal{U}_p) \to (s_{q,p}(g), A(u) - B(u)E_p[A(u)])$, defined in the collection of open subsets $\mathcal{U}_p \times \mathcal{V}_p$ of $\mathfrak{M}_\mu \times L^\infty(p \cdot \mu)$.

Transition mappings for this manifold are given for $(u_1, v_1) \in \mathcal{V}_p \times L^\infty(p \cdot \mu)$ by

$$(u_1, v_1) \to \left((s_{q,z} \circ \mathrm{e}_{q,p})_{(u_1)}, A(v_1) - B(v_1)E_z[A(v_1)]\right) \in \mathcal{V}_z \times L^\infty(z \cdot \mu),$$

which gives the local representation of the bundle.

6. *q*-divergence

In this section, we will show the relationship between the Tsallis relative entropy functional and the manifold constructed in Section 4.

Let $p, z \in \mathfrak{M}_\mu$; the q-divergence (Tsallis's divergence) of z with respect to p is given by

$$I^{(q)}(z \parallel p) = \int_\Omega p f\left(\frac{z}{p}\right) d\mu, \tag{17}$$

where f is a function defined for all $t \neq 0$ and $0 < q < 1$ as

$$f(t) = -t \ln_q\left(\frac{1}{t}\right). \tag{18}$$

Some properties of this functional are well known, for example that it is equal to the α-divergence functional up to a constant factor where $\alpha = 1 - 2q$, satisfying the invariance criterion; also, when $q \to 0$ then $I^{(q)}(z \parallel p) = 0$ and if $q \to 1$ then $I^{(q)}(z \parallel p) = K(z \parallel p)$ which is the Kullback–Leibler divergence functional [19]. For further properties of this functional see [12].

The proof of the next proposition is analogous to the one given in [12].

Proposition 16. *Given $p, z \in \mathfrak{M}_\mu$ then:*

1. $I^{(q)}(z \parallel p) \geq 0$ *and equality holds iff* $p = z$.
2. $I^{(q)}(z \parallel p) \leq \int_\Omega (z - p) f'\left(\frac{z}{p}\right) d\mu$.

Proof. Let $p, z \in \mathfrak{M}_\mu$;

1. $I^{(q)}(z \parallel p) = \int_\Omega p f\left(\frac{z}{p}\right) d\mu = \int_\Omega f\left(\frac{z}{p}\right) dp \cdot \mu$, and by the Jensen inequality and properties of f, it follows that

$$\int_\Omega f\left(\frac{z}{p}\right) dp \cdot \mu \geq f\left[\int_\Omega \left(\frac{z}{p}\right) dp \cdot \mu\right] = 0,$$

so $I^{(q)}(z \parallel p) \geq 0$. Equality follows immediately.

2. Since f is convex it is easy to show that

$$f'\left(\frac{z}{p}\right)\left(\frac{z}{p} - 1\right) \geq f\left(\frac{z}{p}\right) - f(1) \geq f'(1)\left(\frac{z}{p} - 1\right);$$

if we multiply by p and integrate on Ω, and since $f(1) = 0$, we can deduce that

$$\int_\Omega (z - p) f'\left(\frac{z}{p}\right) d\mu \geq I^{(q)}(z \parallel p) \geq \int_\Omega (z - p) f'(1)\, d\mu,$$

and since $f'(1) = 1$, the desired result follows. $\square$

Note that the upper bound in part 2 of this proposition simplifies to

$$\int_{\Omega}(z-p)\,f'\left(\frac{z}{p}\right)\,d\mu = \frac{q}{1-q}\int_{\Omega}(p-z)\left(\frac{p}{z}\right)^{1-q}\,d\mu$$
$$= \frac{q}{1-q}\left\{E_p\left[\left(\frac{p}{z}\right)^{1-q}\right]-E_z\left[\left(\frac{p}{z}\right)^{1-q}\right]\right\}.$$

Proposition 17. *Let $p \in \mathfrak{M}_\mu$, $z \in \mathcal{U}_p$ and $u = s_{q,p}(z)$. If we take $^*u = \frac{z}{p} - 1$ and have the mapping $H_p(^*u) = E_p\big[(1+{}^*u)\ln_q\big(\frac{1}{1+{}^*u}\big)\big]$, then*

$$I^{(q)}(p \parallel z) = \frac{K_p(u)}{1+(1-q)K_p(u)} \quad \text{and} \quad H_p(^*u) = -I^{(q)}(z \parallel p).$$

Proof. The definition of q-divergence implies that $I^{(q)}(p \parallel z) = -E_p\big[\ln_q\big(\frac{z}{p}\big)\big]$. Since $z = e_q^{[u \ominus_q K_p^q(u)]}p$, we get

$$I^{(q)}(p \parallel z) = -E_p\left[\ln_q\left(\frac{z}{p}\right)\right] = -\left(E_p\left[\frac{u-K_p(u)}{1+(1-q)K_p(u)}\right]\right)$$
$$= -\left(\frac{E_p[u]-K_p(u)}{1+(1-q)K_p(u)}\right) = \frac{K_p(u)}{1+(1-q)K_p(u)}.$$

Finally, we have

$$H_p(^*u) = E_p\left[(1+{}^*u)\ln_q\left(\frac{1}{1+{}^*u}\right)\right] = E_p\left[\left(\frac{z}{p}\right)\ln_q\left(\frac{p}{z}\right)\right] = -I^{(q)}(z \parallel p). \quad \square$$

There are two important aspects to note. One is that the coordinate mappings

$$s_{q,p}(z) = \ln_q\left(\frac{z}{p}\right) \ominus_q E_p\left[\ln_q\left(\frac{z}{p}\right)\right],$$

can be written as

$$s_{q,p}(z) = \left(\frac{1}{1+(q-1)I^{(q)}(p \parallel z)}\right)\left(\ln_q\left(\frac{z}{p}\right) + I^{(q)}(p \parallel z)\right);$$

so the relation between the q-divergence and the manifold constructed is clear in the sense that the coordinate representation of the density $z \in \mathcal{U}_p$ contains the information on p with respect to z. On the other hand, since $s_{q,p}(z) = u$, we get

$$E_z[u] = \left(\frac{1}{1+(q-1)I^{(q)}(p \parallel z)}\right)\left(I^{(q)}(p \parallel z) - D^{(q)}(z \parallel p)\right)$$

where $D^{(q)}(z \parallel p)$ is a q-deformation of the Kullback–Leibler divergence functional.

7. Conclusions

We have presented a non-parametric construction of a statistical Banach manifold where positive densities are represented with the Tsallis q-exponential for the case $0 < q < 1$. This representations of densities, using deformed exponentials, has been considered by many authors in the finite dimensional setting [8,17]; moreover, the infinite dimensional case is interesting from the conceptual point of view and the methodological point of view. Some examples of such constructions can be found in [20]. The formalism for the q-exponential Banach manifold is derived in section [4], noting that the standard case is recovered when $q \to 1$; nevertheless the manifolds constructed differ in the modeling spaces. Since $L^\infty(p \cdot \mu)$ is a subset of $L^\Psi(p \cdot \mu)$ for any Young function [3,21], it is interesting to find such a Young function for which the modeling spaces, of the q-exponential and exponential manifolds, are related.

The use of $L^\infty(p\cdot\mu)$ as a modeling space is interesting itself, since $(u_1-u_2) \in L^\infty(p\cdot\mu)$ for $q_1 = e_{q,p}(u_1)$ and $q_2 = e_{q,p}(u_2)$; and this is a necessary condition for defining an isomorphism between tangent spaces of the exponential manifold, which can be understood as a parallel transport on the convex mixture of two densities—see [22]; so we expect that the explicit construction of such an isomorphism in the q-exponential manifold should be well defined.

Many applications of statistical models and, particularly, parametric exponential models are known for when the state space Ω is finite; moreover, it has been shown that certain parametric and infinitely many parametric models are submanifolds of the exponential manifold, even if Ω is not finite—see [20]—so it should be investigated which statistical models are submanifolds of the q-exponential manifold, i.e., which cases are covered by this new manifold.

Since tangent vectors of this manifold are related to one-dimensional q-exponential models (regular curves) by Proposition 15, and these reduce to exponential ones, an explicit calculation of the tangent bundle should show the one-dimensional exponential models which are embedded in this manifold, and also the q-exponential family constructed in [8].

The functional $I^{(q)}(z \parallel p)$ given in (17) induces a Riemannian structure on $\mathfrak{M}_\mu$; we expect this to be a flat one, and to characterize the geodesic curves and parallel transports.

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