Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 154 (2007) 2786–2794
Topology and its Applications
www.elsevier.com/locate/topol

On a minimal factorization conjecture

Carlos A. Cadavid [a,*], Juan D. Vélez [b]

[a] *Departamento de Ciencias Básicas, Universidad EAFIT, Medellín, Colombia*
[b] *Departamento de Matemáticas, Universidad Nacional de Colombia, Medellín, Colombia*

Received 9 February 2007; received in revised form 6 June 2007; accepted 6 June 2007

Abstract

Let $\phi : S \to D$ be a proper holomorphic map from a connected complex surface S onto the open unit disk $D \subset \mathbb{C}$, with $0 \in D$ as its unique singular value, and having fiber genus $g > 0$. Assume that in case $g \geqslant 2$, $\phi : S \to D$ admits a deformation $\phi' : S' \to D$ whose singular fibers are all of simple Lefschetz type. It has been conjectured that the factorization of the monodromy $f \in \mathcal{M}_g$ around $\phi^{-1}(0)$ in terms of right-handed Dehn twists induced by the monodromy of $\phi' : S' \to D$ has the least number of factors among all possible factorizations of f as a product of right-handed Dehn twists in the mapping class group (see [M. Ishizaka, One parameter families of Riemann surfaces and presentations of elements of mapping class group by Dehn twists, J. Math. Soc. Japan 58 (2) (2006) 585–594]). In this article, the validity of this conjecture is established for $g = 1$.

MSC: 14D05; 32S30; 14B07; 11F06

Keywords: Elliptic fibration; Singular fiber; Deformation; Monodromy; Modular group; Dehn twist

1. Introduction

Let $\phi : S \to D$ be a proper holomorphic map from a connected complex surface S onto the open unit disk $D \subset \mathbb{C}$, with $0 \in D$ as its unique singular value. The restriction $\phi : S - \phi^{-1}(0) \to D - \{0\}$ is therefore a locally trivial fiber bundle whose fiber is diffeomorphic to a closed connected orientable surface of genus g. From now on we assume that $g \geqslant 1$ and that S is minimal (i.e. S does not contain smooth rational curves having selfintersection -1). In some special cases, it is possible to *morsify* this map, i.e. to deform it into a new map $\phi' : S' \to D$ with a finite number of singular values, and such that each singular fiber is either of simple Lefschetz type (i.e. it contains exactly one critical point and $\phi'(z, w) = z^2 + w^2$ in appropiate local coordinates near the critical point and its image in the disk), or of smooth multiple type. We remark that maps with fiber genus $g = 1$ can always be morsified (see [10]). The map $\phi' : S' \to D$ is called a *morsification of* $\phi : S \to D$, and a morsification will be called *special* if either (i) $g \geqslant 2$ and it does not contain smooth multiple fibers, or (ii) $g = 1$. It is easily seen that the number of simple Lefschetz fibers in a special morsification of $\phi : S \to D$ (if it exists) equals $E_\phi = \chi(S) - (2 - 2g) = \chi(\text{singular fiber}) - \chi(\text{regular fiber})$ where χ denotes the Euler characteristic, and is therefore independent of the (special) morsification.

* Corresponding author.
E-mail address: ccadavid@eafit.edu.co (C.A. Cadavid).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.06.003

Now, each special morsification $\phi' : S' \to D$ induces a factorization of the *total monodromy* of $\phi : S \to D$ (i.e. the monodromy along a loop surrounding positively and once the unique critical value $0 \in D$) as a product of E_ϕ right-handed Dehn twists. The fact that the number of factors is independent of the choice of special morsification suggests that these factorizations should be somehow distinguished among the possible factorizations of the total monodromy in terms of right-handed Dehn twists, in the corresponding mapping class group. It has been conjectured by one of the authors and by M. Ishizaka [3] that any factorization of the total monodromy of $\phi : S \to D$ in terms of right-handed Dehn twists has at least E_ϕ factors. In this article we prove that the conjecture holds in the $g = 1$ case, as a result of a careful combinatorial study of the usual presentation of the modular group $PSL(2, \mathbb{Z})$ as the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. A counterexample for each $g \geqslant 2$ has been given by Ishizaka (see [3]).

The problem considered in this article is an instance of a class of problems collectively called *growth problems* in group theory. These can be described as follows. Let G be a group and $S \subset G$ such that every $f \in G$ can be written as $f = s_1 \cdots s_r$ where each s_i belongs to S. The *S-length of f* is defined as

$$l_S(f) := \min\{r\colon\ f = s_1 \cdots s_r \text{ for some } s_i \in S\}$$

and the *stable S-length of f* is defined as

$$\|f\|_S := \lim_{n\to\infty} \frac{l_S(f^n)}{n}.$$

The goal is to calculate or estimate these numbers in terms of some information about the element f. The cases where $G = \mathcal{M}_g$ and S is respectively the collection T_g of torsion elements, the collection C_g of all commutators, the collection D_g^+ of right-handed Dehn twists, and the collection PD_g^+ consisting of the positive powers of right-handed Dehn twists, have been recently studied because they arise naturally in several topological contexts (see [6,5,8,11,1] and the references therein). For instance, the case $S = D_g^+$ is related to questions about the geography of closed symplectic 4-manifolds, and the case $S = PD_g^+$ is related to the differential topological analogues of Szpiro's inequality. (This inequality measures the obstruction to the clustering of critical points in an elliptic semistable fibration over a curve [7].)

This article is organized as follows. Section 2 gives the basic definitions and the key classical results used in the rest of the paper. Section 3 presents the technical combinatorial results necessary for the proof of the main result. The main result is proven in Section 4.

2. Preliminaries

2.1. Kodaira's classification of singular fibers

In this section S will denote a non-compact complex surface and D the open unit disk in the complex plane.

Definition 1. By a *family of curves of genus g* we will mean a triple (ϕ, S, D) where $\phi : S \to D$ is a proper surjective holomorphic map with a finite number of critical values $q_1, \ldots, q_n \in D$, such that the preimage of each regular value is a compact connected Riemann surface of genus g. If $g = 1$ the family is also called a *family of elliptic curves*.

Two families (ϕ_1, S_1, D) and (ϕ_2, S_2, D) of curves of genus g are said to be *topologically* (resp., C^∞) *equivalent* if there exist orientation preserving homeomorphisms (resp. diffeomorphisms) $h : S_1 \to S_2$ and $h' : D \to D$ such that $\phi_2 \circ h = h' \circ \phi_1$.

A family is called *minimal* if S does not contain any (-1)-curve.

For a given family of curves of genus g we choose a closed disk of radius $0 < r < 1$ centered at the origin, D_r, such that all the critical values lie in its interior, and denote by q_0 the point $(r, 0)$. By C_r we will denote the boundary of this disk with its standard counterclockwise orientation. As usual,

$$\rho : \pi_1\bigl(D - \{q_1, \ldots, q_n\}, q_0\bigr) \to \mathcal{M}_g$$

will stand for the *monodromy representation* where $\mathcal{M}_g$ denotes the mapping class group of a closed connected oriented 2-manifold of genus g (model of a regular fiber). The anti-homomorphism ρ is determined by its action on any basis of the rank n free group $\pi_1(D - \{q_1, \ldots, q_n\}, q_0)$. Let $\{[\gamma_1], \ldots, [\gamma_n]\}$ be a standard basis consisting of the classes of clockwise oriented, pairwise disjoint arcs (except for q_0) where each γ_i surrounds exclusively the critical value q_i. We can always choose the γ_i's in such a way that (for an appropiate numbering of the q_i's) the product $[\gamma_1]\dots[\gamma_n]$ equals the class of C_r. In this article we will only be concerned with the case $g = 1$. In this case the mapping class group $\mathcal{M}_1$ is isomorphic to $SL(2,\mathbb{Z})$.

Each fiber $\phi^{-1}(q) = \sum m_i X_i$ will be regarded as a (effective) pull-back divisor, where the X_i's are its irreducible components and the m_i's are their corresponding multiplicities. Let us recall that the *multiplicity* m of $\phi^{-1}(q)$ is defined as the greatest common divisor of the m_i's, and the divisor is called *simple* or *multiple* according as $m = 1$ or $m > 1$.

Definition 2. Following Kodaira [4], by the *type* of the fiber $\phi^{-1}(q)$ we will mean the homeomorphism type of the pair $(\phi^{-1}(q), \sum m_i[X_i])$ consisting of the (triangulable) topological space $\phi^{-1}(q)$ and the homology class in $H_2(\phi^{-1}(q),\mathbb{Z})$ determined by the 2-cycle $\sum m_i[X_i]$. (In general, two pairs (X,α) and (Y,β) where X, Y are topological spaces, and $\alpha \in H_*(X,\mathbb{Z})$ and $\beta \in H_*(Y,\mathbb{Z})$ are said to have the same homeomorphism type if there is a homeomorphism $f : X \to Y$ such that $f_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$ sends α to β.)

We will rely heavily on the following classical result of Kodaira (see [4], Theorem 6.2).

Theorem 3. *Let (ϕ, S, D) be a minimal family of elliptic curves and q a critical value of ϕ. Then the type of the fiber $\phi^{-1}(q)$ is one (and only one) of the following.*

mI_0: mX_0, $m > 1$, *where X_0 is a non-singular elliptic curve.*
mI_1: mX_0, $m \geqslant 1$, *where X_0 is a rational curve with an ordinary double point.*
mI_2: $mX_0 + mX_1$, $m \geqslant 1$, *where X_0 and X_1 are non-singular rational curves with intersection $X_0 \cdot X_1 = p_1 + p_2$.*
II: $1X_0$ *where X_0 is a rational curve with one cusp.*
III: $X_0 + X_1$ *where X_0 and X_1 are non-singular rational curves with $X_0 \cdot X_1 = 2p$.*
IV: $X_0 + X_1 + X_2$, *where X_0, X_1, X_2 are non-singular rational curves and $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_0 = p$.*
The rest of the types are denoted by mI_b, $b \geqslant 3$, I_b^*, II^*, III^*, IV^* and are composed of non-singular rational curves $X_0, X_1, \dots, X_s, \dots$ such that $X_s \cdot X_t \leqslant 1$ (i.e. X_s and X_t have at most one simple intersection point) for $s < t$ and $X_r \cap X_s \cap X_t$ is empty for $r < s < t$. These types are therefore described completely by showing all pairs X_s, X_t with $X_s \cdot X_t = 1$ together with $\sum m_i X_i$.
mI_b: $mX_0 + mX_1 + \dots + mX_{b-1}$, $m = 1, 2, 3, \dots$, $b = 3, 4, 5, \dots$, $X_0 \cdot X_1 = X_1 \cdot X_2 = \dots = X_s \cdot X_{s+1} = \dots = X_{b-2} \cdot X_{b-1} = X_{b-1} \cdot X_0 = 1$.
I_b^*: $X_0 + X_1 + X_2 + X_3 + 2X_4 + \dots + 2X_{4+b}$ *where* $b \geqslant 0$, *and* $X_0 \cdot X_4 = X_1 \cdot X_4 = X_2 \cdot X_{4+b} = X_3 \cdot X_{4+b} = X_4 \cdot X_5 = X_5 \cdot X_6 = \dots = X_{3+b} \cdot X_{4+b} = 1$.
II^*: $X_0 + 2X_1 + 3X_2 + 4X_3 + 5X_4 + 6X_5 + 4X_6 + 3X_7 + 2X_8$, *where* $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_4 = X_4 \cdot X_5 = X_5 \cdot X_7 = X_5 \cdot X_6 = X_6 \cdot X_8 = 1$.
III^*: $X_0 + 2X_1 + 3X_2 + 4X_3 + 3X_4 + 2X_5 + 2X_6 + X_7$, *where* $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_5 = X_3 \cdot X_4 = X_4 \cdot X_6 = X_6 \cdot X_7 = 1$.
IV^*: $X_0 + 2X_1 + 3X_2 + 2X_3 + 2X_4 + X_5 + X_6$, *where* $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_3 = X_2 \cdot X_4 = X_3 \cdot X_5 = X_4 \cdot X_6 = 1$.

It is known that the conjugacy class of $\rho([\gamma_i])$ depends only on the type of the fiber $\phi^{-1}(q_i)$. The following table contains a matrix representative of the monodromy and the Euler characteristic of each type (cf. [12]):

Type	Matrix representative	χ
mI_0	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	0
mI_1	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	1

mI_2	$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	2
II	$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$	2
III	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	3
IV	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	4
mI_b	$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$	b
I_b^*	$\begin{bmatrix} -1 & -b \\ 0 & -1 \end{bmatrix}$	$b+6$
II^*	$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$	10
III^*	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	9
IV^*	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	8

(1)

2.2. Morsification

Definition 4. By a *deformation* of a family of elliptic curves (ϕ, S, D) we will mean a surjective proper holomorphic map $\Phi : \mathcal{S} \to D \times \Delta_\epsilon$, where $\mathcal{S}$ is a three-dimensional complex manifold and $\Delta_\epsilon = \{z \in \mathbb{C}\colon |z| < \epsilon\}$, and such that

1. Its general fibers are elliptic curves.
2. The composition $\mathcal{S} \overset{\Phi}{\to} D \times \Delta_\epsilon \overset{pr_2}{\to} \Delta_\epsilon$ does not have critical points.
3. If $D_t := D \times \{t\}$, $\mathcal{S}_t := \Phi^{-1}(D_t)$ and $\Phi_t := \Phi|_{\mathcal{S}_t} : \mathcal{S}_t \to D_t$ then the families (ϕ, S, D) and $(\Phi_0, \mathcal{S}_0, D_0)$ are topologically equivalent.

Furthermore, the deformation $(\Phi, \mathcal{S}, D \times \Delta_\epsilon)$ is called a *morsification of the family* (ϕ, S, D) if for any $t \neq 0$, each singular fiber of the map $\Phi_t : \mathcal{S}_t \to D_t$ is of type mI_0 $(m > 1)$ or I_1.

Let $(\Phi, \mathcal{S}, D \times \Delta_\epsilon)$ be a deformation. For each $t \in \Delta_\epsilon$ we choose a number in the interval $(0, 1)$, r_t, and a disk D_{r_t} with counterclockwise oriented boundary C_{r_t} and the point $Q_t = (r_t, 0)$, with $r_0 = r$ and such that all the critical values of Φ_t are contained in D_{r_t}. In what follows we will use the fact that for every $t \in \Delta_\epsilon$, $\rho_t([C_{r_t}])$ is a conjugate of $\rho_0([C_{r_0}])$ in $SL(2, \mathbb{Z})$, where ρ_t denotes the monodromy representation of the family $(\Phi_t, \mathcal{S}_t, D_t)$. Furthermore, all the $\mathcal{S}_t$ are diffeomorphic to $\mathcal{S}_0$ which, in turn, is diffeomorphic to S.

According to a result of Moishezon (see Theorems 8 and 8a of [10]), each family of elliptic curves admits a morsification. The number of singular fibers $N(\Phi, t)$ (with $t \neq 0$) of type I_1 of a member $(\Phi_t, \mathcal{S}_t, D_t)$ of a morsification, is actually independent of Φ and t. To see this it is sufficient to prove that $N(\Phi, t)$ only depends on the Euler characteristic of $\mathcal{S}_t$. Indeed, $\chi(\mathcal{S}_t)$ is equal to the sum $\chi(\Phi_t^{-1}(D_t - \bigcup D_i)) + \chi(\Phi_t^{-1}(\bigcup D_i)$ where the D_i's are pairwise disjoint disks centered at the critical values and contained in D_t. Each $\Phi_t^{-1}(D_i)$ deformation retracts to its central (singular) fiber and therefore its Euler characteristic equals 1, if it is of type I_1, or 0 if it is of type mI_0. This is because in the first case the central fiber has the homeomorphism type of a one-pinched torus, and in the second case it has the homeomorphism type of a torus. Now, since $\Phi_t^{-1}(D_t - \bigcup D_i)$ is the total space of a fiber bundle having $D - \bigcup D_i$ as base and fiber a torus, its Euler characteristic is $\chi(D - \bigcup D_i)\chi(\text{torus}) = 0$. On the other hand, $\chi(\Phi_t^{-1}(\bigcup D_i)) = \sum \chi(\Phi_t^{-1}(D_i))$ and this last sum is equal to the number of fibers of type I_1 in the family. This number will be denoted by $N(\phi)$.

3. Lemmas on $SL(2,\mathbb{Z})$

In this section we recall some basic facts about the group $SL(2,\mathbb{Z})$, of all the 2×2 matrices with integral entries and determinant equal to 1, and about the modular group $PSL(2,\mathbb{Z})$, defined as the quotient $SL(2,\mathbb{Z})/\{\pm Id_{2\times 2}\}$ by the subgroup generated by minus the identity matrix. We will also prove some properties about sequences of elements in this group that are essential for the proof of our main theorem (stated below), reprove some of the theorems are used there and prove some new and stronger versions of these. For the sake of completeness we have included proofs and references to all the basic results. Standard references for this material are [10], [2] and [9].

Let u denote the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The main result asserts that the nth power of u cannot be written as a product of $r<n$ conjugates of it in $SL(2,\mathbb{Z})$.

Theorem 5. *Let n be a positive integer. If $u^n = g_1\cdots g_r$, where each g_i is a conjugate of u in $SL(2,\mathbb{Z})$, then $r\geqslant n$.*

This theorem is an immediate corollary of the following slightly more general result about the modular group.

Theorem 6. *Let $\pi : SL(2,\mathbb{Z})\to PSL(2,\mathbb{Z})$ be the canonical homomorphism and n a positive integer. If $\pi(u)^n = g_1\cdots g_r$ where each g_i is a conjugate of $\pi(u)$, then $r\geqslant n$.*

In what follows we will refer to particular elements (classes) in $PSL(2,\mathbb{Z})$ by specifying one of its representatives.

It is a well known fact that there is an isomorphism $\mathbb{Z}_2 * \mathbb{Z}_3 \to PSL(2,\mathbb{Z})$ taking each factor to the subgroup generated by the elements $\omega = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, and $b=\omega u$, respectively. Hence,

$$G = PSL(2,\mathbb{Z}) = \langle \omega, b \mid \omega^2 = b^3 = 1\rangle.$$

From this we see that the abelianization of G is $\mathbb{Z}_2\times\mathbb{Z}_3$ and consequently that the abelianization of $SL(2,\mathbb{Z})$ is $\mathbb{Z}_{12}$, with any conjugate of the matrix u being sent to 1.

It also follows that each element A in this group can be written uniquely as a product $A = t_k\cdots t_1$, where each t_i is either ω, b, or b^2 and no consecutive pair $t_i t_{i+1}$ is formed either by two powers of b or two copies of ω. We call the product $t_k\cdots t_1$ the *reduced expression of* A, and k the *length* of A, and we will denote it by $l(A)$. Let $B = t'_1\cdots t'_l$ be the reduced expression of B. If exactly the first $m-1$ terms of B cancel with those of A, i.e. $t'_i = t_i^{-1}$, for $1\leqslant i\leqslant m-1$, and if $m\leqslant \min(k,l)$, then $AB = t_k\cdots t_m t'_m\cdots t'_l$ and $t_m t'_m$ has to be equal to a non-trivial power of b. This is because if t_m were not a power of b then it would have to be ω and therefore t_{m-1} would be a first or second power of b, and so would be t'_{m-1}. Hence, t'_m would also have to be ω but in this case there would be m instead of $m-1$ cancellations at the juncture of A and B. Thus, t_m and t'_m are both powers of b and since there are exactly $m-1$ cancellations their product must be non-trivial. Thus, the reduced expression for AB is of the form

$$AB = t_k\cdots t_{m+1} b^r t'_{m+1}\cdots t'_l,\quad r = 1 \text{ or } 2,\ \text{if } m\leqslant \min(k,l). \tag{2}$$

Let s_1 denote the element $b\omega b$. The shortest conjugates of s_1 in G are precisely $s_0 = b^2(b\omega b)b = \omega b^2$ and $s_2 = b(b\omega b)b^2 = b^2\omega$. The element s_1 is trivially a conjugate of itself of length 3 and it can be easily seen that if g is a conjugate of greater length its reduced expression is of the form $Q^{-1}s_1Q$, where Q is a reduced word that begins with ω (see [2]), and $l(g) = 2l(Q)+3$. We will call a conjugate of s_1 ("conjugate" will always mean conjugate of s_1 in G) *short* if $g\in\{s_0,s_1,s_2\}$, otherwise it will be called *long*. With this notation we want to observe that Theorem 6 is equivalent to the following assertion.

Theorem 7. *Let n be a positive integer. If $s_1^n = g_1\cdots g_r$, where each g_i is a conjugate of s_1, then $r\geqslant n$.*

Let us demonstrate the equivalence between Theorems 6 and 7.

Proof. Let $c_b(A) = b^{-1}Ab$ denote conjugation by b. This is an automorphism of G that sends $u=\omega b$ to $b^2\omega b^2$. The map $\varphi : \mathbb{Z}_2 * \mathbb{Z}_3 \to \mathbb{Z}_2 * \mathbb{Z}_3$ defined by sending ω to itself, and b to b^2, that is, $\varphi = Id * \psi$, where ψ is the automorphism of $\mathbb{Z}_3$ that sends b to b^2, is an automorphism that maps $b^2\omega b^2$ to s_1. The composite of these two automorphisms transforms any equation $u^n = g_1\cdots g_r$ into an equation of the form $s_1^n = g'_1\cdots g'_r$, where each g'_i is a conjugate of s_1. This clearly implies the equivalence of both assertions. □

The following notion is the key ingredient for understanding the reduced expression of a product of conjugates of s_1.

Definition 8. We will say that two conjugates g and h of s_1 *join well* if $l(gh) \geqslant \max(l(g), l(h))$.

In [2] (Lemma 4.10) the following result is proved.

Lemma 9. *Suppose that $g = t_k \cdots t_1$ and $h = t'_1 \cdots t'_l$ are the reduced expressions of two conjugates of s_1 that join well. When gh is calculated either*:

1. *no cancellation occurs, and in this case $t_k \cdots t_1 t'_1 \cdots t'_l$ is the reduced expression of gh , or*
2. *exactly the first $m-1$ terms of g and h cancel out, in which case $m \leqslant \min(k, l)$. Moreover, if g is short or h is short, then they have to be s_2 and s_0, respectively. If both are long with reduced expressions of the form $g = Q_1^{-1} s_1 Q_1$ and $h = Q_2^{-1} s_1 Q_2$, hence with lengths equal to $2l(Q_i) + 3$, then the reduced expression of gh is of the form*

   $$gh = t_k \cdots t_{m+1} b^r t'_{m+1} \cdots t'_l, \quad r = 1 \text{ or } 2,$$

   and the inequality (2) *can be improved to $m \leqslant \min((k-1)/2, (l-1)/2)$ which implies that $m - 1 \leqslant \min(l(Q_1), l(Q_2))$.*

Now suppose that g and h are two conjugates that do not join well, and are not both short. The next lemma shows that in this case there exist g' and h' conjugates of s_1 such that $gh = g'h'$ and such that $l(g') + l(h') < l(g) + l(h)$ ([2], Proposition 4.15).

Lemma 10. *Suppose that g and h are conjugates of s_1 which satisfy the inequality $l(gh) < \max(l(g), l(h))$, and assume that at least one of them is a long conjugate. Then $l(h) \neq l(g)$. If $l(h) > l(g)$, the elements $g' = ghg^{-1}$, $h' = g$ are conjugates of s_1 and satisfy*:

1. *$gh = g'h'$, and*
2. *$l(g') + l(h') < l(g) + l(h)$.*

If instead, $l(h) < l(g)$, then the same conclusion holds taking $g' = h$, and $h' = h^{-1}gh$.

The replacement of the pair (g, h) by the pair (g', h') is called a *Hurwitz move*. Using this lemma we can prove that a product $g_1 \cdots g_r$ of conjugates of s_1 can always be changed into a product $g'_1 \cdots g'_r$ of conjugates of s_1 in which each pair of consecutive terms joins well.

Proposition 11. *Let $\{g_1, \ldots, g_r\}$ be a set of r conjugates of s_1. Then there exists a set of r conjugates $\{g'_1, \ldots, g'_r\}$ of s_1 such that $g_1 \cdots g_r = g'_1 \cdots g'_r$ and either they are all short, or any two consecutive terms g'_i, g'_{i+1} join well.*

First, we need to know how to handle pairs of consecutive short conjugates that do not join well.

Claim 12. *Let $s_{i_1} s_{i_2} \cdots s_{i_l}$ be a product of short conjugates where there is at least one pair of two consecutive terms that do not join well. Then the product $s_{i_1} s_{i_2} \cdots s_{i_l}$ is equal to another product $s_{j_1} s_{j_2} \cdots s_{j_l}$, with the same number of terms, where the first conjugate s_{j_1}can be chosen arbitrarily from the set $\{s_0, s_1, s_2\}$. In a similar way, $s_{i_1} s_{i_2} \cdots s_{i_l}$ is equal to another product of short conjugates with the same number of terms where the last conjugate can be chosen arbitrarily.*

Proof. We use induction on l. For $l = 2$ a direct computation shows that the pairs $s_2 s_0$, $s_0 s_1$, and $s_1 s_2$ are the only ones that do not join well, and their product is equal to b, and from this the claim readily follows. Now let $l > 2$. If the first pair does not join well, the same argument as before could be applied. Hence, we may assume that there is a pair that does not join well in the product $s_{i_2} \cdots s_{i_l}$. Then, by induction we can change this product by a new product

$s_{j_2} \cdots s_{j_l}$, where s_{j_2} can be made to be any short conjugate. Consequently, if s_{i_1} is s_0 (s_1, s_2, respectively) then we may choose s_{j_2} to be s_1 (s_2, s_0, respectively) so that the first pair does not join well and therefore can be changed again by a pair whose first term can be chosen arbitrarily.

A similar argument shows that $s_{i_1} s_{i_2} \cdots s_{i_l}$ can be changed into another one with the same number of terms where the last conjugate can be chosen arbitrarily. □

Proof of Proposition 11. Among sets of r conjugates $\{g_1', \ldots, g_r'\}$ such that $g_1 \cdots g_r = g_1' \cdots g_r'$ we may choose one such that the sum $\sum_{i=1}^{r} l(g_i')$ is as small as possible. If all g_k' are short we are done. If not, any one g_k' which is long has to join well with any term (if any) before or after it, for otherwise by Lemma 10, the corresponding pair could be changed by another one making the sum $\sum_{i=1}^{r} l(g_i')$ smaller.

On the other hand, let $s_{i_1} \cdots s_{i_l}$ be a product of consecutive short conjugates that appears in $g_1' \cdots g_r'$. If this product precedes a long conjugate g_k', i.e., if $s_{i_1} \cdots s_{i_l} g_k'$ is a segment of the product $g_1' \cdots g_r'$, then, by the previous lemma any pair of consecutive elements of $s_{i_1} \cdots s_{i_l}$ joins well or we can change this product by $s_{j_1} s_{j_2} \cdots s_{j_l}$, where s_{j_l} can be chosen arbitrarily. If the reduced expression of g_k' is of the form $\omega b^e t_3 \cdots t_m$, $e = 1, 2$, we may choose $s_{j_l} = s_2$ so that s_{j_l} and g_k' do not join well. Applying again Lemma 10 we could change the pair g_{k-1}', g_k' by another one making the sum $\sum_{i=1}^{r} l(g_i')$ smaller. Similarly, if the reduced expression of g_k' is of the form $b^e \omega t_3 \cdots t_m$, then if $e = 1$, we could choose s_{j_l} equal to s_2 so that s_{j_l} and g_k' do not join well, and if $e = 2$, s_{j_l} can be chosen as s_1. Hence, we conclude that any pair of consecutive elements of $s_{i_1} \cdots s_{i_l}$ must join well. The argument is entirely similar if $g_k' s_{i_1} \cdots s_{i_l}$ is a segment of the product $g_1' \cdots g_r'$. Thus, we see that if a set $\{g_1', \ldots, g_r'\}$ with minimal sum $\sum_{i=1}^{r} l(g_i')$ contains at least one long conjugate then all consecutive pairs in it must join well. This proves the proposition. □

Let us define the *left end* of each conjugate g, left(g), as follows: If g is long of the form $g = Q^{-1} s_1 Q$, define left$(g) = Q^{-1} s_1$. If g is $s_0 = \omega b^2$, $s_1 = b\omega b$, or $s_2 = b^2 \omega$, we define its left end as ω, b, b^2, respectively.

Lemma 13. *If in the product $P = g_1 \cdots g_r$ of conjugates any two consecutive terms g_i, g_{i+1} join well then the reduced expression of P is of the form* left$(g_1) t_2 \cdots t_l$, *with each t_i is one of b, b^2 or ω.*

Proof. We prove this by induction on r, the assertion being trivial for $r = 1$. We distinguish several cases.

1. $g_1 = s_0$. Since g_1 and g_2 join well the conjugate g_2, if short, should be equal to s_0 or to s_2. In the first case, by the induction hypothesis we must have that the reduced expression for $g_2 \cdots g_r$ is of the form $\omega t_2 \cdots t_l$, and consequently
$$g_1 \cdots g_r = \omega b^2 \omega t_2 \cdots t_l = \text{left}(s_0) t_2' \cdots t_k',$$
and the result holds. On the other hand, if g_2 equals s_2 then $g_2 \cdots g_r = b^2 t_2 \cdots t_l$ and the results also holds, since
$$g_1 \cdots g_r = \omega b t_2 \cdots t_l = \text{left}(s_0) t_2' \cdots t_k'.$$
Finally, if g_2 is long, let us say $g_2 = Q^{-1} s_1 Q$ then $g_2 \cdots g_r = Q^{-1} s_1 t_2 \cdots t_l$. But since g_1 and g_2 join well the reduced expression of Q^{-1} cannot start with $b\omega$. Hence, it starts with $b^2\omega$ or with ω. In either case the reduced expression for $g_1 \cdots g_r$ has the form
$$\omega b \omega t_4 \cdots t_k = \text{left}(s_0) t_2' \cdots t_k', \quad \text{or} \quad \omega b^2 \omega t_4 = \text{left}(s_0) t_2' \cdots t_v'.$$
2. $g_1 = s_2$. In this case, if g_2 if short, it must be s_2 or s_1 and in either case the reduced expression of $g_1 \cdots g_r$ is of the form $b^2 \omega b^2 t_4 \cdots t_l$, $b^2 \omega b t_4 \cdots t_l$, respectively, and the result holds. If g_2 is long, by Lemma 9 (2) and since g_1 and g_2 join well, no cancellation can occur and the reduced expression of $g_1 \cdots g_r$ has the form $b^2 \omega t_2 \cdots t_v$ and the result also holds.
3. The case $g_1 = s_1$ can be treated in exactly the same way as the previous case.
4. g_1 is long. If g_2 is short, by Lemma 9 either it has to be s_2, or no cancellation occurs. In the latter case the result follows immediately. In the former, the reduced expression of Q cannot end in ωb, since in this case g_1 and g_2 would not join well. Thus $Q = R\omega b^2$, or $Q = R b^r \omega$, $r = 1, 2$. By induction $g_2 \cdots g_r = b^2 t_2 \cdots t_s$, and consequently $P = Q^{-1} s_1 R \omega b t_2 \cdots t_s$, or $P = Q^{-1} s_1 R b^r \omega b^2 t^2 \cdots t_s$. In each case the reduced expression of the product starts with left(g_1).

Finally, we suppose g_2 is also long. If $g_1 = Q_1^{-1}s_1Q_1$ and $g_2 = Q_2^{-1}s_1Q_2$ then by Lemma 9 (2) either no cancellation occurs, or the number of terms that cancel out in the product g_1g_2 is $\leqslant \min(l(Q_1), l(Q_2))$. By induction $g_2\cdots g_r = Q_2^{-1}s_1t_2\cdots t_s$, and in either case the reduced expression of $P = g_1\cdots g_r$ starts with $Q_1^{-1}s_1$. □

Proof of Theorem 7. Let us suppose by contradiction that there exists $n > 0$ and a factorization of s_1^n as a product of less that n conjugates of s_1. Let us choose n as small as possible such that there is a counterexample. By Proposition 11, after a finite number of Hurwitz moves we may assume that $s_1^n = g_1\cdots g_r$ where either all the g_i's are short, or they all join well. In the first case, let $N_\omega : G \to \mathbb{N}$ be the function that counts the number of times ω appears in the reduced expression of A. From (2) it readily follows that $N_\omega(AB) \leqslant N_\omega(A) + N_\omega(B)$, and by a trivial induction that

$$N_\omega(A_1\cdots A_r) \leqslant N_\omega(A_1) + \cdots + N_\omega(A_r). \tag{3}$$

But for each s_i, $i = 0, 1, 2$, we have that $N_\omega(s_i) = 1$. A direct computation shows that $N_\omega(s_1^n) = n$. Hence,

$$n = N_\omega(s_1^n) = N_\omega(g_1\cdots g_r) \leqslant N_\omega(g_1) + \cdots + N_\omega(g_r) = r,$$

a contradiction.

Hence, we may assume that all pairs g_i, g_{i+1} join well. If $g_1 = s_1$, we could cancel out a power of s_1 on each side of equation $s_1^n = g_1\cdots g_r$ and obtain in this way a new counterexample with a smaller n. We notice that the reduced expression of s_1^n is $b\omega b^2\omega\cdots b^2\omega b$. By the previous lemma the reduced expression of $g_1\cdots g_r$ has the form $\text{left}(g_1)t_2\cdots t_l$, therefore g_1 cannot be equal to s_0 or s_2 since $\text{left}(s_0) = \omega$ and $\text{left}(s_2) = b^2$. Thus, g_1 has to be long and in this case $\text{left}(g_1) = Q^{-1}b\omega b$. But this would imply that in the reduced expression of s_1^n the string $b\omega b$ would appear which happens only if $n = 1$, a contradiction. □

4. Proof of the main result

Let $\phi : S \to D$ be a family of elliptic curves having $\phi^{-1}(0)$ as its unique singular fiber and $\Phi : \mathcal{S} \to D \times \Delta_\epsilon$ be a morsification for this family. Let $\rho_t : \pi_1(D - \{q_{1,t}, \ldots, q_{k,t}\}, Q_t) \to SL(2,\mathbb{Z})$ be the monodromy representation of the member $\Phi_t : \mathcal{S}_t \to D$ of the family with $t \neq 0$. Then we have

$$\rho_t([C_{r_t}]) = \rho_t([\gamma_{1,t}]\cdots[\gamma_{k,t}]) = \rho_t([\gamma_{k,t}])\cdots\rho_t([\gamma_{1,t}]) = \rho_t([\gamma_{i_N,t}])\cdots\rho_t([\gamma_{i_1,t}])$$

where $N = N(\phi)$ and $\gamma_{i_1}, \ldots, \gamma_{i_N}$ are the paths surrounding the fibers of type I_1. Each $\rho_t([\gamma_{i_j,t}])$ is a conjugate of u, and corresponds to the class of a right-handed Dehn twist along some essential simple closed curve in the model fiber. With this notation we have the following result.

Theorem 14. *Let $\phi : S \to D$ be a family of elliptic curves having $\phi^{-1}(0)$ as its unique singular fiber. Then the shortest factorization of $\rho([C_r])$ as a product of (classes) of right-handed Dehn twists has $N(\phi)$ terms.*

Proof. By Theorem 3 the singular fiber is of one of the types: mI_b (with $m = 1$ and $b > 0$ or $m > 1$ and $b \geqslant 0$), I_b^* (with $b \geqslant 0$), II, II^*, III, III^*, IV, IV^*. We divide the proof into the following cases.

1. The singular fiber has type II, II^*, III, III^*, IV or IV^*. According to Table (1), the Euler characteristic $\chi(\phi^{-1}(0))$ in these cases is a number bigger than 1 and strictly smaller than 12. Suppose by contradiction that there exists a factorization $\rho([C_r]) = g_1\cdots g_s$ in terms of conjugates of the matrix u, with $0 \leqslant s < N(\phi)$, and let $\Phi_t : \mathcal{S}_t \to D$ with $t \neq 0$ be a member of some morsification of ϕ. The already mentioned fact that $\rho([C_r])$ and $\rho_t([C_{r_t}])$ are conjugate of each other, implies that there exists a factorization $\rho_t([C_{r_t}]) = g_1'\cdots g_s'$ in terms of conjugates of the matrix u. But we also know that $\rho_t([C_{r_t}]) = \rho_t([\gamma_{i_N,t}])\cdots\rho_t([\gamma_{i_1,t}])$ where each factor is a conjugate of u. Since each conjugate of u corresponds to 1 in the abelianization $\mathbb{Z}_{12}$ of $SL(2,\mathbb{Z})$, $\rho_t([C_{r_t}])$ would correspond simultaneously to (the classes of) s and $N(\phi)$ in $\mathbb{Z}_{12}$, implying that $s \equiv N(\phi) \pmod{12}$. But this is impossible because $0 \leqslant s < N(\phi) < 12$.
2. The singular fiber has type mI_b with $m = 1$ and $b > 0$ or $m > 1$ and $b \geqslant 0$. According to table (1) the Euler characteristic $\chi(\phi^{-1}(0))$ in these cases is equal to b making $N(\phi) = b$, and $\rho([C_r])$ is conjugate to the matrix

$$u^b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}. \tag{4}$$

Suppose by contradiction that there exists a factorization $\rho([C_r]) = g_1 \cdots g_s$ in terms of conjugates of u with $0 \leqslant s < N(\phi)$. This would imply that u^b is conjugate to the product $g_1 \cdots g_s$ with $s < b$. This in turn implies that a relation of the form $u^b = g'_1 \cdots g'_s$ where each g'_i is a conjugate of u and $s < b$, holds in $SL(2, \mathbb{Z})$. But this contradicts Theorem 5.

3. The singular fiber has type I_b^* with $b \geqslant 0$. According to table (1) the Euler characteristic $\chi(\phi^{-1}(0))$ in these cases is equal to $b + 6$ making $N(\phi) = b + 6$, and $\rho([C_r])$ is conjugate to the matrix $-u^b$. Suppose by contradiction that there exists a factorization $\rho([C_r]) = g_1 \cdots g_s$ in terms of conjugates of u with $0 \leqslant s < N(\phi)$. This implies that a relation of the form
$$-u^b = g'_1 \cdots g'_s \tag{5}$$
where each g'_i is a conjugate of u and $s < b + 6$, holds in $SL(2, \mathbb{Z})$. If we apply the natural map from $SL(2, \mathbb{Z})$ to its abelianization $\mathbb{Z}_{12}$, to both sides of Eq. (5) we see that $b + 6 \equiv s \pmod{12}$. This is due to the fact that the matrix $-I$ can be written as a product of 6 conjugates of u. On the other hand, applying Theorem 6 to the relation $\pi(u)^b = \pi(g'_1) \cdots \pi(g'_s)$, we see that $b \leqslant s$. So $b \leqslant s < b + 6$ and this is incompatible with the fact that $b + 6$ and s are congruent modulo 12. □

References

[1] J. Amorós, F. Bogomolov, L. Katzarkov, T. Pantev, Symplectic Lefschetz fibrations with arbitrary fundamental groups. With an appendix by Ivan Smith, J. Differential Geom. 54 (3) (2000) 489–545.

[2] R. Friedman, J.W. Morgan, Smooth Four-Manifolds and Complex Surfaces, Ergebnise der Mathematik und ihrer Grenzgebiete 3. Folge, Band 27, Springer-Verlag, 1994.

[3] M. Ishizaka, One parameter families of Riemann surfaces and presentations of elements of mapping class group by Dehn twists, J. Math. Soc. Japan 58 (2) (2006) 585–594.

[4] K. Kodaira, On compact analytic surfaces: II, Ann. Math. (2) 77 (3) (1963) 563–626.

[5] M. Korkmaz, On stable torsion length of a Dehn twist, Math. Res. Lett. 12 (2–3) (2005) 335–339.

[6] M. Korkmaz, Stable commutator length of a Dehn twist, Michigan Math. J. 52 (1) (2004) 23–31.

[7] D. Kotschick, Quasi-homomorphisms and stable lengths in mapping class groups, Proc. Amer. Math. Soc. 132 (11) (2004) 3167–3175 (electronic).

[8] D. Kotschick, Clustering of critical points in Lefschetz fibrations and the symplectic Szpiro inequality, Trans. Amer. Math. Soc. 355 (8) (2003) 3217–3226 (electronic).

[9] Y. Matsumoto, Diffeomorphism types of elliptic surfaces, Topology 25 (4) (1986) 549–563.

[10] B. Moishezon, Complex Surfaces and Connected Sums of Complex Projective Planes, in: Lecture Notes in Mathematics, vol. 603, Springer-Verlag, 1977.

[11] A. Stipsicz, Singular fibers in Lefschetz fibrations on manifolds with $b_2^+ = 1$, Topology Appl. 117 (1) (2002) 9–21.

[12] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact Complex Surfaces, second ed., Springer-Verlag, 2004.