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# Shape optimisation of continuum structures via evolution strategies and fixed grid finite element analysis*

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**Abstract** Evolution strategies (ES) are very robust and general techniques for finding global optima in optimisation problems. As with all evolutionary algorithms, ES apply evolutionary operators and select the most fit from a set of possible solutions. Unlike genetic algorithms, ES do not use binary coding of individuals, working instead with real variables.

Many recent studies have applied evolutionary algorithms to structural problems, particularly the optimisation of trusses. This paper focuses on shape optimisation of continuum structures via ES. Stress analysis is accomplished by using the fixed grid finite element method, which reduces the computing time while keeping track of the boundary representation of the structure. This boundary is represented by b-spline functions, circles, and polylines, whose control points constitute the parameters that govern the shape of the structure. Evolutionary operations are applied to each set of variables until a global optimum is reached. Several numerical examples are presented to illustrate the performance of the method. Finally, structures with multiple load cases are considered along with examples illustrating the results obtained.

**Key words** evolution strategies, fixed grid, shape optimization, multiload cases

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## 1 Introduction

Shape optimisation problems of continuum structures have been approached using conventional and non-conventional optimisation techniques. Conventional methods have drawbacks, including convergence to local optima and a need for gradient information. Non-conventional techniques include the metamorphic method (Liu *et al.* 1999), the biological simulated growth method (Mattheck and Burkhardt 1990; Baumgartner *et al.* 1992), and the evolutionary structural optimisation method (Xie and Steven 1993) based on the addition/removal of material at sites of high/low stresses. Other non-conventional methods are the evolutionary algorithms. These methods are based on the theory of natural selection, and use a population of individuals or structures to evolve toward the optimum design. Genetic algorithms (GA) are by far the most popular of the evolutionary algorithms. Woon and Querin used a structure whose boundary is represented by polylines (Woon and Querin 1999). Each node of the polylines was related to a gene of the chromosome and was moved up or down according to the value of the gene acquired through genetic operators. Because the boundary was represented by lines, the optimum structure contained sharp and jagged edges. Kita and Tanie (1999) used spline functions to represent the boundary of continuum structures, and a real coded GA and the boundary element method (BEM) to compute the fitness. The main disadvantage of this approach is the high computational cost. The optimisation method developed by Burcsinzky and Kokot (1999) uses a GA as the search algorithm, BEM for the analysis, and NURBS to represent the boundary of the structure. They used the bubble method to allow the formation of cavities in the structure (Eschenauer *et al.* 1994). With this method not only shape but topological optimisation can be accomplished. As the number of bubbles is controlled interactively by the user, the final shape will vary according to user selection.

The present article introduces a shape optimisation method based on the evolution strategies and fixed grid finite element analysis (ESFG-FEA) methods. As opposed to GA, ES use real variables, eliminating the need to code into binary form as required by GA. Also, mutations are normally distributed instead of uniformly distributed, thus allowing the occurrence of small changes with greater frequency than large changes. Fixed grid (FG) finite element analysis is used instead of the traditional finite element method because it reduces the computing time and uses a parametric representation of the structure's geometry via its boundary representation (B-rep). The B-rep uses spline functions and lines so that the shape of the structure does not contain sharp edges and jagged boundaries.

Finally, multiple load cases are factored into the analysis. Finding the optimum structure under these conditions is more valuable for structures operating under real-life conditions. Driven by this importance, several researchers have focused on multi-objective optimisation. Thus, the main concern in dealing with multiload structures is the way in which multiple objectives are evaluated. There are two main approaches for evaluating multiple objectives. First, transforming multiple objectives into a single one, which can be accomplished by measuring the distance between the ideal and optimum solution vectors. This is the case for Euclidean and Chebyshev norms (Khot 1998). The second approach uses the branch of mathematical programming called Edgeworth Pareto optimisation. This method deals with finding the optimal set of weight factors of a single function that contains the multiple objective functions (Haftka and Gurdal 1996). This paper considers solutions based on global criteria that generate the Pareto optima.

## 2 Computational representation

Boundary representation (B-rep) is used by the FG method to describe the geometry of the structures, as it allows fast conversion into the fixed grid structure of elements. Furthermore, the FG method easily incorporates geometry changes via modifications of the parameters that control the B-rep. Natural cubic splines are used to represent the boundaries of the structure for three reasons: the low order of the polynomials allows efficient computations, the resulting curves have $C^2$ continuity, which produces smooth surfaces, and, finally, changes in one control point are locally propagated.

When a boundary representation is used to define the geometry of a structure, polygons, circles, and splines can be used to represent the boundaries. A polygon is defined by a set of points $(x, y)$ that represent an irregular boundary. A circle can be defined by the position of the centre and its radius, i.e. $(x, y, r)$. A spline is defined by a set of control points $(x, y)$. Therefore, a structure with a geometry defined by polygons, circles and splines can be represented by a vector $\mathbf{x}$:

$$\mathbf{x} = \{x_0, x_1, \ldots, x_m\} . \quad (1)$$

These parameters are adopted as the design variables in the structural optimisation procedure.

## 3 Fixed grid finite element method

Unlike the traditional finite element method (FEM), the fixed grid (FG) finite element method does not use a fitted mesh to construct the finite element domain (García and Steven 1999; García 1999). Instead, a rectangular grid is superimposed on the structure's geometry. Figure 1 shows an example of a structure modelled by an FG. An $O$ element is given a material property of a non-interactive medium. In other words, its value is significantly less than that of an $I$ element. $NIO$ elements consist of two types of material and therefore their properties are not constant along the element. The FG easily incorporates geometrical changes in the form of changes of the material properties to the elements in the grid. Consider a change $\Delta\mathbf{s}$ in the shape of the initial design $\mathbf{s}_0$ that results in a change $\Delta\mathbf{K}$ in the stiffness matrix and $\Delta\mathbf{f}$ in the load vector. The equilibrium equations at $\mathbf{s}_0 + \Delta\mathbf{s}$ are

$$(\mathbf{K}_0 + \Delta\mathbf{K})(\mathbf{u}_0 + \Delta\mathbf{u}) = \mathbf{f}_0 + \Delta\mathbf{f} , \quad (2)$$

where $\mathbf{K}_0$ is the stiffness matrix before the perturbation, and $\Delta\mathbf{K}$ matrix is calculated only for the elements for which the ($I$, $O$, $NIO$) state has changed. The change in the stiffness matrix of an element can be expressed in terms of a standard element stiffness matrix $\mathbf{K}^*$ as $\Delta\mathbf{K}^e = (E_{\text{new}} - E_{\text{old}})\mathbf{K}^*$. Thus the re-analysis time is reduced drastically, not only by reducing the re-meshing time, but also through reconstruction of the stiffness matrix. This re-analysis feature makes the FG method optimal for solving problems in which it is necessary to evaluate different variations of the geometry of a single structure (García and Steven 2000).

The advantages of using an FG are simplicity and speed at a permissible level of accuracy. The stress error

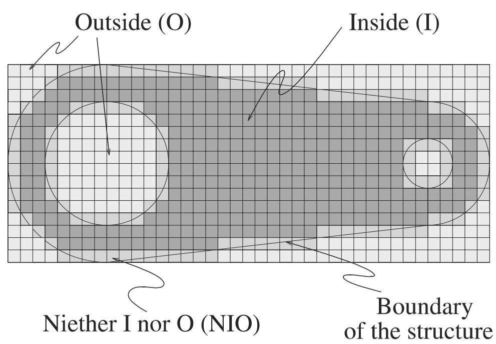


**Fig. 1** Typical mesh in fixed grid finite element method. The elements in the grid are classified as Inside ($I$), Outside ($O$), or Neither Inside nor Outside ($NIO$) of the structure

has been seen to increase near the region of stress concentration, with a maximum stress error being approximately 10% for a mesh of reasonable size. However, the average stress error has been found to be about 5% or below (García and Steven 1999). Thus the FG method was deemed appropriate for interactive design and structural optimisation where no highly accurate analysis is needed.

## 4 Evolution strategies

There are four main variants of evolutionary algorithms: genetic algorithms (GA) (Gen 1997; Mitchell 1996), evolution strategies (ES) (Bäck 1996; Schwefel *et al.* 1995; Bäck and Schwefel 1996), evolutionary programming (EP), (Bäck *et al.* 1993; Bäck and Schwefel 1996), and genetic programming (GP) (Koza 1996). Among these, GA are the most widely used in structural optimisation. The differences among them lie primarily in the type of variables that they use. In GA the chromosomes are represented by binary variables, in ES by real numbers, in EP by finite state machines, and in GP by tree structures. In this work ES are used to find the optimum structure. The optimisation variables are the spline control points defining the structure's boundary. These control points are real variables and move in a real range. An additional advantage of ES is that a normally distributed mutation is used instead of the uniformly distributed mutations used by GA (small changes are more frequent in nature than large changes). The present implementation uses $(\mu+\lambda)$ ES with uniform recombination and normally controlled mutations.

In the $(\mu+\lambda)$ variant of ES, $\mu$ survivors are selected from the union of parents ($\mu$) and offspring ($\lambda$), such that a monotonous course of evolution is guaranteed. In the case of non-correlated standard deviations, an individual, $\mathbf{a}$, comprises a pair of vector-variables: $\mathbf{a} = (\mathbf{x}, \sigma)$, in which $\mathbf{x} \in M$ is the vector of object variables and $\sigma$ is the vector of standard deviations used for mutation. Objective variables are referred to as design variables in structural/shape optimisation. In the present study $\mathbf{x}$ describes the boundary representation of the structure's geometry (see (1)). Shape optimisation requires finding a setting $\mathbf{x} = \{x_1, \ldots, x_n\}$ such that a certain quality criterion $f : M \to \mathbb{R}$ is minimised:

$$f(\mathbf{x}) \to \min . \tag{3}$$

The objective function can be given by a combination of mechanical stiffness, volume, maximum stress, or other properties that characterise a structure. A solution to the global optimum requires finding a vector $\mathbf{x}^*$ such that $\forall \mathbf{x} \in M : f(\mathbf{x}) \geq f(\mathbf{x}^*) = f^*$, where M is the set of restrictions that are related to the topological and mechanical validity of the geometry. A population of individuals undergo probabilistic operations such as mutation, selection, and recombination (crossover) to evolve toward better fitness values of individuals (structures with $f(\mathbf{x}) \to f^*$). Generalised intermediate recombination of the $x$ and $\sigma$ variables is given by:

$$x_i' = x_{S,i} + \lambda(x_{T,i} - x_{S,i}) , \tag{4}$$

$$\sigma_i' = \sigma_{S,i} + \lambda(\sigma_{T,i} - \sigma_{S,i}) , \tag{5}$$

where $S$ and $T$ represent two parent individuals selected at random from the population (the index $i$ in $T_i$ means $T$ is to be sampled anew for each value of $i$), and $\lambda \in [0, 1]$ is a uniform random variable that is sampled only once per creation of one offspring individual (Bäck 1996). The mutation operator is by far the most studied operator in ES. The notation $N(a, b)$ is used here to denote a normally distributed one-dimensional random variable having expectation $a$ and standard deviation $b$. Mutation of both vector variables is given by

$$\sigma_i' = \sigma_i \cdot \exp(\tau' \cdot N(0,1) + \tau \cdot N_i(0,1)) , \tag{6}$$

$$x_i' = x_i + N(0, \sigma_i') . \tag{7}$$

The factors $\tau$ and $\tau'$ are rather robust parameters and can be interpreted in the sense of "learning rates" as in artificial neural networks. For details see Bäck and Schwefel (1993), Bäck (1996), Schwefel *et al.* (1995).

## 5 Examples

Several examples are presented to illustrate the method: a cantilever beam with optimisation of the upper boundary, a Michell-type structure, and a spanner.

### 5.1 Cantilever beam

Linear elastic behaviour is assumed. The structure consists of a cantilever beam. The initial dimensions of the beam are a height of 10 units and a length of 30 units. The left edge of the structure is fixed and a load is applied to the right end, as illustrated in Fig. 2a. The design variables are the control points of a spline curve that describes the shape of the upper boundary of the beam. Control points are allowed to move in the vertical direction. The optimisation goal is to find the location of the control points that minimises the objective function defined in terms of the volume and displacement. The objective is to minimise the volume while maximising the stiffness of the structure. That is,

$$f(\mathbf{x}_i) = w_1 \frac{V(\mathbf{x}_i)}{V(\mathbf{x}_0)} + w_2 \frac{\max(d(\mathbf{x}_i))}{\max(d(\mathbf{x}_0))} , \tag{8}$$

where $\mathbf{x}_i$ represents the current structure, $\mathbf{x}_0$ represents the base structure, $V(\mathbf{x}_i)$ is the volume of the evaluated structure, $\max(d(\mathbf{x}_i))$ is the maximum displacement of

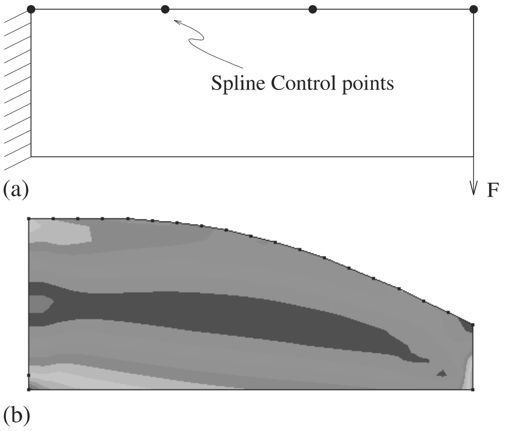


**Fig. 2** Optimisation of a cantilever beam. (a) Initial structure and loads. (b) Optimum structure found by ESFG-FEA

the evaluated structure, and $w_i$ and $w_0$ are weight factors. Parent and offspring populations were selected as: $\mu = 3$ and $\lambda = 12$. The mesh size was 300 elements.

The result of the optimisation is shown in Fig. 2b. The objective function was improved by 19% with respect to the original structure after 54 generations. The total optimisation time was 4 minutes 15 seconds using a Pentium III 900 MHz processor. The result from García (1999) using the steepest gradient optimisation method was found after 12 seconds using the same computer.

## 5.2 Michell-type structure

The initial domain and applied load are shown in Fig. 3a. The initial rectangular domain is 10 units in height and 30 units in length. The design variables are the control points of the spline curves that describe the upper and lower boundaries of the beam (see Fig. 3a). The objective function and population parameters were selected as before.

The optimum topology consist of two 45-degree arms extending from the supports toward a 90-degree centre fan section of an infinite number of beams, which extends upward from the point of application of the force. The infinite number of beams can be interpreted as continuous material (Hemp 1973).

After optimisation, the objective function was reduced by 13%. The total elapsed time was 1 minute 48 seconds. Figure 3b shows the resulting shape. For the same problem using the steepest gradient method (García 1999) the optimum structure was reached after 15 seconds.

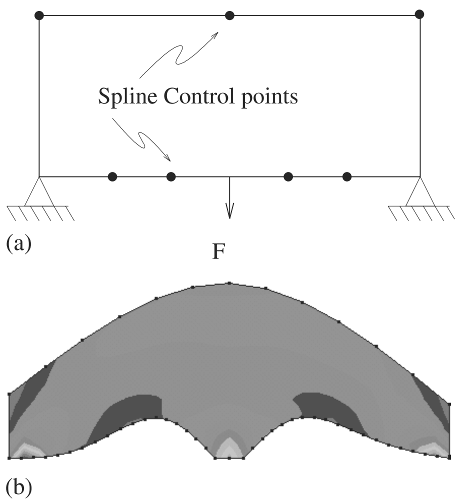


**Fig. 3** Michell-type structure optimisation with seven design variables representing spline control points in the lower and upper boundary

## 5.3 Spanner

The third example consists of the shape optimisation of a spanner-like tool. The initial domain consist of a rectangular domain of dimensions $10 \times 24$ units$^2$, as shown in Fig. 4a. Linear elastic behaviour is assumed. A single downward unit load is applied at the right end, while the left end is restricted from movement. The initial conditions and objective function for this optimisation problem are the same as those used for the examples of the cantilever beam and the Michell-type structure. The mesh size used by the FG was 240 elements. After 88 generations of the ESFG-FEA program the objective function was minimised by 7%. The optimum value was found after 5 minutes, while by the steepest gradient method, the solution of this problem took 33 seconds using the same computer.

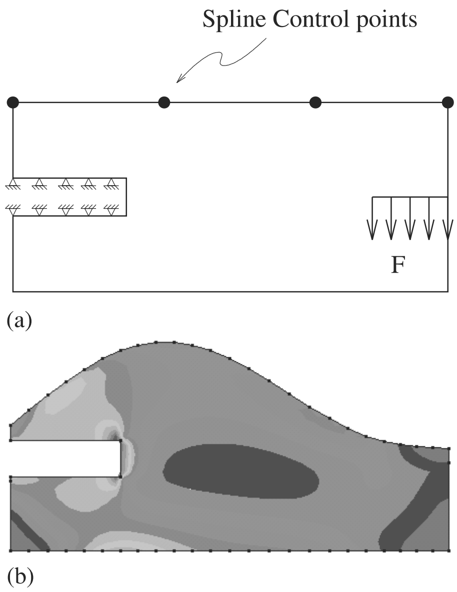


**Fig. 4** Shape optimisation of spanner. (a) Initial shape and loads. (b) Optimum structure by ESFG-FEA

## 6
## Multiple load cases

The optimisation problem for structures subjected to multiple load cases is usually simplified by analysing the most critical case or by symmetry assumptions. Nevertheless, these simplifications may not give the real optima for the problem. A better approach considers the multiple load cases as multiple objective functions to be optimised. The present work considers solutions based on global criteria that generate the Pareto optima. The Euclidean norm of the set of solutions is used to find the fitness value.

When compared with a single load case, multiple load cases require a more detailed description of the shape of the structure, thus increasing the number of design variables. This increase in the number of design variables (spline control points) allows finer details to be modelled by the curves, but requires a finer grid in order to capture these details. A larger number of design variables and a finer resolution of the grid result in a considerably longer computing time. Furthermore convergence of the ES algorithm becomes slow. To overcome this situation a rebirthing procedure has been implemented (Ghasemi *et al.* 1999).

The rebirthing procedure reduces the range of the design variables after some degree of convergence is achieved, after which the optimisation problem is restarted with a smaller search space. Figure 5 shows the rebirthing procedure with bound reduction. This procedure allows the algorithm to reduce the search space by fine-tuning the search while increasing the speed of convergence of the algorithm.

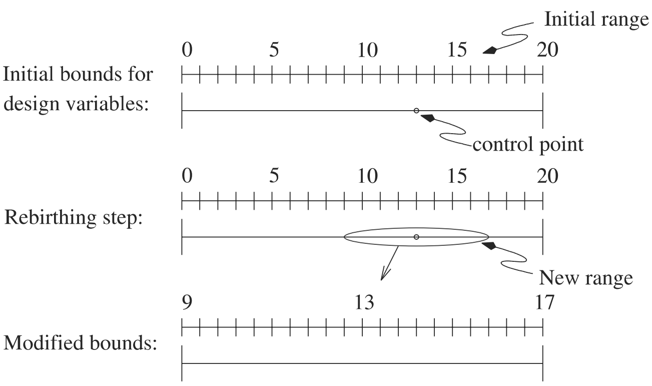


**Fig. 5** Illustration of design-variable rebirthing and range reduction

## 6.1
## Cantilever beam with multiload

As in the single-load cantilever beam, this cantilever beam is optimised assuming elastic behaviour. The initial dimensions of the beam are a height of 10 units and a length of 30 units. The left edge of the structure is fixed

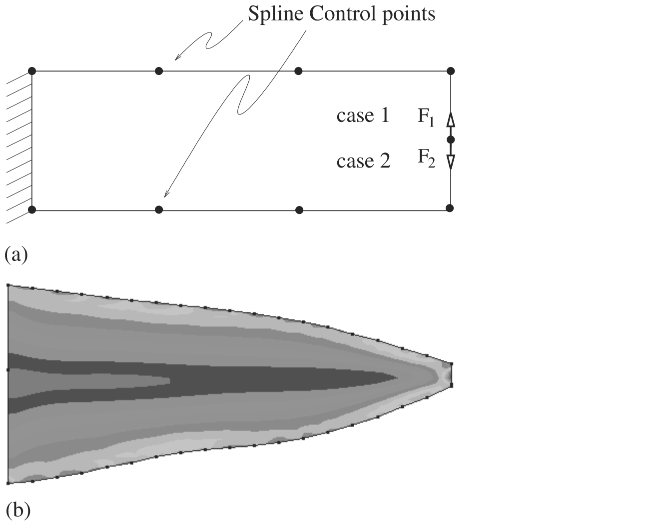


**Fig. 6** Optimisation of a cantilever beam with multiple load cases. (a) Initial shape and loads. (b) Optimum structure by ESFG-FEA

and both upward and downward loads are applied to the right end, as shown in Fig. 6(a).

Parent and offspring populations were selected as: $\mu = 3$ and $\lambda = 11$. The mesh size was 2718 elements. The optimum structure was found after 9 hours 15 minutes. In spite of the symmetry of the load conditions, it was observed that the resulting shape obtained by the ES is not exactly symmetric.

## 6.2
## Spanner-type structure with multiple loads

As in the single-load spanner structure, linear elastic behaviour is assumed. The initial dimensions of the domain

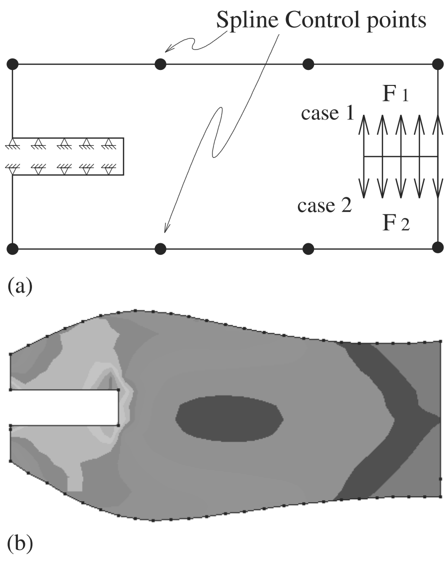


**Fig. 7** Optimisation of a spanner-like geometry with multiple load cases. (a) Initial shape and loads. (b) Optimum structure by ESFG-FEA

are a height of 10 units and a length of 24 units. The left side of the structure is fixed as in the single load case. Both upward and downward loads are applied to the right end, as shown in Fig. 7a.

Parent and offspring populations were selected as: $\mu = 3$ and $\lambda = 13$. The mesh size was 3037 elements. The total elapsed time was 8 hours using a Pentium 900 processor. Figure 7b shows the resulting shape. Once again, it can be seen that the resulting geometry is not perfectly symmetric, as was expected with the two load conditions imposed. Nevertheless the obtained solution improved the results by 36%.

## 7
## Mesh-size dependency of the results

A Michell-type structure was optimised using different mesh sizes in order to study the dependence of the final results on the mesh density (see Fig. 3). Meshes sizes ranging from 300 to 13 333 elements were used to compare the dependence of the final results on the accuracy of the mesh. Parent populations of three individuals and offspring populations of eleven individuals were used. The values of the objective function for each optimal structure are shown in Table 1. Comparing the different objective function values obtained by different mesh densities, it can be seen the mesh size is not an important factor in the optimisation process for two reasons: first, the differences between the results are not high (on the order of 3.2%), and second, the best individual is not the one obtained with the finest grid. In this way, the ESFG-FEA method can be used with a coarser mesh, allowing the method to evaluate the individuals fitness in a shorter time. As the mesh density is related to the accuracy of the solutions, this confirms the hypothesis that a highly accurate analysis is not needed for the structural optimisation process.

**Table 1** Objective function for each optimal individual using different mesh sizes

| Mesh size (elements) | Objective function value |
|---|---|
| 300 | 0.883213 |
| 1200 | 0.880379 |
| 3333 | 0.880705 |
| 7500 | 0.879722 |
| 13 333 | 0.8733459 |

## 8
## Conclusions

Shape optimisation via ES using the fixed grid finite element method gives similar results to those obtained by different optimisation methods. Smooth surfaces that need no future post-processing are obtained by this method. However, the solutions obtained with this method do not coincide with, but approach, the global optimal value.

When compared with genetic algorithms, ESFG-FEA presents faster solution times in all cases (Woon *et al.* 2000). In contrast, solution times by the ESFG-FEA method were larger than those obtained by the steepest gradient plus fixed grid method. However, this cannot be the case when the number of design variables increases, as increasing the number of evaluations required to compute the gradient increases the probability of converging toward a local minima by the steepest gradient method. Nevertheless, a more exhaustive study that includes a large number of variables and complex geometries will help to understand and test the robustness of both methods. The last test in this paper revealed no dependence of the the results on the mesh density. As the mesh density is related to the accuracy of the solutions, the experiments confirm that a highly accurate analysis is not needed for the structural optimisation process. A rebirthing technique improved the quality of the results when applied to structures with multiple load cases.

Finally, the present implementation of ESFG-FEA is limited to size and shape optimisation. No topological optimisation is possible. As an alternative, topological optimisation can be carried out by implementing the bubble method in the ESFG-FEA algorithm.

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