Insurance: Mathematics and Economics 65 (2015) 111–123

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A directional multivariate value at risk

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ARTICLE INFO

Article history:
Received January 2015
Received in revised form
July 2015
Accepted 3 September 2015
Available online 11 September 2015

MSC:
60E15
62N05
90B25
60E05
62H12

Keywords:
Multivariate risks
Value at risk
Directional approach
Multivariate quantile
Copula

ABSTRACT

In economics, insurance and finance, value at risk (VaR) is a widely used measure of the risk of loss on a specific portfolio of financial assets. For a given portfolio, time horizon, and probability α, the $100\alpha\%$ VaR is defined as a threshold loss value, such that the probability that the loss on the portfolio over the given time horizon exceeds this value is α. That is to say, it is a quantile of the distribution of the losses, which has both good analytic properties and easy interpretation as a risk measure. However, its extension to the multivariate framework is not unique because a unique definition of multivariate quantile does not exist. In the current literature, the multivariate quantiles are related to a specific partial order considered in $\mathbb{R}^n$, or to a property of the univariate quantile that is desirable to be extended to $\mathbb{R}^n$. In this work, we introduce a multivariate value at risk as a vector-valued directional risk measure, based on a directional multivariate quantile, which has recently been introduced in the literature. The directional approach allows the manager to consider external information or risk preferences in her/his analysis. We derive some properties of the risk measure and we compare the univariate VaR over the marginals with the components of the directional multivariate VaR. We also analyze the relationship between some families of copulas, for which it is possible to obtain closed forms of the multivariate VaR that we propose. Finally, comparisons with other alternative multivariate VaR given in the literature, are provided in terms of robustness.
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1. Introduction

Value at risk (VaR) has become a benchmark for risk management, and it is defined as the threshold quantity that does not exceed a certain probability level which is considered to be dangerous. It is commonly implemented by investment banks to measure the market risk of their asset portfolios. Although (VaR) has been broadly criticized from the work of Artzner et al. (1999) since it does not verify the diversification property, it has also been defended by Heyde et al. (2009) for its robustness. For univariate risks, the VaR is simply the α-quantile of the loss distribution function. Thus, the VaR is a risk measure easily interpretable, and it still remains the most popular measure used by risk managers. However, there is not a unique definition of VaR in the multivariate context because there are different possible definitions of multidimensional quantiles which are related to a specific partial order considered in $\mathbb{R}^n$, or to a property of the univariate quantile that is desirable to be extended to $\mathbb{R}^n$. Therefore, each definition of quantile could provide a potential definition of multivariate VaR. For instance, the proposals given by Koltchinskii (1997) of multivariate quantiles as inversions of mappings, multivariate quantiles in terms of norm minimization as in Chaudhuri (1996), multivariate quantiles as level-sets given by Fernández-Ponce and Suárez-Llorens (2002), multivariate quantiles based on depth functions developed in Serfling (2002), and finally, multivariate quantiles based on projections as in Fraiman and Pateiro-López (2012), Hallin et al. (2010) and Kong and Mizera (2012).

Currently business and financial activities generate data for which it has been shown that it is insufficient to consider single real-value measures over marginal aspects, in order to quantify risks jointly associated to the data. For instance, one of the drawbacks detected in the global banking regulatory *Basel* II is the solvency and liabilities dependence among the financial institution branches, or even the domino effect in the markets that could be generated by dependence among filial products. Thus, the solvability of each individual branch may strongly be affected, not only

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http://dx.doi.org/10.1016/j.insmatheco.2015.09.002
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by its activities, but also by the level of dependence among all the branches. In consequence, it is necessary to quantify the risk, considering both the multivariate nature of the data and the dependence among the marginal risks.

In *Basel* III, a new liquidity regulation was proposed in order to avoid the weakness detected in the 2007–2009 crisis; but these regulations have to be complemented by internal models in the institutions, in order to obtain better hedge results. These models have to include multivariate risk measures computable in high dimensions and also, to consider possible internal and external risks, even if the nature of those risks is strongly heterogeneous.

In Insurance, there is also interest in analyzing joint risks considering claims from different types of policies offered by the company, e.g. life, fire or health insurances, among others. Thus, allocated loss adjustment expenses play an important role in determining the expenses that are due to the processing of a specific insurance claim and they are part of the insurer expense reserves. It is one of the largest expenses that an insurer has to set aside funds for, along with contingent commissions. Insurers set aside reserves for these expenses so that they can ensure that claims are not being fraudulently made, and to process legitimate claims quickly. Since the joint behavior of the different types of policies have to be taken into consideration to determine the reserves for the insurance company, multivariate risk measures are necessary (e.g. see Frees and Valdez, 1998).

In recent decades, literature devoted to extend the VaR measure to the multivariate setting has been published. For instance, bivariate versions have been studied in Arbia (2002), Tibiletti (2001) and Nappo and Spizzichino (2009). Also, for multivariate distributions in general, some notions of VaR have been introduced (e.g. Lee and Prékopa, 2012; Embrechts and Puccetti, 2006; Cousin and Di Bernardino, 2013). Embrechts and Puccetti (2006) linked the risk measure to the level surface defined when the distribution function of risk $\mathbf{X}$, or the survival function, accumulate some α-value, which is considered as a quantile surface. Recently, Cousin and Di Bernardino (2013) introduced a new notion of multivariate VaR based on those level surfaces studied in Embrechts and Puccetti (2006). They commented that considering the whole surface as a risk measure could result in interpretation problems. Therefore, they defined the multivariate VaR as the mean of the points belonging to the surface considered in Embrechts and Puccetti (2006) and hence, the focus should be a point with the same dimension as the random vector of losses. Specifically, they define the *upper-orthant Value-at-Risk* (*lower-orthant Value-at-Risk*) at α-level ($(1-\alpha)$-level) as the conditional expectation of $\mathbf{X}$, given that $\mathbf{X}$ belongs to the α-set of its distribution (survival) function.

In this paper, we introduce a *directional multivariate Value at Risk*, based on the extremality level sets introduced in Laniado et al. (2012), which permit the concept of directional multivariate quantile to be defined. The extremality level sets are surfaces defined by following the same idea as in Embrechts and Puccetti (2006) but linked to rotations of the multivariate distribution; that is, we consider a directional approach. We share with Cousin and Di Bernardino (2013) the idea that a multivariate VaR seen as a surface could bring problems with its interpretation. Hence, we introduce the idea of considering the multivariate VaR as a vector-valued point that defines the vertex of an oriented orthant in the direction of analysis accumulating a probability α. The vertex is obtained using the mean of $\mathbf{X}$ to establish a reference system.

The risk measure that we propose, considers the high dimension nature of the real problems, and the dependence among the risks is implied in the analysis. Finally, we admit the possibility of various manager preferences, introducing a parameter of direction $\mathbf{u}$. For instance, directions like the maximum variability given for the principal components in the portfolio, or the assets weight composition could be more interesting to analyze than the classic directions assumed in the information summarized in the cumulative or survival distribution functions. Besides, the directional approach allows us to give bounds for the VaR related to linear combination of random variables, mainly when they are statistically dependent.

We prove properties of the directional VaR that we consider as relevant for a multivariate risk measure, such as consistency with respect to a particular stochastic order, tail subadditivity in the mean loss direction, as well as some invariance properties. We compare the components of the directional multivariate VaR with the univariate VaR on the marginals, in order to show that the vector given by the VaR on the marginals provides incomplete information on the joint risk. Some of these properties can be viewed as an extension to the multivariate field of the axiomatic given by Artzner et al. (1999). Some of the properties that we prove, are implicitly related with the axiomatics introduced recently in the literature (see Balbas et al., 2012; Jouini et al., 2004; Hamel and Heyde, 2010) for coherent multivariate measures.

We also obtain closed expressions of the VaR when bivariate copulas are considered or when multivariate Archimedean copulas govern the dependence among the components of the portfolio. Finally, we present comparisons in terms of robustness with the alternative vector-valued multivariate VaR, introduced by Cousin and Di Bernardino (2013).

The paper is structured as follows. In Section 2, we introduce some preliminary concepts and notation necessary in order to understand the main contributions of the paper. In Section 3, the *directional multivariate Value at Risk* ($VaR_\alpha^{\mathbf{u}}(\mathbf{X})$) is introduced and we provide analytic properties for this risk measure. Section 4 provides the comparisons between the univariate VaR over the marginals and the components of the directional multivariate VaR. Section 5 is devoted to theoretical results and closed forms of the multivariate VaR when particular families of copulas are considered. In Section 6, we develop the robustness analysis. Finally, some conclusions are summarized as well as some possible extensions are suggested for future work.

2. Preliminaries

The main objective of this paper is to introduce a directional multivariate Value at Risk, based on the notion of directional multivariate quantile given in Laniado et al. (2010). In order to make the paper self contained, we devote this section to introduce and review main concepts necessary to properly define the risk measure.

Definition 2.1. An oriented orthant in $\mathbb{R}^n$ with vertex $\mathbf{x}$ in the direction $\mathbf{u}$ is defined as,

$$\mathfrak{C}_{\mathbf{x}}^{R_{\mathbf{u}}} = \{\mathbf{z} \in \mathbb{R}^n : R_{\mathbf{u}}(\mathbf{z}-\mathbf{x}) \geq 0\} \tag{2.1}$$

where $\mathbf{u} \in \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| = 1\}$ and $R_{\mathbf{u}}$ is an orthogonal matrix such that $R_{\mathbf{u}}\mathbf{u} = \mathbf{e}$, with $\mathbf{e} = \frac{\sqrt{n}}{n}[1, \ldots, 1]'$.

Note that given $\mathbf{u}$, $R_{\mathbf{u}}$ is not unique for $n \geq 3$ and thus, Definition 2.1 generates a family of oriented orthants. In order to simplify the definition of the risk measure introduced in this paper, we impose conditions on the possible $R_{\mathbf{u}}$ to guarantee uniqueness in the transformation. From now on, let $\mathbf{u}$ be a unit vector with non-null components and let $M_{\mathbf{u}}$ and $M_{\mathbf{e}}$ be matrices defined as,

$$\begin{aligned} M_{\mathbf{u}} &= [\mathbf{u},\ sgn(u_2)\mathbf{e}_2, \ldots, sgn(u_n)\mathbf{e}_n], \\ M_{\mathbf{e}} &= [\mathbf{e},\ \mathbf{e}_2, \ldots, \mathbf{e}_n], \end{aligned} \tag{2.2}$$

where u_i, $i = 1, \ldots, n$ is the ith component of $\mathbf{u}$, $sgn(\cdot)$ is the scalar sign function and $\mathbf{e}_i$ is the vector with all its components equal to zero except the ith component equal to one. Note that the hypothesis of $u_i \neq 0,\ i = 1, \ldots, n$ guarantees that $M_{\mathbf{u}}$ always is a matrix of rank n. Now, we consider the QR decomposition of $M_{\mathbf{u}}$ and $M_{\mathbf{e}}$ (see e.g. Horn and Johnson, 2013, Ch. 2),

$$M_{\mathbf{u}} = Q_{\mathbf{u}} T_{\mathbf{u}}, \qquad M_{\mathbf{e}} = Q_{\mathbf{e}} T_{\mathbf{e}},$$

such that $T_{\mathbf{u}}$ and $T_{\mathbf{e}}$ are triangular matrices with positive diagonal elements, and $Q_{\mathbf{u}}$ and $Q_{\mathbf{e}}$ are orthogonal matrices. Note that these decompositions are unique due to both the full rank of $M_{\mathbf{u}}$ and $M_{\mathbf{e}}$ and the structure of the decomposition (see e.g. Horn and Johnson, 2013, Theorem 2.1.14, p.g. 89). Also, the first columns on $Q_{\mathbf{u}}$ and $Q_{\mathbf{e}}$ are the same as in $M_{\mathbf{u}}$ and $M_{\mathbf{e}}$; that is, $\mathbf{u}$ and $\mathbf{e}$ respectively. Therefore, $Q_{\mathbf{e}}\mathbf{e}_1 = \mathbf{e}$ and $Q_{\mathbf{u}}\mathbf{e}_1 = \mathbf{u}$ and thus, $\left(Q_{\mathbf{e}} Q'_{\mathbf{u}}\right)\mathbf{u} = \mathbf{e}$, which motivates the following definition.

Definition 2.2. The QR oriented orthant with vertex $\mathbf{x}$ in direction $\mathbf{u}$ is the oriented orthant as in Definition 2.1 but using $R_{\mathbf{u}} = Q_{\mathbf{e}} Q'_{\mathbf{u}}$. It is denoted by $\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}}$.

Based on Definition 2.2, a partial data order in $\mathbb{R}^n$ (denoted by $\preceq_{\mathbf{u}}$) can be defined by,

$$\mathbf{x} \preceq_{\mathbf{u}} \mathbf{y}, \quad \text{if and only if,} \quad \mathfrak{C}^{\mathbf{u}}_{\mathbf{x}} \supseteq \mathfrak{C}^{\mathbf{u}}_{\mathbf{y}}, \tag{2.3}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}}$ is as in (2.2). Equivalently,

$$\mathbf{x} \preceq_{\mathbf{u}} \mathbf{y}, \quad \text{if and only if,} \quad R_{\mathbf{u}}\mathbf{x} \leq R_{\mathbf{u}}\mathbf{y},$$

where $R_{\mathbf{u}} = Q_{\mathbf{e}} Q'_{\mathbf{u}}$ and the order on the right side is component-wise.

Throughout the paper we will use the following notation related to subsets in $\mathbb{R}^n$. Given $\mathbf{b} \in \mathbb{R}^n$, $c \in \mathbb{R}$, and $A \subset \mathbb{R}^n$, the sets $\mathbf{b} + A$ and cA are defined as,

$$\mathbf{b} + A := \{\mathbf{b} + \mathbf{a} : \mathbf{a} \in A\}, \qquad cA := \{c\mathbf{a} : \mathbf{a} \in A\}. \tag{2.4}$$

We recall some results on *QR oriented orthants* that will be useful in the main sections of the paper. The proofs are given in the Appendix.

Lemma 2.3. *Given a direction* $\mathbf{u}$ *and a vertex* $\mathbf{x}$*, then*

$$\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}} = -\mathfrak{C}^{-\mathbf{u}}_{-\mathbf{x}}. \tag{2.5}$$

Lemma 2.4. *Given* $c > 0$ *and* $\mathbf{b} \in \mathbb{R}^n$*, then*

$$\mathfrak{C}^{\mathbf{u}}_{c\mathbf{x}+\mathbf{b}} = c\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}} + \mathbf{b}. \tag{2.6}$$

We also recall some definitions of useful stochastic orders; see Shaked and Shanthikumar (2007), for more details.

Definition 2.5. Given two random vectors $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in:

(i) Usual stochastic order (denoted by $\mathbf{X} \leq_{st} \mathbf{Y}$) if $\mathbb{E}[\phi(\mathbf{X})] \leq \mathbb{E}[\phi(\mathbf{Y})]$, for any increasing function $\phi(\cdot)$ with finite expectations.
(ii) Upper orthant order (denoted by $\mathbf{X} \leq_{uo} \mathbf{Y}$) if $\bar{F}_{\mathbf{X}}(x_1, \ldots, x_n) \leq \bar{F}_{\mathbf{Y}}(x_1, \ldots, x_n)$, for all $\mathbf{x}$, where $\bar{F}_{\mathbf{X}}$, $\bar{F}_{\mathbf{Y}}$ denote the *survival* functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively.
(iii) Lower orthant order (denoted by $\mathbf{X} \leq_{lo} \mathbf{Y}$) if $F_{\mathbf{X}}(x_1, \ldots, x_n) \geq F_{\mathbf{Y}}(x_1, \ldots, x_n)$, for all $\mathbf{x}$, where $F_{\mathbf{X}}$, $F_{\mathbf{Y}}$ denote the *cumulative distribution* functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively.

It is easy to verify that both orders, the upper orthant and the lower orthant, are implied by the usual stochastic order. The following stochastic order defined in Laniado et al. (2012) is a key tool in providing some properties of the multivariate VaR that we introduce in the next section.

Definition 2.6. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors with associated probability distribution $\mathbb{P}$, $\mathbf{X}$ is said smaller than $\mathbf{Y}$ in the extremality order in the direction $\mathbf{u}$ (denoted by $\mathbf{X} \leq_{\mathcal{E}_{\mathbf{u}}} \mathbf{Y}$) if,

$$\mathbb{P}\left[R_{\mathbf{u}}(\mathbf{X} - \mathbf{z}) \geq 0\right] \leq \mathbb{P}\left[R_{\mathbf{u}}(\mathbf{Y} - \mathbf{z}) \geq 0\right], \quad \text{for all } \mathbf{z} \text{ in } \mathbb{R}^n.$$

It is easy to show that $\mathbf{X} \leq_{\mathcal{E}_{\mathbf{u}}} \mathbf{Y} \Leftrightarrow R_{\mathbf{u}}\mathbf{X} \leq_{uo} R_{\mathbf{u}}\mathbf{Y}$. Moreover, if $\mathbf{X} \leq_{\mathcal{E}_{\mathbf{u}}} \mathbf{Y}$ then $\mathbb{E}[\mathbf{X}] \preceq_{\mathbf{u}} \mathbb{E}[\mathbf{Y}]$, as it is proven in Laniado et al. (2012, *Property* 3.4). Since the multivariate VaR is based on the definition of a quantile, we also need to introduce the directional multivariate quantile given in Laniado et al. (2010).

Definition 2.7. Let $\mathbf{X}$ be a random vector with associated probability distribution $\mathbb{P}$. Then the directional multivariate quantile at level α, in direction $\mathbf{u}$ is defined as

$$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) := \partial\{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}(\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}}) \leq \alpha\}, \tag{2.7}$$

where ∂ denoted the boundary of the subset considered into brackets and $0 \leq \alpha \leq 1$.

From now on, we focus on an absolutely-continuous random vector $\mathbf{X}$ (with respect to the Lebesgue measure ν on $\mathbb{R}^n$) with increasing marginal distribution functions and such that $\mathbb{E}[X_i] < \infty$, for $i = 1, \ldots, n$. These conditions are called *regularity conditions*.

We also recall the two versions of the vector-valued VaR introduced in Cousin and Di Bernardino (2013). They are the benchmarks of the risk measure introduced in this paper as it is shown in Sections 5 and 6.

- The *lower multivariate VaR* at level α is defined as,

$$\underline{VaR}_{\alpha}(\mathbf{X}) = \mathbb{E}\left[\mathbf{X} | F(\mathbf{x}) = \alpha\right]. \tag{2.8}$$

- The *upper multivariate VaR* at level α is defined as,

$$\overline{VaR}_{\alpha}(\mathbf{X}) = \mathbb{E}\left[\mathbf{X} | \bar{F}(\mathbf{x}) = 1 - \alpha\right]. \tag{2.9}$$

Note that (2.8) and (2.9) are the expected value of the hypersurfaces defined as *Upper-Orthant VaR* and *Lower-Orthant VaR* in Embrechts and Puccetti (2006).

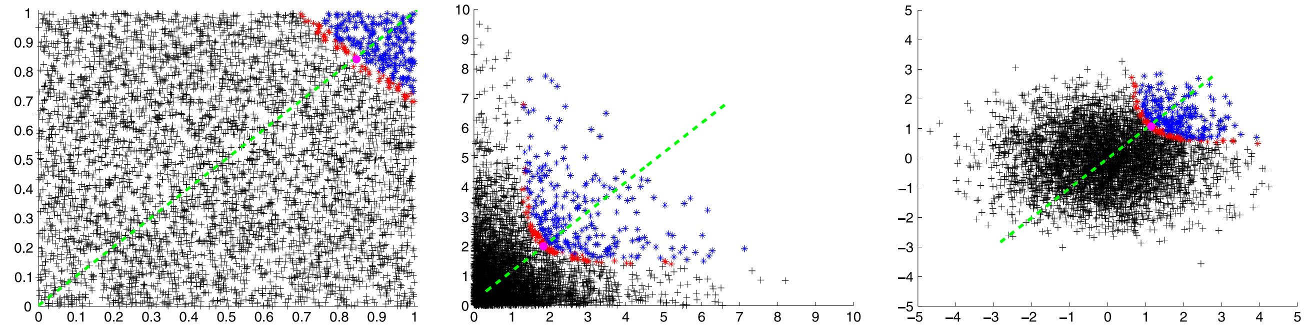
3. Directional multivariate value at risk

In the univariate setting, the relationship between the quantiles related to the loss distribution and the VaR is obvious. In this section, we propose a definition of multivariate VaR for a portfolio of n-dependent risks, linked with the directional multivariate quantile defined in (2.7). As well, the result is a point in $\mathbb{R}^n$; that is, a vector of the same dimension as the considered portfolio of risks. Specifically, as in the univariate case, this point defines the vertex of an oriented orthant that accumulates a probability α, but in the direction that the investor or the risk manager considers more convenient.

Definition 3.1. Let $\mathbf{X}$ be a random vector satisfying the regularity conditions and $0 \leq \alpha \leq 1$. Then the directional multivariate Value at Risk of $\mathbf{X}$ in direction $\mathbf{u}$ at probability level α is given by

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) = \left(\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) \bigcap \{\lambda\mathbf{u} + \mathbb{E}[\mathbf{X}] : \ \lambda \in \mathbb{R}\}\right). \tag{3.1}$$

We must highlight that given a direction $\mathbf{u}$, the $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ is the intersection between the directional quantile at level α, and the line defined by both the direction $\mathbf{u}$ and the mean of $\mathbf{X}$; that is, $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ is a point in $\mathbb{R}^n$. Note that the *regularity conditions* on $\mathbf{X}$ ensure that the intersection in (3.1) is non void. We want to point out that the central tool is chosen to be the mean as a reference point for the random vector space, i.e., for the support of the associated probability distribution. As we demonstrate, the choice of the mean in Definition (3.1) allows us to derive desirable and interpretable analytic properties related to the risk measure. However, other central reference points can be possible; for example the median seen as the deepest point associated with some multivariate depth measure, which may provide a more robust risk measure (e.g. Zuo and Serfling, 2000; Cascos et al., 2011).



(A) Bivariate uniform. (B) Bivariate exponential. (C) Bivariate normal.

Fig. 1. $VaR^{-\mathbf{e}}_{0.7}(\mathbf{X})$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Remark 3.2. Definition 3.1 assumes that $\mathbf{u}$ is a vector with non-null components in order to the associated *QR oriented orthant* be properly defined (see Definition 2.2). However, this is not a restrictive condition in multivariate risk analysis since a null component in $\mathbf{u}$ is equivalent to ignore/depreciate the information related to that specific component. Therefore, the advisable is to reduce the dimension of the problem avoiding the null components before the evaluation of the directional risk measure.

Fig. 1 displays some examples of the risk measure defined in (3.1), for three different bivariate distributions in the direction $-\mathbf{e}$ with $\alpha = 0.7$. This direction makes reference to the distribution function of $\mathbf{X}$. Fig. 2 presents examples with the same bivariate distributions, but in the direction $\mathbf{e}$ and for $\alpha = 0.3$; that is, regarding the information contained in the survival function of $\mathbf{X}$. We call these two directions classical directions, but the aim of this work is to show that it could be interesting to consider other directions in the analysis of risk.

Observe that in the figures, the line in direction $\mathbf{u}$ crossing the mean in green is displayed while the quantile curve is displayed in red. The VaR that we propose is just the intersection between the line and the quantile curve. On the other hand, the points in blue are the points "below" the level of risk α in the corresponding direction; meanwhile the black points are those "exceeding" the level risk. Observe Fig. 1, if you take any point on the blue region as a vertex of an oriented orthant in direction $-\mathbf{e}$, then the probability of that orthant will be greater than α. It will be equal to α or smaller than α if the point is taken from the red curve or black region, respectively. From Fig. 2 in direction $\mathbf{e}$, the same conclusion can be drawn. Fig. 3 displays the risk measure for a bivariate normal distribution, but considering alternative directions. Specifically, we consider the directions corresponding to the second and fourth orthants in $\mathbb{R}^2$, which are the complementary orthants of those used by the distribution and survival functions. These orthants result interesting when it is necessary to analyze the relationships between random variables of the type $\mathbb{P}[X_1 > x_1, X_2 \le x_2]$ or $\mathbb{P}[X_1 \le x_1, X_2 > x_2]$, or when the bivariate distribution in consideration has negative dependence.

It is desirable that the classical univariate VaR agrees with our definition of VaR in the case $n = 1$; this fact is shown in the following. Recall that the univariate VaR is defined as,

$$VaR_{1-\alpha}(X) := \inf\{x \in \mathbb{R} : \mathbb{P}[X \ge x] \le \alpha\}, \tag{3.2}$$

where $1-\alpha$ is usually considered closed to 1. Moreover, the VaR may also be defined in terms of the distribution function as,

$$VaR_{1-\alpha}(X) := \inf\{x \in \mathbb{R} : \mathbb{P}[X \le x] \ge 1-\alpha\}. \tag{3.3}$$

As $\mathbb{P}[X \le x] = 1 - \mathbb{P}[X \ge x]$ in the univariate setting under continuity, then (3.2) and (3.3) are the same. To be consistent with the univariate VaR, our definition of multivariate VaR agrees with the classical definition for $n = 1$. That is, we have, in terms of $VaR^{\mathbf{u}}_{\alpha}(X)$, that:

$$VaR^{1}_{\alpha}(X) = VaR_{1-\alpha}(X) = VaR^{-1}_{1-\alpha}(X),$$

where $VaR^{1}_{\alpha}(X)$ is related to definition (3.2) and $VaR^{-1}_{1-\alpha}(X)$ is related to definition (3.3). However, this fact does not hold in the multivariate context where $F(\mathbf{x}) + \bar{F}(\mathbf{x}) = 1$ is not true in general, being

$$F(\mathbf{x}) = \mathbb{P}[\mathfrak{C}^{-\mathbf{e}}_{\mathbf{x}}] = \mathbb{P}[\mathbf{X} \le \mathbf{x}], \tag{3.4}$$

$$\bar{F}(\mathbf{x}) = \mathbb{P}[\mathfrak{C}^{\mathbf{e}}_{\mathbf{x}}] = \mathbb{P}[\mathbf{X} \ge \mathbf{x}]. \tag{3.5}$$

The remainder of this section is devoted to providing some properties of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ which are similar to those properties considered in the risk literature; (see Artzner et al., 1999; Burgert and Ruschendorf, 2006; Cardin and Pagani, 2010; Rachev et al., 2008). Specifically, we provide properties of the multivariate $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ in terms of Artzner et al. (1999)'s properties related to coherent risk measures in the univariate setting. In a similar way, Jouini et al. (2004), Cascos and Molchanov (2007), Hamel and Heyde (2010) and Molchanov and Cascos (2014) propose some properties to coherent versions of multivariate risk measures defined as set-value measures. Balbas et al. (2012) also include properties referred to vector-value measures, but we have explored other properties inherent to the vector-value output in our proposal, such as the invariance under orthogonal transformations.

Property 3.3 (*Non-Negative Loading*)**.** *For $\alpha > 0$ small,*

$$\mathbb{E}[\mathbf{X}] \preceq_{\mathbf{u}} VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}). \tag{3.6}$$

This property reflects that the risk measure is an upper-bound of the mean value of the losses, with respect to the partial order given in (2.3).

Property 3.4 (*Quasi-Odd Measure*)**.** *$VaR^{\mathbf{u}}_{\alpha}(\cdot)$ satisfies the property:*

$$VaR^{\mathbf{u}}_{\alpha}(-\mathbf{X}) = -VaR^{-\mathbf{u}}_{\alpha}(\mathbf{X}). \tag{3.7}$$

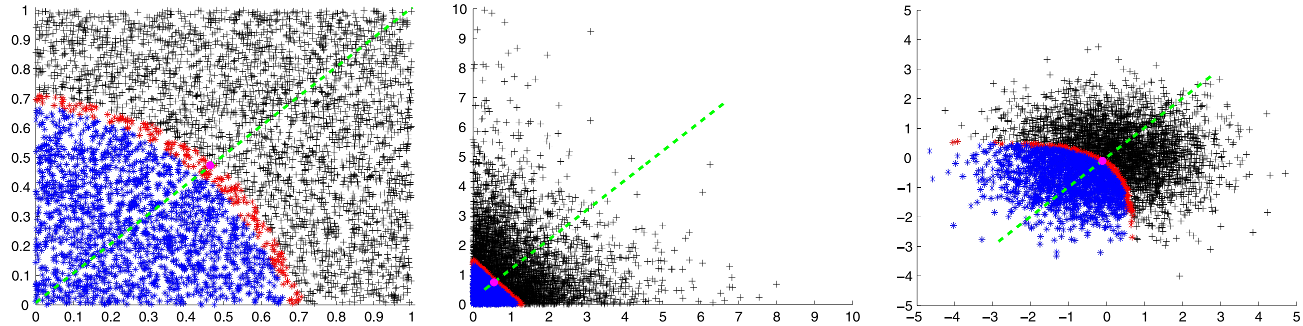
This property shows *symmetry* with respect to the analysis of risk for positive random losses, or the analysis of negative random returns.

Property 3.5 (*Positive Homogeneity and Translation Invariance*)**.** *Let $c \in \mathbb{R}_+$, $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{Y} = c\mathbf{X} + \mathbf{b}$, then,*

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{Y}) = cVaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) + \mathbf{b}. \tag{3.8}$$

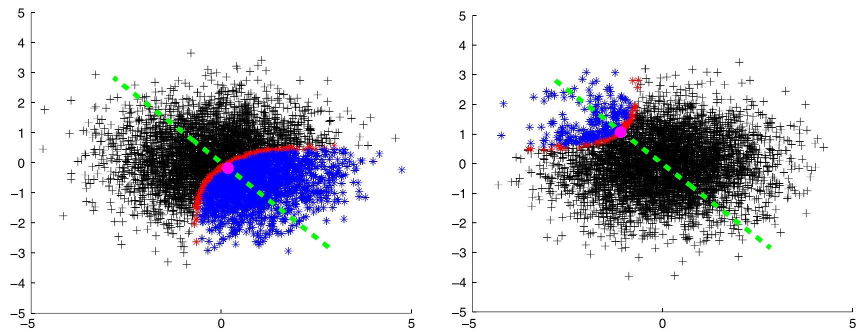
Property 3.6 (*Consistency w.r.t. Extremality Stochastic Order*)**.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors satisfying the regularity conditions. If $\mathbb{E}[\mathbf{Y}] = c\mathbf{u} + \mathbb{E}[\mathbf{X}]$ with $c > 0$, and $\mathbf{X} \le_{\mathcal{E}_u} \mathbf{Y}$, then:*

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) \preceq_u VaR^{\mathbf{u}}_{\alpha}(\mathbf{Y}). \tag{3.9}$$



(A) Bivariate uniform. (B) Bivariate exponential. (C) Bivariate normal.

Fig. 2. $VaR^{\mathbf{e}}_{0.3}(\mathbf{X})$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(A) $\mathbf{u} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\alpha = 0.3$. (B) $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\alpha = 0.7$.

Fig. 3. $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$.

Now, we introduce a type of orthogonal transformations before the following property.

Definition 3.7. A QR rotation of a unit vector $\mathbf{u}$ over another unit vector $\mathbf{w}$ is characterized by the matrix $Q = Q_{\mathbf{w}}Q'_{\mathbf{u}}$, where the matrices $Q_{\mathbf{w}}$, $Q_{\mathbf{u}}$ correspond to the orthogonal parts in the QR decompositions of the matrices $M_{\mathbf{u}}$ and $M_{\mathbf{w}}$, defined in (2.2).

Note that a QR rotation of $\mathbf{u}$ over $\mathbf{w}$ implies that $Q\,\mathbf{u} = \mathbf{w}$.

Property 3.8 (*Orthogonal Quasi-Invariance*)**.** *Let* $\mathbf{u}$ *and* $\mathbf{w}$ *be two unit vectors. If* Q *is the QR rotation of* $\mathbf{u}$ *over* $\mathbf{w}$*. Then,*

$$VaR^{\mathbf{w}}_{\alpha}(Q\mathbf{X}) = Q\,VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}). \tag{3.10}$$

Property 3.9 (*Non-Excessive Loading*)**.** *Let* $R_{\mathbf{u}}$ *be the orthogonal matrix described in* (2.2)*. Then,*

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) \preceq_{\mathbf{u}} R'_{\mathbf{u}} \sup_{\omega\in\Omega}\{R_{\mathbf{u}}\mathbf{X}(\omega)\}. \tag{3.11}$$

This property shows that $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ is upper bounded by the supremum of the losses in the direction considered. Another desirable property in the literature for risk measures is the subadditivity. As it is well-known, the classical univariate VaR is not a subadditivity measure. However, there are conditions that ensure the tail region subadditivity property (see Artzner et al., 1999; Heyde et al., 2009; Daníelsson et al., 2013). In the same way, we stress that the $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ is not subadditive in general, but we prove that this property holds under some conditions. First another definition is necessary.

Definition 3.10. A random vector $\mathbf{X}$ is multivariate regularity varying with tail index β if there is a real-value function $\phi(t) > 0$ that is[1] regularly varying at infinity with exponent $\frac{1}{\beta}$ and a non-zero measure $\mu(\cdot)$ on the Borel σ-field $\mathcal{B}([0,\infty]^n \setminus \{\mathbf{0}\})$ such that,

$$t\mathbb{P}[(\phi(t))^{-1}\mathbf{X} \in \cdot\,] \xrightarrow{v} \mu(\cdot), \tag{3.12}$$

where $\xrightarrow{v}$ means vague convergence and $t \to \infty$ (see e.g. Jessen and Mikosh, 2006; Resnick, 1987).

In this case, the measure has the property

$$\mu(cB) = c^{-\beta}\mu(B), \tag{3.13}$$

for all $c > 0$ and every Borel set B. In Mikosch (2003, pg. 25), it is possible to see the proof of the property. Illustrative examples of Definition 3.10 can be found in Resnick (2007, pg. 192).

As it is noted in Daníelsson et al. (2013), the previous definition allows to introduce a notion of a fat-tailed multivariate distribution that induces the tail region subadditivity property of the $VaR^{\mathbf{u}}_{\alpha}(\cdot)$.

Property 3.11 (*Tail Region Subadditivity*)**.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be random vectors, with the same mean* $\mathbf{m}$*. If* $(\mathbf{X},\mathbf{Y})$ *is a regularly varying random vector with index* $\beta > 1$ *and non-degenerate tails then, the* $VaR^{\mathbf{u}}_{\alpha}(\cdot)$ *is subadditive in the tail region in direction* $\mathbf{u} = \frac{\mathbf{m}}{\|\mathbf{m}\|}$*, i.e.,*

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}+\mathbf{Y}) \preceq_{\mathbf{u}} VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) + VaR^{\mathbf{u}}_{\alpha}(\mathbf{Y}). \tag{3.14}$$

The proof is provided in the Appendix following a similar approach as in Daníelsson et al. (2013). Note that Property 3.11 extends to the multivariate case the Proposition 1 given in Daníelsson et al. (2013) for the univariate case. As you can see, the

[1] If a function $\phi(\cdot)$ holds $\lim_{x\to\infty}\frac{\phi(tx)}{\phi(x)} = t^{\frac{1}{\beta}}$, for all $t > 0$ is called regularly varying at infinity with exponent $\frac{1}{\beta}$.

property ensures that at least in the direction of the mean loss, it is useful to merge two risky activities in order to diversify the risk. Property 3.11 could be extended to random vectors with means satisfying $\mathbb{E}[\mathbf{X}] = c\mathbb{E}[\mathbf{Y}]$ for $c > 0$.

4. Comparison of the univariate VaR componentwise and the directional multivariate VaR

The aim of this section is to compare the components of $VaR_\alpha^{\mathbf{u}}(\mathbf{X})$ with the univariate VaR related to each marginal distribution of $\mathbf{X}$. But prior to this we need to recall the definition of a multivariate quasi-concave function.

Definition 4.1. A multivariate function $g : \mathbb{R}^n \to \mathbb{R}$ is a quasi-concave function if the upper-level set $U_q := \{\mathbf{x} \in \mathbb{R}^n : g(x) \geq q\}$ is a convex set for all $q \in \mathbb{R}$. Or equivalently, the complementary of the lower set $L_q := \{\mathbf{x} \in \mathbb{R}^n : g(x) \leq q\}$ is a convex set for all $q \in \mathbb{R}$.

We point out that both the distribution and survival functions, satisfy Definition 4.1 under regularity conditions. Specifically, this result is proven by Tibiletti (1995) but for elliptical distribution and Archimedean copula families.

Let us denote by X_i the ith marginal of the random vector $\mathbf{X}$ and by $[\cdot]_i$ the ith component related to a point in $\mathbb{R}^n$. The following result provides comparisons between the components of the multivariate VaR introduced in this work and the classical univariate VaR on the marginals.

Proposition 4.2. *Consider a random vector* $\mathbf{X}$ *satisfying the regularity conditions. Assume that its survival function* $\bar{F}$ *is quasi-concave. Then, for all* $\alpha \in (0, 1)$*:*

$$VaR_{1-\alpha}(X_i) \geq \left[VaR_\alpha^{\mathbf{e}}(\mathbf{X})\right]_i, \quad \text{for all } i = 1, \ldots, n.$$

If its multivariate distribution function F is also quasi-concave, then, for all $\alpha \in (0, 1)$, *we have that*

$$\left[VaR_{1-\alpha}^{-\mathbf{e}}(\mathbf{X})\right]_i \geq VaR_{1-\alpha}(X_i), \quad \text{for all } i = 1, \ldots, n.$$

The proof is given in the Appendix. As you can see, the preceding result can be extended in other directions as follows.

Corollary 4.3. *Let* $\mathbf{X}$ *be a random variable satisfying the regularity conditions and let* $\mathbf{u}$ *be a specified direction. If the survival function of* $R_{\mathbf{u}}\mathbf{X}$ *is a quasi-concave function, then, for all* $0 \leq \alpha \leq 1$,

$$VaR_{1-\alpha}([R_{\mathbf{u}}X]_i) \geq \left[R_{\mathbf{u}} VaR_\alpha^{\mathbf{u}}(\mathbf{X})\right]_i, \quad \text{for all } i = 1, \ldots, n.$$

Besides, if $R_{\mathbf{u}}X$ *has a quasi-concavity cumulative distribution, then*

$$\left[R_{\mathbf{u}} VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X})\right]_i \geq VaR_{1-\alpha}([R_{\mathbf{u}}X]_i), \quad \text{for all } i = 1, \ldots, n,$$

with $R_{\mathbf{u}}$ *as in Definition 2.2.*

The proof is straightforward from Property 3.8 and Proposition 4.2. Therefore, by linking the previous results we have the following inequality for all pairs $(\mathbf{u}, \alpha)$, $(-\mathbf{u}, 1-\alpha)$.

$$VaR_\alpha^{\mathbf{u}}(\mathbf{X}) \preceq_{\mathbf{u}} VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X}). \tag{4.1}$$

This relationship allows us to define a *directional upper VaR* and a *directional lower VaR* in a similar way to Embrechts and Puccetti (2006) and Cousin and Di Bernardino (2013), but with a unified notation that takes into consideration the directional parameter. Specifically, we introduce the following by redenoting our measure in the pairs $(\mathbf{u}, \alpha)$ and $(-\mathbf{u}, 1-\alpha)$:

The *upper VaR in direction* $\mathbf{u}$ is defined as,

$$\overline{VaR}_\alpha^{\mathbf{u}}(\mathbf{X}) = VaR_\alpha^{\mathbf{u}}(\mathbf{X}). \tag{4.2}$$

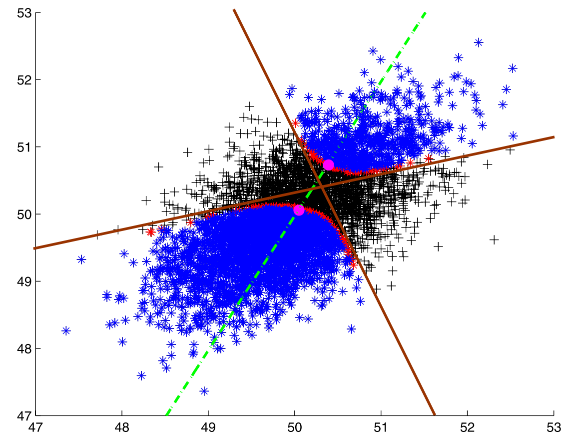


Fig. 4. Lower and upper $VaR_\alpha^{\mathbf{u}}(\mathbf{X})$ with $\mathbf{u} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $\alpha = 0.3$ for a bivariate normal.

The *lower VaR in a direction* $\mathbf{u}$ is defined as,

$$\underline{VaR}_\alpha^{\mathbf{u}}(\mathbf{X}) = VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X}). \tag{4.3}$$

An example of these concepts is displayed in Fig. 4, where we can see in a bivariate normal distribution, the *upper VaR in direction* $\mathbf{u} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ for a level of risk $\alpha = 0.3$, and the corresponding *lower VaR in direction* $-\mathbf{u}$ and level risk $1-\alpha$. Note that we describe on the figure types of asymptotes for the quantile curves, that represent the univariate quantiles for each marginal of the rotated random vector $R_{\mathbf{u}}\mathbf{X}$ at the same α, where the rotation matrix $R_{\mathbf{u}}$ is the same as in (2.2). These asymptotes can be seen as a generalization of those defined in Belzunce et al. (2007) for the quantile curves in the classical directions.

There is another practical application where the link between the multivariate VaR and the univariate VaR is interesting (see e.g. Embrechts and Puccetti, 2006; Wang et al., 2013; Bernard et al., 2014). It is when is necessary to give upper-bounds of the univariate VaR over a linear transformation of the marginal losses. For instance, when the risk over the transformation given by the portfolio weights vector is considered, i.e., when the objective random variable is the return function given by

$$Z = \mathbf{w}'\mathbf{X},$$

where $\mathbf{w}$ is the portfolio weights vector chosen by the investor. Since it is difficult to obtain the VaR of Z mainly when the components of the portfolio cannot be assumed independent, there is special interest in obtaining at least a bound for $VaR_\alpha(Z)$. Fortunately, we can give an upper-bound using the directional approach.

Proposition 4.4. *Let* $\mathbf{u} = -\frac{\mathbf{w}}{\|\mathbf{w}\|}$ *be the unit vector in direction of the portfolio weights. If* $\mathbf{x} \in \mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u})$, *then* $\mathbf{w}'\mathbf{x} \geq VaR_\alpha(Z)$.

The proof is given in the Appendix. As a consequence of Proposition 4.4 we can consider the bound given by,

$$\mathbf{w}' VaR_\alpha^{-\frac{\mathbf{w}}{\|\mathbf{w}\|}}(\mathbf{X}) \geq VaR_\alpha(Z), \tag{4.4}$$

which is another justification to consider a directional approach of the multivariate VaR, as well as its utility in financial applications.

5. Directional multivariate VaR and copulas

Researchers refer to copulas as "the multivariate distribution functions whose one-dimensional marginal distributions are uniform in [0, 1]". For an extensive discussion of copulas, we refer the reader to Nelsen (2006). This powerful tool allows the definition of scale-free measures of dependence and families of multivariate distributions. Two aspects are important in multivariate distributions, the distribution of the marginals and the dependence structure among them. The concept of copula fully describes the overall structure of dependence between the marginal variables and provides a global model for their stochastic behavior. The important result that links these two aspects is Sklar's theorem, which allows in terms of a copula, to write the multivariate distribution function as,

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \tag{5.1}$$

where F is the joint distribution function, $F_1, \dots, F_n$ its marginal distributions and C the copula, which according to Sklar's theorem always exists. The copulas provide a powerful tool to find closed expressions of multivariate quantiles for special families of copulas. For example, in finance where losses are modeled in percentage terms, losses will be in the unitary hyper cube of dimension n.

Hence, the objective of this section is to analyze how the $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ can be obtained in terms of some families of copulas. The first result shows the representation of the $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ restricted to bivariate copulas. Let $\mathbf{X}$ be a bivariate random vector with marginals uniformly distributed in the interval $[0, 1]$. In this case, the distribution function of $\mathbf{X}$ is a copula $C(\cdot, \cdot)$ with density $c(\cdot, \cdot)$. It is well known that $E[\mathbf{X}] = (\frac{1}{2}, \frac{1}{2})$. Note that assuming $n = 2$, a direction $\mathbf{u} = (u_1, u_2)$ can be characterized by an angle θ such that $\tan\theta = u_2/u_1$, and then, $\mathbf{u} = (\cos\theta, \sin\theta)$. Following with the notation given by the angles, the $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ must be a point on the line l_θ defined by,

$$l_\theta := \begin{cases} \left\{(w_1, w_2) : \ w_2 - \dfrac{1}{2} = \left(w_1 - \dfrac{1}{2}\right)\tan(\theta)\right\}, \\ \quad \text{if } \cos(\theta) \neq 0, \\ \left\{(w_1, w_2) : \ w_1 \in [0, 1], w_2 = \dfrac{1}{2}\right\}, \\ \quad \text{if } \cos(\theta) = 0. \end{cases} \tag{5.2}$$

Therefore, given a direction θ, $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ is characterized by its first component and the second one is obtained using (5.2). Now, the first component can be obtained by solving the following equation on the domain of the integral,

$$\iint_{D_\theta(w_1)} c(s, t)dtds = \alpha, \tag{5.3}$$

where $D_\theta(w_1)$ is given by the intersection of the unit square $[0, 1] \times [0, 1]$ and the oriented quadrant with direction determined by θ and vertex $(w_1, l_\theta(w_1))$. Specifically, $D_\theta(w_1)$ can be expressed in terms of the unknown w_1 by using the semi-lines $l^1_\theta(w_1), l^2_\theta(w_1)$ that bound the corresponding quadrant which are defined as,

$$\begin{aligned} l^1_\theta(w_1) \ :=& \ \left\{(z_1, z_2) : z_2\cos\left(\theta - \frac{\pi}{4}\right) - z_1\sin\left(\theta - \frac{\pi}{4}\right)\right. \\ =& \ w_1\left(\tan(\theta)\cos\left(\theta - \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\right) \\ & \left. - \frac{1}{2}\left(\tan(\theta) - 1\right)\cos\left(\theta - \frac{\pi}{4}\right)\right\} \\ l^2_\theta(w_1) \ :=& \ \left\{(z_1, z_2) : z_2\sin\left(\theta - \frac{\pi}{4}\right) + z_1\cos\left(\theta - \frac{\pi}{4}\right)\right. \\ =& \ w_1\left(\tan(\theta)\sin\left(\theta - \frac{\pi}{4}\right) + \cos\left(\theta - \frac{\pi}{4}\right)\right) \\ & \left. - \frac{1}{2}(\tan(\theta) - 1)\sin\left(\theta - \frac{\pi}{4}\right)\right\}. \end{aligned}$$

For instance, if $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, we can write the equation as follows:

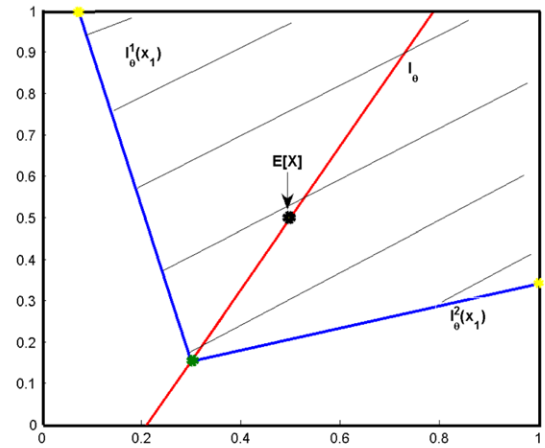


Fig. 5. Quadrant given by $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ and vertex over the line l_θ.

$$\int_{\min\{l^2_\theta(w_1)\bigcap\{z_1=0\},0\}}^{w_1} \int_{l^2_\theta(w_1)}^{1} c(s, t)dtds + \int_{w_1}^{\min\{l^1_\theta(w_1)\bigcap\{z_1=1\},1\}} \int_{l^1_\theta(w_1)}^{1} c(s, t)dtds = \alpha. \tag{5.4}$$

Fig. 5 shows a case of the region $D_\theta(w_1)$ with $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ being the solution to (5.4), a point over the line l_θ. In summary, we can obtain $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ for a given bivariate vector with copula density $c(\cdot, \cdot)$.

Now, we focus on the Archimedean family of copulas widely used in the literature whose definition is the following:

Definition 5.1 (*Archimedean Copulas*)**.** Let $\phi : [0, 1] \to [0, \infty)$ be a continuous, convex and strictly decreasing function with $\phi(\mathbf{1}) = 0$. Let $\phi^{-1}(\cdot)$ be a pseudo-inverse function of $\phi(\cdot)$. Then an Archimedean copula $C(v_1, \dots, v_n)$ is generated by

$$C(v_1, \dots, v_n) = \phi^{-1}(\phi(v_1) + \dots + \phi(v_n)). \tag{5.5}$$

In this case, for an n-dimensional random variable with distribution function belonging to the Archimedean family of copulas with generator $\phi(\cdot)$, $VaR^{-\mathbf{e}}_{\alpha}(\mathbf{X})$ is given by the vector with all components equal to

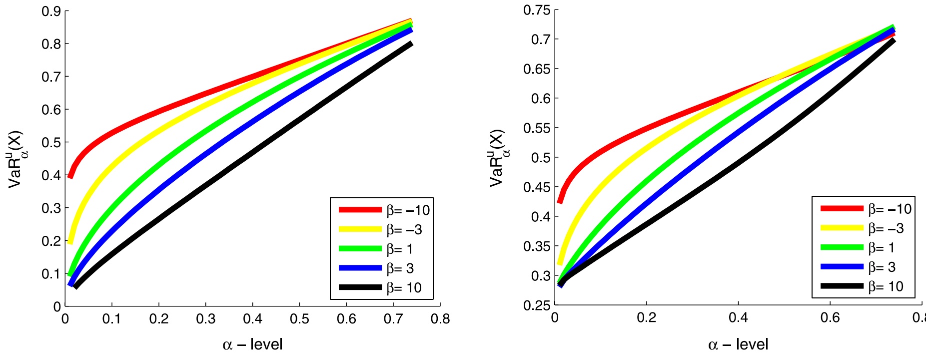
$$[VaR^{-\mathbf{e}}_{1-\alpha}(\mathbf{X})]_i = \phi^{-1}\left(\frac{\phi(1-\alpha)}{n}\right). \tag{5.6}$$

Moreover, if $\mathbf{X}$ has a survival copula $\breve{C}$ belonging to the Archimedean family with generator $\breve{\phi}(\cdot)$, the equivalent Sklar's representation gives the relation $\bar{F}_{\mathbf{X}}(x_1, \dots, x_n) = \breve{C}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n))$, where $\bar{F}$ is the join survival function and $\bar{F}_1, \dots, \bar{F}_n$ its marginal survival functions. Hence, we obtain that:

$$[VaR^{\mathbf{e}}_{\alpha}(\mathbf{X})]_i = 1 - \breve{\phi}^{-1}\left(\frac{\breve{\phi}(\alpha)}{n}\right). \tag{5.7}$$

Recall that if a vector $\mathbf{X}$ has a copula C, then the survival copula of $\mathbf{1} - \mathbf{X}$ will also be C. Therefore, if $\mathbf{X} \overset{d}{=} \mathbf{1} - \mathbf{X}$, then the copula of $\mathbf{X}$ and its survival copula are the same; for example, Frank's copula in the Archimedean family, as considered below, holds this property, as well as the elliptical family of copulas. Then, in this case the closed expression for $VaR^{\mathbf{e}}_{\alpha}(\mathbf{X})$ is the reflection point of $VaR^{-\mathbf{e}}_{1-\alpha}(\mathbf{X})$ with respect to the point $(\frac{1}{2}, \dots, \frac{1}{2})$.

Now we present some examples using some Archimedean copulas. Firstly, we use Frank's subclass to present an example of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ for any direction $\mathbf{u}$ in the bivariate case. Later we present



(a) Direction $-\mathbf{e} = -\frac{\sqrt{2}}{2}[1, 1]'$.

(b) Direction $\mathbf{u} = -\frac{3\sqrt{5}}{5}[\frac{1}{3}, \frac{2}{3}]'$.

Fig. 6. Behavior for the first component in $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ varying α.

some comparisons between our proposal and the notions reviewed in (2.8) and (2.9) but considering an n-dimensional copula belonging to Clayton's subclass. Let us define these two subclasses of copulas.

(i) **Frank copula**: The generator function of this copula is

$$\phi_{\beta}(r) = -\ln\left(\frac{e^{-\beta r} - 1}{e^{-\beta} - 1}\right) \quad \text{and} \qquad (5.8)$$
$$\phi_{\beta}^{-1}(s) = -\frac{1}{\beta}\ln(1 - (1 - e^{-\beta})e^{-s}),$$

$$C_{\beta}(v_1, v_2) = -\frac{1}{\beta}\ln\left(1 + \frac{(e^{-\beta v_1} - 1)(e^{-\beta v_2} - 1)}{e^{-\beta} - 1}\right), \qquad (5.9)$$

$$c_{\beta}(v_1, v_2) = -\frac{\beta(1 - e^{-\beta})e^{-\beta(v_1+v_2)}}{\left((e^{-\beta v_1} - 1)(e^{-\beta v_2} - 1) - (e^{-\beta} - 1)\right)^2}, \qquad (5.10)$$

where $\beta \in \mathbb{R} \setminus \{0\}$.

(ii) **Clayton copula**: This family is generated by

$$\phi_{\beta}(r) = \frac{1}{\beta}(r^{-\beta} - 1) \quad \text{and} \quad \phi_{\beta}^{-1}(s) = (1 + \beta s)^{-1/\beta}, \qquad (5.11)$$

$$C_{\beta}(v_1, v_2) = \max\left\{(v_1^{-\beta} + v_2^{-\beta} - 1)^{1/\beta}, 0\right\}, \qquad (5.12)$$

where $\beta \in [-1, 0) \cup (0, +\infty]$.

In Fig. 6 we have drawn the first component of the directional $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ for a bivariate random vector with density given by the Frank copula. The left plot is related to $\mathbf{u} = -\mathbf{e}$ and the right plot is related to $\mathbf{u} = -\frac{1}{\sqrt{5}}(1, 2)$. Both plots present the behavior for $0 \leq \alpha \leq 1$ but considering different values of the parameter β in the copula density.

In the left plot where $\mathbf{u} = -\mathbf{e}$, note that if $\beta \to \pm\infty$, we obtain the cases known as comonotonic and counter-monotonic, respectively. Also, it can be seen that in the comonotonic case, the component reaches the value given by the VaR on the marginals, which is α in this case. In addition, it is well known that rotations over random vectors do not preserve the dependence structure in the rotated distribution. This fact is captured in the right plot.

Let $\mathbf{X}$ be a random vector having as distribution function a Clayton copula. Hence, the survival function of $\mathbf{1} - \mathbf{X}$ is a Clayton survival copula. Now, we compare the first components of $VaR^{-\mathbf{e}}_{\alpha}(\mathbf{X})$ and $\underline{VaR}_{\alpha}(\mathbf{X})$ in (2.8), and the same for the first components of $VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{1} - \mathbf{X})$ and $\overline{VaR}_{\alpha}(\mathbf{1} - \mathbf{X})$ in (2.9). Table 1 contains the explicit expressions of $\underline{VaR}_{\alpha}(\mathbf{X})$ and $\overline{VaR}_{\alpha}(\mathbf{1} - \mathbf{X})$ in dimension 2, and the generalized expressions for our proposal in terms of α and β in any dimension. Fig. 7 shows the graphical comparison for $n = 2$; the left plot presents the results for $VaR^{-\mathbf{e}}_{\alpha}(\mathbf{X})$ in solid line and $\underline{VaR}_{\alpha}(\mathbf{X})$ in dashed line, while the right plot presents the results for $VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{1} - \mathbf{X})$ in solid line and $\overline{VaR}_{\alpha}(\mathbf{1} - \mathbf{X})$ in dashed line.

Table 1
Clayton's copula case.

	Directional $VaR^{\mathbf{u}}_{\alpha}(\cdot)$	Cousin and Di Bernardino (2013)'s VaR
$\mathbf{X}$	$\left(\frac{1+\alpha^{-\beta}}{n}\right)^{-\frac{1}{\beta}}$	$\frac{\beta}{\beta-1}\frac{\alpha^{\beta}-\alpha}{\alpha^{\beta}-1}$
$\mathbf{1} - \mathbf{X}$	$1 - \left(\frac{1+(1-\alpha)^{-\beta}}{n}\right)^{-\frac{1}{\beta}}$	$1 - \frac{\beta}{\beta-1}\frac{(1-\alpha)^{\beta}-(1-\alpha)}{(1-\alpha)^{\beta}-1}$

The results in Fig. 7 also show us that in the case of random vectors with Clayton copula class, $VaR^{-\mathbf{e}}_{\alpha}(\mathbf{X})$ increases with respect to the parameter α and decreases in the parameter β. On the other side, $VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{1} - \mathbf{X})$ is an increasing function of the parameter α, but also an increasing function of the dependence parameter β. These features for this class of copulas were commented and proved by Cousin and Di Bernardino (2013) and for our risk measure can be easily proved following the same scheme. In addition, we need to highlight that for each fixed pair (α, β), the following relationships hold,

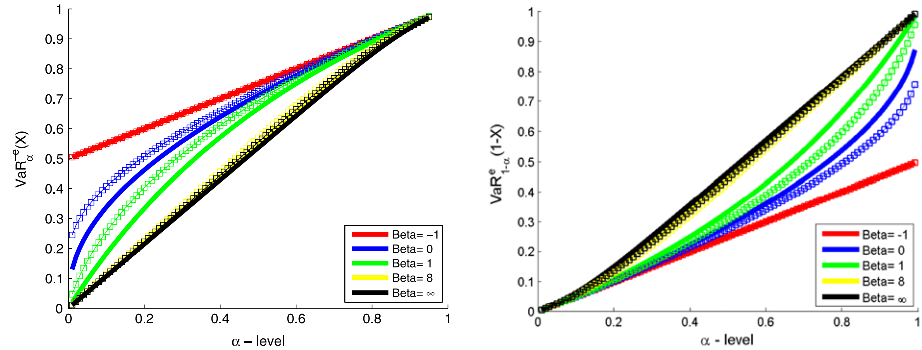
$$\underline{VaR}_{\alpha}(\mathbf{X}) \leq VaR^{-\mathbf{e}}_{\alpha}(\mathbf{X}) \quad \text{and} \qquad (5.13)$$
$$VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{1} - \mathbf{X}) \leq \overline{VaR}_{\alpha}(\mathbf{1} - \mathbf{X}),$$

where the inequalities are componentwise. Hence, we can say that our measurement is more conservative in the upper case and it is more optimistic in the lower case. This can be taken into consideration by the manager according to her/his preferences.

6. Robustness

The previous section presents the analytic results for random vectors with [0, 1]-uniform marginals distributions. However, in practical situations, it is necessary to obtain $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ for any random vector $\mathbf{X}$. In this case, we use a computational approach summarized in the following routine:

Input: $\mathbf{u}, \alpha, h$ and the multivariate sample $\mathbf{X}_m$.
 for $i = 1$ to m
  $P_i = \mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}_i}\right]$,
  If $|P_i - \alpha| \leq h$
   $\mathbf{x}_i \in \hat{\mathcal{Q}}^{h}_{\mathbf{X}_m}(\alpha, \mathbf{u})$,
  end
  for $\mathbf{x}_j \in \hat{\mathcal{Q}}^{h}_{\mathbf{X}_m}(\alpha, \mathbf{u})$
   $d_j = dist(\mathbf{x}_j, \{\boldsymbol{\mu}_{\mathbf{X}_m} + \lambda\mathbf{u}\})$,
  end
 end
 $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}_m) = \{\mathbf{x}_k | d_k = \min\{d_j\}\}$,



(a) Lower case. (b) Upper case.

Fig. 7. Comparison for Clayton's family of copulas.

where $\mathbf{X}_m := \{\mathbf{x_1}, \ldots, \mathbf{x_m}\}$ is the sample of the random vector $\mathbf{X}$, $\boldsymbol{\mu}_{\mathbf{X}_m}$ the sample mean, $\hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u}) := \left\{\mathbf{x_j} : |\mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_j}}\right] - \alpha| \le h\right\}$ the sample quantile hyper-surface with a slack h and $\mathbb{P}_{\mathbf{X}_m}[\cdot]$ is the empirical probability distribution of $\mathbf{X}_m$. Using this procedure, we are able to deal with high dimensional random vectors. We are aware that this procedure can be improved using more sophisticated tools of the non-parametric statistics, but they are outside the scope of this paper.

On the other hand, it is well known that in risk theory, it is desirable that a measure be robust, (see Artzner et al., 1999; Burgert and Ruschendorf, 2006; Cardin and Pagani, 2010; Rachev et al., 2008). But in general, most of the measures are sensitive to extreme outlying observations. In this section, we present a simulation study to show the sensitivity of our *upper VaR in direction* $\mathbf{e}$, using as a benchmark the *upper VaR* in (2.9) introduced in Cousin and Di Bernardino (2013). Setting $\alpha = 90\%$, we compare $VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{X})$ and $\overline{VaR}_\alpha(\mathbf{X})$ in terms of robustness using the following contamination model:

$$\mathbf{X}^\omega \overset{d}{=} \begin{cases} \mathbf{X}_1 & \text{with probability } p = 1-\omega, \\ \mathbf{X}_2 & \text{with probability } p = \omega, \end{cases} \tag{6.1}$$

where $\mathbf{X}_1 \overset{d}{=} N_1(\boldsymbol{\mu}_1, \Sigma_1)$, $\mathbf{X}_2 \overset{d}{=} N_2(\boldsymbol{\mu}_1 + \Delta_{\boldsymbol{\mu}}, \Sigma_1 + \Delta_{\boldsymbol{\Sigma}})$ and $0 \le \omega \le 1$. The parameters of $\mathbf{X}_1$ are,

$$\boldsymbol{\mu}_1 = [50, 50]', \qquad \Sigma_1 = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}.$$

$\mathbf{X}_1$ remains fixed in the analysis, but the parameters of the normal distribution of $\mathbf{X}_2$ are changed in various ways to generate outliers. As a measure to quantify the effect of the outliers, we define the percentage variation,

$$PV^\omega = \frac{\|Measure(\mathbf{X}^\omega) - Measure(\mathbf{X}^0)\|_2}{\|Measure(\mathbf{X}^0)\|},$$

where $Measure(\mathbf{X}^0)$ is the risk measure evaluated in the sample without contamination, $\omega = 0$, and $Measure(\mathbf{X}^\omega)$ is a risk measure evaluated for the sample with a level of contamination $\omega\%$. We have considered the scenarios for $\mathbf{X}_2$, described in Table 2.

The procedure is the following: firstly, we have generated a non-contaminated sample $\mathbf{X}^0$ with 5000 observations and we calculate both $VaR^{\mathbf{e}}_{0.1}(\mathbf{X})$ and $\overline{VaR}_{0.1}(\mathbf{X})$.

Secondly, we have used the contamination model (6.1) taking values for ω from 1% to 10%. Then, we have generated for each ω, 5000 observations of $\mathbf{X}_1$ containing an expected value of outliers $\omega\%$. We have evaluated the risk measure as well as the percentage variation for each level of contamination, performing this procedure 100 times and we have reported the average of PV^ω in the following plots.

Table 2
Simulation stages and parameters.

Scenarios	Parameters of $\mathbf{X}_2$ distribution
Variance analysis	$\boldsymbol{\mu}_1,\ \Sigma_1 + \begin{bmatrix} 4.5 & 0 \\ 0 & 6.5 \end{bmatrix}$
Covariance matrix analysis	$\boldsymbol{\mu}_1,\ \Sigma_1 + \begin{bmatrix} 4.5 & 0.2 \\ 0.3 & 6.5 \end{bmatrix}$
Mean analysis	$\boldsymbol{\mu}_1 + \Delta_{\boldsymbol{\mu}},\ \Sigma_1$
Join analysis	$\boldsymbol{\mu}_1 + \Delta_{\boldsymbol{\mu}},\ \Sigma_1 + \begin{bmatrix} 4.5 & 0.2 \\ 0.3 & 6.5 \end{bmatrix}$

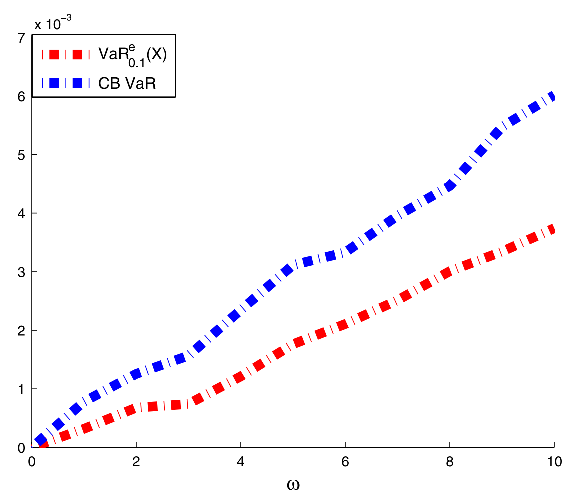


Fig. 8. Percentage variation of the measures varying the variances.

The first scenario suggests outliers result from changes on the variance of the marginals, which are difficult to detect in practice. We can see in Fig. 8 that the behavior of $VaR^{\mathbf{e}}_{0.1}(\mathbf{X})$ is better than that corresponding to *upper-VaR* in Cousin and Di Bernardino (2013) for any level of contamination. "better", in this context, means that PV^ω is smaller.

The second scenario considers changes in all the components of the covariance matrix. The results are depicted in Fig. 9, which shows again the better behavior of $VaR^{\mathbf{e}}_{0.1}(\mathbf{X})$ with respect to robustness.

The last scenario consists of changes in the mean. Firstly, we vary the first component of the mean and then we affected the second one and finally both of them simultaneously. Fig. 10 summarizes the results. As we can see, $VaR^{\mathbf{e}}_{0.1}(\mathbf{X})$ shows robustness in presence of a high percentage of outliers, but an extra-sensitivity under outliers in a unique component (outliers of shape type). The use of the mean of the random loss as the central point in the definition of our VaR could be the cause of this sensitivity.

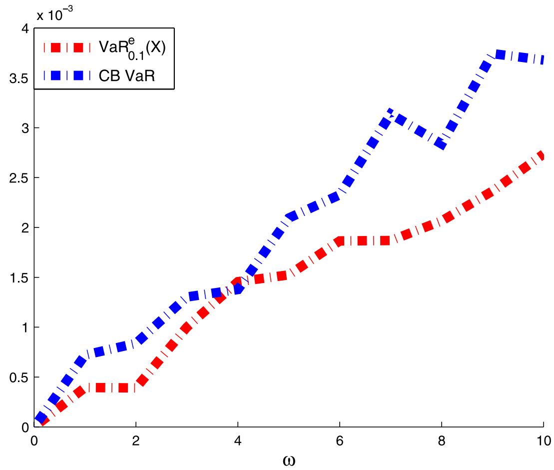


Fig. 9. Percentage variation of the measures varying the covariance matrix.

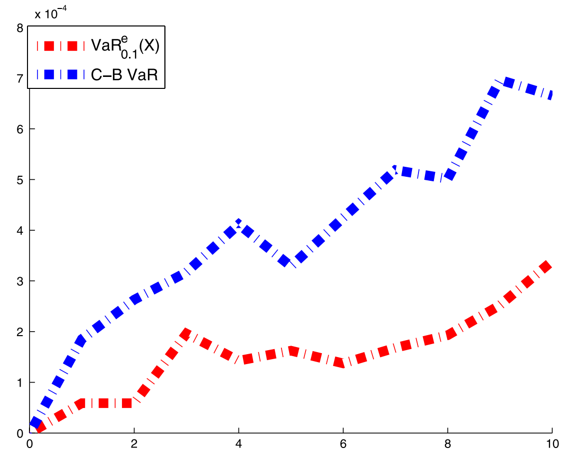


Fig. 11. Percentage variation for a 3D-contamination model.

To evaluate the impact of the dimension in the robustness analysis, we have carried out simulations with normal random vectors in high dimensions obtaining similar conclusions to the previous one. For instance, Fig. 11 displays the results of the percentage variation when we consider $n = 3$, and the covariance matrix is modified while the mean remains fixed.

Obviously, the robustness study can be extended by varying other aspects such as type of distributions, or changes in the "*level of risk*" given by the parameter α.

7. Conclusions

In this paper we have defined a multivariate extension of the classical risk measure VaR based on a directional multivariate quantile recently introduced in the literature. Specifically, we have proposed the *directional multivariate Value at Risk* ($VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$) as a tool to analyze a portfolio of n heterogeneous and dependent risks considering external information or manager preferences.

We have given analytic properties of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ in the same way as Artzner et al. (1999)'s axiomatic. We have provided some invariance properties as well as consistency and tail subadditivity properties, which are desirable in a risk measure. We have demonstrated relations between the components of the output of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ with respect to the corresponding univariate VaR over the marginals. A interesting link between the univariate VaR over the linear transformation using the portfolio weights vector $\mathbf{w}$, and the value of this transformation over $VaR_{\alpha}^{-\frac{\mathbf{w}}{\|\mathbf{w}\|}}(\mathbf{X})$ is provided. We have also presented closed forms for $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ in terms of some families of copulas, considering particular dimensions or particular directions.

Finally we have presented a simulation study of robustness comparing the behavior of $VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})$ with respect to the risk measure proposed in Cousin and Di Bernardino (2013). The simulations show the advantages of our proposal in relation to the presence of outliers.

We have also detected in this study an open question to be taken into consideration in future work. The idea is to consider another central point instead of the mean as the center of the reference system, in order to improve the robustness of the risk measure, but, at the same time, preserving the desirable properties demonstrated. A possibility that could bring us more robustness is to use a multivariate depth measure.

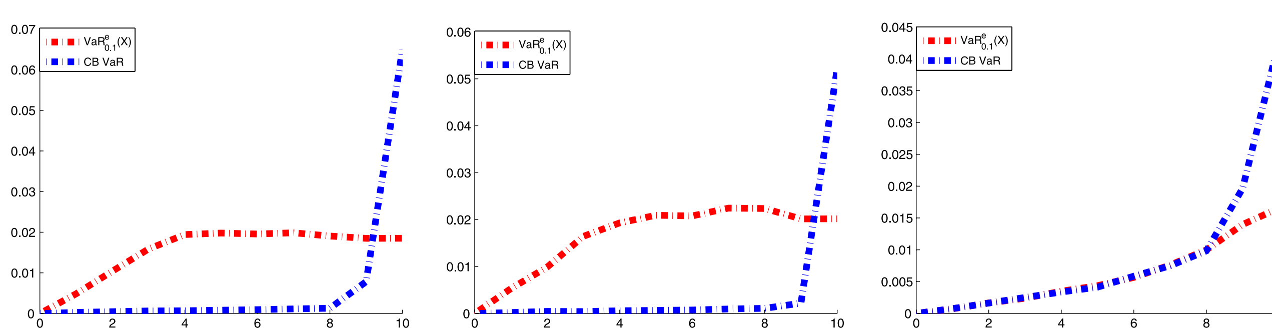
Acknowledgments

The authors would like to thank two referees for their very helpful comments. This research was partially supported by a Spanish Ministry of Economy and Competition grant ECO2012-38442. The third author wishes to thank to El Departamento Administrativo de Ciencia y Tecnología, COLCIENCIAS, for the financial support under the Convocatoria 656, 2014, Es Tiempo de Volver as well as to express his thanks to the hospitality of El Departamento de Ingeniería de la Organización, Universidad Nacional de Colombia sede Medellín and to the research group, Modelamiento y Análisis Energa Ambiente Economía (MAEAE) led by Professor Sergio Botero-Botero.

Appendix

Proof of Lemma 2.3. Without loss of generality, we may assume that $\mathbf{u} > \mathbf{0}$. From (2.2), we have that:

$$M_{\mathbf{u}} = [\mathbf{u}\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n] \quad \text{and} \quad M_{-\mathbf{u}} = [-\mathbf{u}\ \ -\mathbf{e}_2\ \cdots\ \ -\mathbf{e}_n] = -M_{\mathbf{u}}.$$



(a) $\Delta_{\mu} = [0, 25]'$. (b) $\Delta_{\mu} = [25, 0]'$. (c) $\Delta_{\mu} = [25, 25]'$.

Fig. 10. Percentage variation of the measures varying the mean.

Then, under the constraint of positive diagonal elements in the corresponding triangular matrices in the QR decompositions, we have that:

$$M_{\mathbf{u}} = Q_{\mathbf{u}} T_{\mathbf{u}}, \qquad M_{-\mathbf{u}} = -Q_{\mathbf{u}} T_{\mathbf{u}}.$$

Thus $R_{-\mathbf{u}} = -R_{\mathbf{u}}$, which implies,

$$\begin{aligned}\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} &= \{\mathbf{z} \in \mathbb{R}^n : R_{\mathbf{u}}(\mathbf{z}-\mathbf{x}) \geq 0\} \\ &= \{\mathbf{z} \in \mathbb{R}^n : R_{-\mathbf{u}}(-\mathbf{z}-(-\mathbf{x})) \geq 0\} \\ &= -\mathfrak{C}_{-\mathbf{x}}^{-\mathbf{u}}. \quad \square\end{aligned}$$

Proof of Lemma 2.4. The proof is straightforward using the definitions given in (2.4). □

Proof of Property 3.3. The proof is straightforward using the hypothesis of $\alpha > 0$ small, because this implies that $\lambda > 0$ in definition (3.1) and hence the result. □

Proof of Property 3.4. Due to Lemma 2.3, it is easy to prove that

$$\mathcal{Q}_{-\mathbf{X}}(\alpha, \mathbf{u}) = -\mathcal{Q}_{\mathbf{X}}(\alpha, -\mathbf{u}), \tag{A.1}$$

and hence,

$$\begin{aligned}&\mathcal{Q}_{-\mathbf{X}}(\alpha, \mathbf{u}) \bigcap \{\lambda \mathbf{u} + \mathbb{E}[-\mathbf{X}]\} \\ &\equiv (-\mathcal{Q}_{\mathbf{X}}(\alpha, -\mathbf{u})) \bigcap (-\{\lambda(-\mathbf{u}) + \mathbb{E}[\mathbf{X}]\}) \\ &\equiv -\left(\mathcal{Q}_{\mathbf{X}}(\alpha, -\mathbf{u}) \bigcap \{\lambda(-\mathbf{u}) + \mathbb{E}[\mathbf{X}]\}\right).\end{aligned}$$

Then,

$$VaR_{\alpha}^{\mathbf{u}}(-\mathbf{X}) = -VaR_{\alpha}^{-\mathbf{u}}(\mathbf{X}). \quad \square$$

Proof of Property 3.5. This property is derived using Lemma 2.4. □

Proof of Property 3.6. Since $\mathbf{X} \leq_{\mathcal{E}_u} \mathbf{Y} \Leftrightarrow R_{\mathbf{u}}\mathbf{X} \leq_{uo} R_{\mathbf{u}}\mathbf{Y}$, we get:

$$\begin{aligned}L_{\mathbf{X}}(\alpha, \mathbf{u}) &:= \{\mathbf{z} \in \mathbb{R}^n : \mathbb{P}_{\mathbf{X}}(\mathfrak{C}_{\mathbf{z}}^{\mathbf{u}}) \leq \alpha\} \\ &\supseteq \{\mathbf{z} \in \mathbb{R}^n : \mathbb{P}_{\mathbf{Y}}(\mathfrak{C}_{\mathbf{z}}^{\mathbf{u}}) \leq \alpha\} := L_{\mathbf{Y}}(\alpha, \mathbf{u}).\end{aligned}$$

Besides, $VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) = \partial L_{\mathbf{X}}(\alpha, \mathbf{u}) \bigcap \{\lambda \mathbf{u} + \mathbb{E}[\mathbf{X}]\}$ and $VaR_{\alpha}^{\mathbf{u}}(\mathbf{Y}) = \partial L_{\mathbf{Y}}(\alpha, \mathbf{u}) \bigcap \{\lambda \mathbf{u} + \mathbb{E}[\mathbf{Y}]\}$. Therefore, using the partial order defined in (2.3) there are three possibilities for $\mathbf{s}, \mathbf{t} \in \mathbb{R}^n$:

(i) $\mathbf{s} \succ_{\mathbf{u}} \mathbf{t}$,
(ii) $\mathbf{s} \npreceq_{\mathbf{u}} \mathbf{t}$ and $\mathbf{t} \npreceq_{\mathbf{u}} \mathbf{s}$,
(iii) $\mathbf{s} \preceq_{\mathbf{u}} \mathbf{t}$.

We can prove that the two first options are not possible for the points $VaR_{\alpha}^{\mathbf{u}}(\mathbf{X})$ and $VaR_{\alpha}^{\mathbf{u}}(\mathbf{Y})$. Suppose to the contrary that

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) \succ_{\mathbf{u}} VaR_{\alpha}^{\mathbf{u}}(\mathbf{Y}),$$

which implies that,

$$\mathfrak{C}_{\mathbf{z_X}}^{\mathbf{u}} \subset \mathfrak{C}_{\mathbf{z_Y}}^{\mathbf{u}}.$$

Hence,

$$\mathbb{P}_{\mathbf{Y}}(\mathfrak{C}_{\mathbf{z_Y}}^{\mathbf{u}}) \geq \mathbb{P}_{\mathbf{X}}(\mathfrak{C}_{\mathbf{z_Y}}^{\mathbf{u}}) > \mathbb{P}_{\mathbf{X}}(\mathfrak{C}_{\mathbf{z_X}}^{\mathbf{u}}) = \alpha.$$

Which is a contradiction, if we assume the *regularity conditions*. Moreover, the hypothesis $\mathbb{E}[\mathbf{Y}] = c\mathbf{u} + \mathbb{E}[\mathbf{X}]$, for all $c > 0$ and the result $\mathbb{E}[\mathbf{X}] \preceq_{\mathbf{u}} \mathbb{E}[\mathbf{Y}]$ derived in Laniado et al. (2012, Property 3.4), permit us to reject the second possibility of ordering between the two points. Thus, the only option possible is,

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) \preceq_{u} VaR_{\alpha}^{\mathbf{u}}(\mathbf{Y}). \quad \square$$

Proof of Property 3.8. First, note that:

$$\{\lambda(Q\mathbf{u}) + \mathbb{E}[Q\mathbf{X}]\} = Q\{\lambda \mathbf{u} + \mathbb{E}[\mathbf{X}]\}.$$

In addition, given the QR rotation of $\mathbf{u}$ over $\mathbf{w}$, we have $Q = Q_{\mathbf{w}} Q_{\mathbf{u}}'$. Then,

$$\begin{aligned}R_{\mathbf{u}} &= Q_{\mathbf{e}} Q_{\mathbf{w}}' Q_{\mathbf{w}} Q_{\mathbf{u}}' \\ &= R_{\mathbf{w}} Q = R_{Q\mathbf{u}} Q.\end{aligned}$$

Therefore, $\mathfrak{C}_{Q\mathbf{x}}^{Q\mathbf{u}} = Q\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}}$, and $\mathbb{P}_{Q\mathbf{X}}(\mathfrak{C}_{Q\mathbf{x}}^{Q\mathbf{u}}) = \mathbb{P}_{\mathbf{X}}(\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}})$. Thus, we get

$$\mathcal{Q}_{Q\mathbf{X}}(\alpha, Q\mathbf{u}) = Q\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}), \tag{A.2}$$

which proves the result. □

Proof of Property 3.9. Property 3.8 implies that,

$$R_{\mathbf{u}} VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) = VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X}),$$

where $\mathbf{e} = \frac{\sqrt{n}}{n}[1, \dots, 1]'$. Then,

$$R_{\mathbf{u}} VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) \leq \sup_{\omega \in \Omega}\{R_{\mathbf{u}}\mathbf{X}(\omega)\},$$

and the proof is complete. □

Proof of Property 3.11. It is easy to see that the equality in the mean implies that the vectors $\mathbb{E}[R_{\mathbf{u}}\mathbf{X}]$, $\mathbb{E}[R_{\mathbf{u}}\mathbf{Y}]$ and $\mathbb{E}[R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y})]$ lie on the same line, the line with direction vector $\mathbf{e}$. Then, we can write:

$$\begin{aligned}\mathfrak{C}_{VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})}^{\mathbf{e}} &= n[VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})]_1 \mathfrak{C}_{\mathbf{w}}^{\mathbf{e}} \\ &= n[VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})]_1 [\mathbf{w}, \infty)^n,\end{aligned} \tag{A.3}$$

where $\mathbf{w}$ is the vector whose components are the value $\frac{1}{n}$ and $[\cdot]_1$ denotes the first component of the vector. Following a similar approach as in the proof of tail subadditivity of Daníelsson et al. (2013) for the univariate case, we develop a multivariate version. Then, for $\alpha > 0$ small, $\frac{1}{\alpha} \to \infty$, and then,

$$\frac{1}{\alpha}\mathbb{P}\left[(R_{\mathbf{u}}\mathbf{X}, R_{\mathbf{u}}\mathbf{Y}) \in \phi\left(\frac{1}{\alpha}\right) B\right] \to \mu(B).$$

On the other hand, the Borel set $\left(\phi\left(\frac{1}{\alpha}\right)\right)^{-1} \mathfrak{C}_{VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})}^{\mathbf{u}} \times (0, \infty)^n$ satisfies the following property:

$$\frac{1}{\alpha}\mathbb{P}\left[(R_{\mathbf{u}}\mathbf{X}, R_{\mathbf{u}}\mathbf{Y}) \in \left(\phi\left(\frac{1}{\alpha}\right)\right)\left(\phi\left(\frac{1}{\alpha}\right)\right)^{-1} \times \left(\mathfrak{C}_{VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})}^{\mathbf{e}} \times (0, \infty)^n\right)\right] \to 1.$$

Or equivalently,

$$\mu\left\{\left(\phi\left(\frac{1}{\alpha}\right)\right)^{-1}\left(\mathfrak{C}_{VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})}^{\mathbf{u}} \times (0, \infty)^n\right)\right\} \sim 1.$$

Hence using (A.3), we have:

$$[VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{X})]_1 \sim \left(\mu\left\{\left[\frac{1}{n}, \infty\right)^n \times (0, \infty)^n\right\}\right)^{\frac{1}{\beta}} \phi\left(\frac{1}{\alpha}\right) n. \tag{A.4}$$

In the same way,

$$[VaR_{\alpha}^{\mathbf{e}}(R_{\mathbf{u}}\mathbf{Y})]_1 \sim \left(\mu\left\{(0, \infty)^n \times \left[\frac{1}{n}, \infty\right)^n\right\}\right)^{\frac{1}{\beta}} \phi\left(\frac{1}{\alpha}\right) n. \tag{A.5}$$

Now, in the case of the random variable $R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y})$, we have;

$$\begin{aligned}
&\mathfrak{C}^{\mathbf{e}}_{VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))}\\
&= \{(\mathbf{x},\mathbf{y}) \in (0,\infty)^{2n} : (\mathbf{x}+\mathbf{y}) > VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))\}\\
&= n[VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))]_1 \cdot \{(\mathbf{x},\mathbf{y}) \in (0,\infty)^{2n} : (\mathbf{x}+\mathbf{y}) > \mathbf{w}\}\\
&= n[VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))]_1 \cdot \mathfrak{C}^{\mathbf{e}}_{\mathbf{w}} \qquad (A.6)
\end{aligned}$$

where the inequalities in the expression hold componentwise. As a consequence we get,

$$\mu\left\{\left(\phi\left(\frac{1}{\alpha}\right)\right)^{-1}\{(\mathbf{x},\mathbf{y}) \in (0,\infty)^{2n} : (\mathbf{x}+\mathbf{y}) > VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))\}\right\} \sim 1.$$

Then using the last equality in (A.6), we finally get,

$$[VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))]_1 \sim \left(\mu\left\{\{(\mathbf{x},\mathbf{y}) \in (0,\infty)^{2n} : (\mathbf{x}+\mathbf{y}) > \mathbf{w}\}\right\}\right)^{\frac{1}{\beta}} \times \phi\left(\frac{1}{\alpha}\right) n. \qquad (A.7)$$

It is well known that in $\mathbb{R}^n$ all the norms are equivalent, i.e., for two norms $\|\cdot\|$ and $\|\cdot\|^*$, there are positive constants c_1, c_2 such that $c_1\|\cdot\| \le \|\cdot\|^* \le c_2\|\cdot\|$. Then, whatever norm is taken, we use the transformation (Resnick, 1987, pg. 267), $\mathbf{x} \to (\|\mathbf{x}\|, \|\mathbf{x}\|^{-1}\mathbf{x})$ and rewrite $\mu(\cdot)$ in terms of a new measure $\eta(\cdot)$ in $\mathcal{D} := \{\mathbf{z} \in [0,\infty]^{2n}\backslash\{\mathbf{0}\} : \|\mathbf{z}\| = 1\}$ as $r^{-\beta}\eta(\cdot)$, due to the property of the measure in (3.13). The relationship satisfying both measures for a Borel set A in $\mathcal{D}$, it is given by,

$$\mu(A) = \int_{\mathcal{D}}\int_0^{\infty} \mathbf{1}(r(\mathbf{u},\mathbf{v}) \in A)\beta r^{-(1+\beta)} dr \eta(d\mathbf{u}, d\mathbf{v}). \qquad (A.8)$$

Then the measure of the Borel sets in (A.4), (A.5) and (A.7) can be expressed using $\|\cdot\|_1$ as:

$$\mu\left(\left(\frac{1}{n},\infty\right)^n \times (0,\infty)^n\right) = \int_{\mathcal{D}}\left(\sum_i u_i\right)^{\beta} \eta(d\mathbf{u}, d\mathbf{v}), \qquad (A.9)$$

$$\mu\left((0,\infty)^n \times \left(\frac{1}{n},\infty\right)^n\right) = \int_{\mathcal{D}}\left(\sum_i v_i\right)^{\beta} \eta(d\mathbf{u}, d\mathbf{v}), \qquad (A.10)$$

$$\mu(\{(\mathbf{x},\mathbf{y}) \in (0,\infty)^{2n} : (\mathbf{x}+\mathbf{y}) > \mathbf{w}\}) = \int_{\mathcal{D}}\left(\sum_i (u_i+v_i)\right)^{\beta} \eta(d\mathbf{u}, d\mathbf{v}). \qquad (A.11)$$

Now using the Minkowski inequality we obtain:

$$\left(\int_{\mathcal{D}}\left(\sum_i (u_i+v_i)\right)^{\beta} \eta(d\mathbf{u}, d\mathbf{v})\right)^{\frac{1}{\beta}} \le \left(\int_{\mathcal{D}}\left(\sum_i u_i\right)^{\beta} \eta(d\mathbf{u}, d\mathbf{v})\right)^{\frac{1}{\beta}} + \left(\int_{\mathcal{D}}\left(\sum_i v_i\right)^{\beta} \eta(d\mathbf{u}, d\mathbf{v})\right)^{\frac{1}{\beta}}. \qquad (A.12)$$

Hence combining (A.4), (A.5), (A.7) and (A.12), we have the result

$$[VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}(\mathbf{X}+\mathbf{Y}))]_1 \le [VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}\mathbf{X})]_1 + [VaR^{\mathbf{e}}_{\alpha}(R_{\mathbf{u}}\mathbf{Y})]_1,$$

or equivalently, from Property 3.8 and the partial order defined in (2.3), we have for $\mathbf{u} = \frac{\mathbf{m}}{\|\mathbf{m}\|}$ that:

$$VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}+\mathbf{Y}) \preceq_{\mathbf{u}} VaR^{\mathbf{u}}_{\alpha}(\mathbf{X}) + VaR^{\mathbf{u}}_{\alpha}(\mathbf{Y}). \quad \square \qquad (A.13)$$

Proof of Proposition 4.4. By Definition 2.7, if $\mathbf{x} \in \mathcal{Q}_{\mathbf{X}}(\alpha,\mathbf{u})$, we have $\mathbb{P}[R_{\mathbf{u}}(\mathbf{X}-\mathbf{x}) \ge 0] = \alpha$. Therefore,

$$\mathbb{P}[\mathbf{1}'R_{\mathbf{u}}(\mathbf{X}-\mathbf{x}) \ge 0] \ge \alpha \quad \text{where } \mathbf{1} = [1,\ldots,1]'. \qquad (A.14)$$

Since $R_{\mathbf{u}}\mathbf{u} = \mathbf{e}$, we obtain,

$$\begin{aligned}
\mathbb{P}[\mathbf{1}'R_{\mathbf{u}}(\mathbf{X}-\mathbf{x}) \ge 0] &= \mathbb{P}[\sqrt{n}(R_{\mathbf{u}}\mathbf{u})'R_{\mathbf{u}}(\mathbf{X}-\mathbf{x}) \ge 0]\\
&= \mathbb{P}\left[\sqrt{n}\left(-\frac{\mathbf{w}}{\|\mathbf{w}\|}\right)'(\mathbf{X}-\mathbf{x}) \ge 0\right]\\
&= \mathbb{P}[\mathbf{w}'\mathbf{X} \le \mathbf{w}'\mathbf{x}] = \mathbb{P}[Z \le \mathbf{w}'\mathbf{x}].
\end{aligned}$$

Thus, (A.14) and (3.2) imply $\mathbf{w}'\mathbf{x} \ge VaR_{\alpha}(Z)$. $\square$

Proof of Proposition 4.2. The proof follows the same outline as that of Cousin and Di Bernardino (2013, Proposition 2.4). Note that in direction $\mathbf{u} = \mathbf{e}$,

$$\mathfrak{C}^{\mathbf{e}}_{\mathbf{x}} = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \ge \mathbf{x}\}.$$

Then we can write,

$$\begin{aligned}
L_{\alpha} &= \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}(\mathfrak{C}^{\mathbf{e}}_{\mathbf{x}}) \le \alpha\}\\
&= \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}(\mathbf{X} \ge \mathbf{x}) \le \alpha\}.
\end{aligned}$$

The convexity of $[L_{\alpha}]^c$ follows from the quasi-concavity of the survival function $\bar{F}$, where $[\cdot]^c$ denotes the complement of a set. Now, as $\mathcal{Q}_{\mathbf{X}}(\alpha,\mathbf{e}) = \partial L_{\alpha} \equiv \partial[L_{\alpha}]^c$, $VaR^{\mathbf{e}}_{\alpha}(\mathbf{X})$ belongs to the set $\partial[L_{\alpha}]^c$. Moreover, from the definition of survival function we have that,

$$\bar{F}(\infty,\ldots,x_i,\ldots,\infty) \ge \bar{F}(\mathbf{x}) = \bar{F}(x_1,\ldots,x_i,\ldots,x_n)$$

for all $\mathbf{x} \in \mathbf{R}^n$ and $i = 1,\ldots,n$.

Then each component of a vector belonging to $\partial[L_{\alpha}(\mathbf{e})]^c$ is upper bounded by the univariate VaR at level $p = 1-\alpha$ of the corresponding marginal. As a consequence, each component of $VaR^{\mathbf{e}}_{\alpha}(\mathbf{X})$ is upper bounded by the univariate VaR at level $p = 1-\alpha$ of the corresponding marginal and hence, the first inequality holds. Now for the second inequality,

$$\mathfrak{C}^{-\mathbf{e}}_{\mathbf{x}} = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \le \mathbf{x}\}.$$

Then, we have,

$$\begin{aligned}
L_{1-\alpha} &= \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}(\mathfrak{C}^{-\mathbf{e}}_{\mathbf{x}}) \le 1-\alpha\}\\
&= \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}(\mathbf{X} \le \mathbf{x}) \le 1-\alpha\}.
\end{aligned}$$

But, if F is a quasi-concave function, we have that $[L_{1-\alpha}]^c$ is a convex set and $\mathcal{Q}_{\mathbf{X}}(1-\alpha,-\mathbf{e}) = \partial L_{1-\alpha} \equiv \partial[L_{1-\alpha}]^c$. Therefore $VaR^{\mathbf{e}}_{1-\alpha}(\mathbf{X})$ belongs to the set $[L_{1-\alpha}]^c$. Additionally, from the definition of distribution function, it is easy to show that each component of an element in $[L_{1-\alpha}]^c$ is lower bounded by the univariate VaR at level $p = 1-\alpha$ of the corresponding marginal; hence, we obtain the result to be proved. $\square$

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