J. Math. Anal. Appl. 364 (2010) 473–482

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some embedding theorems for Hörmander–Beurling spaces

Joaquín Motos [a,*,1], María Jesús Planells [a], Jairo Villegas G [b]

[a] Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Valencia, Spain
[b] Departamento de Ciencias Básicas, Universidad Eafit, Medellín, Colombia

ARTICLE INFO

Article history:
Received 27 March 2009
Available online 27 November 2009
Submitted by J. Xiao

Dedicated to Professor Manuel Valdivia on the occasion of his 80th birthday

Keywords:
Beurling ultradistributions
Hörmander spaces
Hörmander–Beurling spaces
ω-Hypoelliptic differential operators

ABSTRACT

In this paper we prove a number of results on sequence space representations and embedding theorems of Hörmander–Beurling spaces. As a consequence and using sharp results of Meise, Taylor and Vogt, a result of Kaballo on short sequences and hypoelliptic operators is extended to ω-hypoelliptic differential operators and to the vector-valued setting.

1. Introduction and notations

It is well known that the Hörmander spaces $\mathcal{B}_{p,k}$, $\mathcal{B}^{\text{loc}}_{p,k}(\Omega)$ and $\mathcal{B}^{c}_{p,k}(\Omega)$ play a crucial role in the theory of linear partial differential operators (see [2,15,16]). Our research pursues the study on Hörmander spaces and Hörmander spaces in the sense of Beurling and Björck [2] (=Hörmander–Beurling spaces) carried out in [2,8,14–16,19,40,45,5,29–31,36,37,44] (see also [18]). In this paper we prove a number of results on sequence space representations and embedding theorems of Hörmander–Beurling spaces (extending corresponding results of [29–31]) and as a consequence, and using results of Meise, Taylor and Vogt [24], a result of Kaballo [19] on short sequences and hypoelliptic differential operators is extended to ω-hypoelliptic differential operators and to the vector-valued setting.

The organization of the paper is as follows. Section 2 contains some basic facts about scalar and vector-valued Beurling ultradistributions and the definitions of the spaces which are considered in the paper. In Section 3 we generalize to UMD spaces Theorem 4.6 of [31], we prove an embedding (and sequence spaces representation) theorem for vector-valued Hörmander–Beurling spaces, we give a result of Rosenthal type [38] (every weakly compact subset of $\mathcal{B}^{\text{loc}}_{\infty,k}(\Omega, E)$ is separable when E is a closed subspace of $l_{\infty}^{\mathbb{N}}$) (see Remark 3.1.1), we prove an embedding theorem when E is non-separable Fréchet space and we pose the following question: Is $\mathcal{B}^{\text{loc}}_{\infty,k}(\Omega, l_{\infty})$ isomorphic to a complemented subspace of $l_{\infty}^{\mathbb{N}}$? (See Remark 3.1.3.) In Section 4 we show that, in general, the topology induced by $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ on $\mathcal{B}^{\text{loc}}_{p,k}(\Omega) \otimes E$ is strictly finer than the ε topology and strictly coarser than the π topology (our example extends to $1 < p < \infty$, by using a different technique, the example studied in [31, Remark 4.7.2]) and we pose another question: Are the spaces $\mathcal{B}^{\text{loc}}_{\infty,k}(\Omega, l_{\infty})$ and $\mathcal{B}^{\text{loc}}_{\infty,k}(\Omega) \hat{\otimes}_{\varepsilon} l_{\infty}$ isomorphic? We also give a sequence space representation theorem when E is a nuclear Fréchet space (for example it is

* Corresponding author.
E-mail addresses: jmotos@mat.upv.es (J. Motos), mjplanells@mat.upv.es (M.J. Planells), javille@eafit.edu.co (J. Villegas G).

[1] The author is partially supported by Project MTM2008-04594, Spain.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.11.030

shown that if $E \simeq s$ or $s^{\mathbb{N}}$ then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is isomorphic to $(\mathcal{D}_{L^p})^{\mathbb{N}}$). Then, using results of Meise, Taylor and Vogt [24], we extend a result of Kaballo [19] to ω-hypoelliptic differential operators.

Notations. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $\mathcal{L}_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The (topological) dual of E is denoted by E' and is given the strong topology so that $E' = \mathcal{L}_b(E, \mathbb{C})$. $E \hat{\otimes}_\varepsilon F$ (resp. $E \hat{\otimes}_\pi F$) is the completion of the injective (resp. projective) tensor product of E and F. If E and F are (topologically) isomorphic we put $E \simeq F$. If E is isomorphic to a subspace (resp. complemented subspace) of F we write $E \subset F$ (resp. $E < F$). We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous (we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F). If $(E_n)_{n=1}^\infty$ is a sequence of locally convex spaces, $\prod_{n=1}^\infty E_n$ ($E^{\mathbb{N}}$ if $E_n = E$ for all n) is the topological product of the spaces E_n; $\bigoplus_{n=1}^\infty E_n$ ($E^{(N)}$ if $E_n = E$ for all n) is the locally convex direct sum of the spaces E_n. The Fréchet space defined by the projective sequence of Fréchet spaces E_n and linking maps A_n will be denoted by $\operatorname{proj}(E_n, A_n)$ (or $\operatorname{proj} E_n$, for short). This projective limit is said to be reduced if $\overline{\operatorname{Im} P_j} = E_j$ for $j = 1, 2, \ldots,$ being $P_j : \operatorname{proj}(E_n, A_n) \to E_j : (e_n)_1^\infty \to e_j$. If the E_n are Banach spaces and the maps A_n are surjective then $\operatorname{proj}(E_n, A_n)$ is said to be a quojection (see e.g. [28]).

Let $1 \leqslant p \leqslant \infty$, $k : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function, and E be a Fréchet space. Then $L_p(E)$ is the set of all (equivalence classes of) Bochner measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = (\int_{\mathbb{R}^n} \|f(x)\|^p \, dx)^{1/p}$ is finite (with the usual modification when $p = \infty$) for all $\|\cdot\| \in \operatorname{cs}(E)$ (see, e.g. [10]). $L_{p,k}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for all $f \in L_{p,k}(E)$ and for all $\|\cdot\| \in \operatorname{cs}(E)$, $L_{p,k}(E)$ becomes a Fréchet space isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x} \, dx$. If f is a function on $\mathbb{R}^n$ then $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$.

Finally we recall the definition of A_p^* functions. A positive, locally integrable function ω on $\mathbb{R}^n$ is in A_p^* provided, for $1 < p < \infty$,

$$\sup_R \left(\frac{1}{|R|} \int_R \omega \, dx \right) \left(\frac{1}{|R|} \int_R \omega^{-p'/p} \, dx \right)^{p/p'} < \infty,$$

where R runs over all bounded n-dimensional intervals. The basic properties of these functions can be found in [9].

2. Spaces of Beurling ultradistributions. Hörmander–Beurling spaces

In this section we collect some basic facts about vector-valued (Beurling) ultradistributions and we recall the definitions of the vector-valued Hörmander–Beurling spaces. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [2,13,20,21]. Our notations are based on [2,41].

Let $\mathcal{M}$ (or $\mathcal{M}_n$) be the set of all functions ω on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties:

(i) $\sigma(0) = 0$,
(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2} \, dt < \infty$ (Beurling's condition),
(iii) there exist a real number a and a positive number b such that

$$\sigma(t) \geqslant a + b \log(1 + t) \quad \text{for all } t \geqslant 0.$$

The assumption (ii) is essentially the Denjoy–Carleman non-quasianalyticity condition (see [2]). The two most prominent examples of functions $\omega \in \mathcal{M}$ are given by $\omega(x) = \log(1 + |x|)^d$, $d > 0$, and $\omega(x) = |x|^\beta$, $0 < \beta < 1$.

If $\omega \in \mathcal{M}$ and E is a Fréchet space, we denote by $\mathcal{D}_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|\hat{f}(\xi)\| e^{\lambda\omega(\xi)} \, d\xi < \infty$, for all $\lambda > 0$ and for all $\|\cdot\| \in \operatorname{cs}(E)$. For each compact subset K of $\mathbb{R}^n$, $\mathcal{D}_\omega(K, E) = \{f \in \mathcal{D}_\omega(E)\colon \operatorname{supp} f \subset K\}$, equipped with the topology induced by the family of seminorms $\{\|\cdot\|_\lambda\colon \|\cdot\| \in \operatorname{cs}(E), \lambda > 0\}$, is a Fréchet space and $\mathcal{D}_\omega(E) = \operatorname{ind}_{\to K} \mathcal{D}_\omega(K, E)$ becomes a strict (LF)-space. If Ω is any open set in $\mathbb{R}^n$, $\mathcal{D}_\omega(\Omega, E)$ is the subspace of $\mathcal{D}_\omega(E)$ consisting of all functions f with $\operatorname{supp} f \subset \Omega$. $\mathcal{D}_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology: $\mathcal{D}_\omega(\Omega, E) = \operatorname{ind}_{\to K} \mathcal{D}_\omega(K, E)$. Let $\mathcal{S}_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\| < \infty$ and $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha \hat{f}(x)\| < \infty$ for all multi-indices α and all positive numbers λ and all $\|\cdot\| \in \operatorname{cs}(E)$. $\mathcal{S}_\omega(E)$ with the topology induced by the above family of seminorms is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}_\omega(E)$. If $E = \mathbb{C}$ then $\mathcal{D}_\omega(E)$ and $\mathcal{S}_\omega(E)$ coincide with the spaces $\mathcal{D}_\omega$ and $\mathcal{S}_\omega$ (see [2]). Let us recall that, by Beurling's condition, the space $\mathcal{D}_\omega$ is non-trivial and the usual procedure of the resolution of unity can be established with $\mathcal{D}_\omega$-functions (see [2]). Furthermore $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{D}$ (see [2]) and $\mathcal{D}_\omega$ is nuclear [45]. On the other hand, $\mathcal{D}_\omega = \mathcal{D} \cap \mathcal{S}_\omega$, $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{S}_\omega \overset{d}{\hookrightarrow} \mathcal{S}$ (see [2]) and $\mathcal{S}_\omega$ is nuclear too (see [13]). If $\mathcal{E}_\omega$ is the set of multipliers on $\mathcal{D}_\omega$, i.e., the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\varphi f \in \mathcal{D}_\omega$, for all $\varphi \in \mathcal{D}_\omega$, then $\mathcal{E}_\omega$

with the topology generated by the seminorms $\{f \to \|\varphi f\|_\lambda = \int_{\mathbb{R}^n} |\widehat{\varphi f}(\xi)| e^{\lambda\omega(\xi)}\, d\xi\colon\ \lambda > 0,\ \varphi \in \mathcal{D}_\omega\}$ becomes a nuclear Fréchet space (see [45]) and $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{E}_\omega$. Using the above results and [21, Theorem 1.12] we can identify $\mathcal{S}_\omega(E)$ with $\mathcal{S}_\omega \hat{\otimes}_\varepsilon E$. However, though $\mathcal{D}_\omega \otimes E$ is dense in $\mathcal{D}_\omega(E)$, in general $\mathcal{D}_\omega(E)$ is not isomorphic to $\mathcal{D}_\omega \hat{\otimes}_\varepsilon E$ (cf., e.g. [12]). A continuous linear operator from $\mathcal{D}_\omega$ into E is said to be a (Beurling) ultradistribution with values in E. We write $\mathcal{D}'_\omega(E)$ for the space of all E-valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $\mathcal{D}'_\omega(E) = \mathcal{L}_b(\mathcal{D}_\omega, E)$. $\mathcal{D}'_\omega(\Omega, E) = \mathcal{L}_b(\mathcal{D}'_\omega(\Omega), E)$ is the space of all (Beurling) ultradistributions on Ω with values in E. A continuous linear operator from $\mathcal{S}_\omega$ into E is said to be an E-valued tempered ultradistribution. $\mathcal{S}'_\omega(E)$ is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology, i.e., $\mathcal{S}'_\omega(E) = \mathcal{L}_b(\mathcal{S}_\omega, E)$. The Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}'_\omega(E)$.

If $\omega \in \mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a positive constant N such that $k(x+y) \leqslant e^{N\omega(x)} k(y)$ for all x and y in $\mathbb{R}^n$, cf. [2] (when $\omega(x) = \log(1+|x|)$ the functions k of the corresponding class $\mathcal{K}_\omega$ are called temperate weight functions, see [16]). If $k, k_1, k_2 \in \mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1 k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [2]). If $u \in L_1^{\mathrm{loc}}$ and $\int_{\mathbb{R}^n} \varphi(x) u(x)\, dx = 0$ for all $\varphi \in \mathcal{D}_\omega$, then $u = 0$ a.e. (see [2]). This result, the Hahn–Banach theorem and [7, Chapter II, Corollary 7] prove that if $k \in \mathcal{K}_\omega$, $p \in [1, \infty]$ and E is a Fréchet space, we can identify $f \in L_{p,k}(E)$ with the E-valued tempered ultradistribution $\varphi \to \langle \varphi, f\rangle = \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx$, $\varphi \in \mathcal{S}_\omega$, and $L_{p,k}(E) \hookrightarrow \mathcal{S}'_\omega(E)$. If $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $p \in [1, \infty]$ and E is a Fréchet space, we denote by $\mathcal{B}_{p,k}(E)$ the set of all E-valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\langle \varphi, \hat{T}\rangle = \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx$, $\varphi \in \mathcal{S}_\omega$. $\mathcal{B}_{p,k}(E)$ with the seminorms $\{\|T\|_{p,k} = ((2\pi)^{-n} \int_{\mathbb{R}^n} \|k(x)\hat{T}(x)\|^p\, dx)^{1/p}\colon\ \|\cdot\| \in \mathrm{cs}(E)\}$ (usual modification if $p = \infty$), becomes a Fréchet space isomorphic to $L_{p,k}(E)$. Spaces $\mathcal{B}_{p,k}(E)$ are called Hörmander–Beurling spaces with values in E (see [2] for the scalar case and [44] for the vector-valued case). We denote by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ (see [30]) the space of all E-valued ultradistributions $T \in \mathcal{D}'_\omega(\Omega, E)$ such that, for every $\varphi \in \mathcal{D}_\omega(\Omega)$, the map $\varphi T : \mathcal{S}_\omega \to E$ defined by $\langle u, \varphi T\rangle = \langle u\varphi, T\rangle$, $u \in \mathcal{S}_\omega$, belongs to $\mathcal{B}_{p,k}(E)$. The space $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is a Fréchet space with the topology generated by the seminorms $\{\|\cdot\|_{p,k,\varphi}\colon\ \varphi \in \mathcal{D}_\omega(\Omega),\ \|\cdot\| \in \mathrm{cs}(E)\}$, where $\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k}$ for $T \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$, and $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \hookrightarrow \mathcal{D}'_\omega(\Omega, E)$. We shall also use the spaces $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E)$ which generalize the scalar spaces $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega)$ considered by Hörmander in [16], by Vogt in [45] and by Björck in [2]. If ω, k, p, Ω and E are as above, then $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E) = \bigcup_{j=1}^\infty [\mathcal{B}_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)]$ (here (Kj) is any fundamental sequence of compact subsets of Ω and $\mathcal{E}'_\omega(K_j, E)$ denotes the set of all $T \in \mathcal{D}_\omega(E)$ such that $\operatorname{supp} T \subset K_j$). Since for every compact $K \subset \Omega$, $\mathcal{B}_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)$ is a Fréchet space with the topology induced by $\mathcal{B}_{p,k}(E)$, it follows that $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E)$ becomes a strict (LF)-space (strict (LB)-space if E is a Banach space): $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E) = \operatorname*{ind}_{\to j}[\mathcal{B}_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)]$. These spaces are studied in [36,31].

3. An embedding theorem

In this section we generalize to UMD spaces Theorem 4.6 of [31], we prove an embedding theorem for vector-valued Hörmander–Beurling spaces (Theorem 3.1, see also Remark 3.1.2) and we give a result of Rosenthal type [38] (every weakly compact subset of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is separable when E is a closed subspace of $l_\infty^{\mathbb{N}}$; see Remark 3.1.1).

We shall need the following technical result.

Lemma 3.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \leqslant p \leqslant \infty$. Let $E = \mathrm{proj}(E_j, A_j)$ be the reduced projective limit of the projective sequence of Fréchet spaces E_j and linking maps A_j. Then the map*

$$P : \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \to \mathrm{proj}\big(\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j), \overline{A}_j\big) : T \to (P_j \circ T)_1^\infty$$

is an isomorphism ($\overline{A}_j$ is the map $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_{j+1}) \to \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j) : T \to A_j \circ T$) and this projective limit is reduced if $p < \infty$. If $E = \prod_{j=1}^\infty E_j$ then the space $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is isomorphic to $\prod_{j=1}^\infty \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j)$.

Proof. Although the proof of the lemma is straightforward, for the sake of completeness we give here the proof of the surjectivity of P: Let $(T_j)_1^\infty$ be any element in $\mathrm{proj}(\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j), \overline{A}_j)$. For each $\varphi \in \mathcal{D}_\omega(\Omega)$ and each $j \geqslant 1$, we have $A_j(\langle \varphi, T_{j+1}\rangle) = \langle \varphi, A_j \circ T_{j+1}\rangle = \langle \varphi, T_j\rangle$ and so $(\langle \varphi, T_j\rangle)_1^\infty \in \mathrm{proj}(E_j, A_j)$. Let $T : \mathcal{D}_\omega \to E$ be defined by $\langle \varphi, T\rangle := (\langle \varphi, T_j\rangle)_1^\infty$ for $\varphi \in \mathcal{D}_\omega(\Omega)$. Let us prove that $T \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$, i.e., that for every $\varphi \in \mathcal{D}_\omega(\Omega)$ there is an $f \in L_{p,k}(E)$ such that $\langle \theta, \widehat{\varphi T}\rangle = \int_{\mathbb{R}^n} \theta(x) f(x)\, dx$ for all $\theta \in \mathcal{S}_\omega$. Given such a φ let $f_j \in L_{p,k}(E_j)$, $j = 1, 2, \ldots,$ such that $\langle \theta, \widehat{\varphi T_j}\rangle = \int_{\mathbb{R}^n} \theta(x) f_j(x)\, dx$ for all $\theta \in \mathcal{S}_\omega$. Then, for every $\theta \in \mathcal{S}_\omega$, we have $\int_{\mathbb{R}^n} \theta(x) A_j \circ f_{j+1}(x)\, dx = A_j(\int_{\mathbb{R}^n} \theta(x) f_{j+1}(x)\, dx) = A_j(\langle \theta, (\varphi T_{j+1})^\wedge\rangle) = \langle \theta, A_j \circ (\varphi T_{j+1})^\wedge\rangle = \langle \theta, [\varphi(A_j \circ T_{j+1})]^\wedge\rangle = \langle \theta, (\varphi T_j)^\wedge\rangle = \int_{\mathbb{R}^n} \theta(x) f_j(x)\, dx$, that is, $\int_{\mathbb{R}^n} \theta(x)[A_j \circ f_{j+1}(x) - f_j(x)]\, dx = 0$. Hence it follows (see Section 2) that $A_j \circ f_{j+1}(x) = f_j(x)$ for almost all $x \in \mathbb{R}^n$. Then, modifying the functions f_j in a nullset if necessary, we get $(f_j(x))_1^\infty \in \mathrm{proj}(E_j, A_j)$ for all $x \in \mathbb{R}^n$. It is easy to verify that the function $f(x) = (f_j(x))_1^\infty$ is Bochner measurable. In fact, if $\phi \in E'$ we can find $N \geqslant 1$ and $(e'_1, \ldots, e'_N) \in E'_1 \times \cdots \times E'_N$ (see, e.g. [25]) such that $\langle (e_j)_1^\infty, \phi\rangle = \sum_{j=1}^N \langle e_j, e'_j\rangle$, $(e_j)_1^\infty \in E$. Thus $\phi \circ f = \sum_{j=1}^N e'_j \circ f_j$ is measurable. Moreover, if N_j is a nullset such that $f_j(\mathbb{R}^n \setminus N_j)$ is separable, then $f(\mathbb{R}^n \setminus \bigcup N_j)$ is also separable. Hence by the Pettis's measurability theorem (in Fréchet spaces, see e.g. [10]) it follows that f is Bochner measurable. Then, by using the properties of the f_j, $j = 1, 2, \ldots,$ we conclude that $f \in L_{p,k}(E)$. Finally, since $\int_{\mathbb{R}^n} \theta(x) f(x)\,dx = (\int_{\mathbb{R}^n} \theta(x) f_j(x)\,dx)_1^\infty = (\langle \theta, \widehat{\varphi T_j} \rangle)_1^\infty = (\langle \hat{\theta}\varphi, T_j \rangle)_1^\infty = \langle \hat{\theta}\varphi, T \rangle = \langle \theta, \widehat{\varphi T} \rangle$ for all $\theta \in \mathcal{S}_\omega$, it follows that $T \in \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$. Thus P is surjective. □

The next lemma generalizes to UMD spaces Theorem 4.6 of [31]. We will reason as we did in [31] but we will use Theorem 4.2 of [29] instead of Corollary 4.2 of [29]. For convenience of the reader we will give a complete proof. The following elementary fact will be used: "Let $F = \operatorname{ind}_{\to j} F_j$ be the strict inductive limit of a properly increasing sequence $F_1 \subset F_2 \subset \cdots$ of Banach spaces. Assume that every F_j is a complemented subspace of F_{j+1} and that G_j is a topological complement of F_j in F_{j+1}. Then the mapping $F_1 \oplus G_1 \oplus G_2 \oplus \cdots \to F : (f_1, g_1, g_2, \ldots) \to f_1 + g_1 + g_2 + \cdots$ is an isomorphism." We will also need the weighted L_p-spaces of vector-valued entire analytic functions $L_{p,k}^K(E)$ and the operators $S_K(f) = \mathcal{F}^{-1}(\chi_K \hat{f})$ (see [29,41]).

Lemma 3.2. *Let Ω be an open set in $\mathbb{R}^n$, $p \in (1, \infty)$ and k a temperate weight function on $\mathbb{R}^n$ with $k^p \in A_p^*$. Let E be a Banach space with the UMD-property. Then the space $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ is isomorphic to $\prod_{j=0}^\infty H_j$ where H_0 is isomorphic to $l_p(E)$ and H_j is isomorphic to a complemented subspace of $l_p(E)$ for $j = 1, 2, \ldots$.*

Proof. Let (K_j) be a covering of Ω consisting of compact sets such that $K_j \subset \mathring{K}_{j+1}$, $K_j = \overline{\mathring{K}_j}$ and $\mathring{K}_j$ has the segment property (we may also assume, without loss of generality, that each K_j is a finite union of n-dimensional compact intervals). Then $\mathcal{B}_{p,k}^{\mathrm{c}}(\Omega, E) = \operatorname{ind}_{\to j}[\mathcal{B}_{p,k}(E) \cap \mathcal{E}'(K_j, E)]$. In this inductive limit, the step $\mathcal{B}_{p,k}(E) \cap \mathcal{E}'(K_j, E)$ is isomorphic (via Fourier transform) to $L_{p,k}^{-K_j}(E)$ and this space is isomorphic, by Theorem 4.2 and Corollary 5.1 of [29], to $l_p(E)$. Furthermore, $L_{p,k}^{-K_j}(E)$ is a complemented subspace of $L_{p,k}^{-K_{j+1}}(E)$: $L_{p,k}^{-K_{j+1}}(E) = L_{p,k}^{-K_j}(E) \oplus [\ker S_{-K_j} \cap L_{p,k}^{-K_{j+1}}(E)]$. Thus, the space $G_j = \ker S_{-K_j} \cap L_{p,k}^{-K_{j+1}}(E)$ is isomorphic to an infinite-dimensional complemented subspace of $l_p(E)$. Then, by using the former result, we obtain $\mathcal{B}_{p,k}^{\mathrm{c}}(\Omega, E) \simeq L_{p,k}^{-K_1}(E) \oplus G_1 \oplus G_2 \oplus \cdots \simeq l_p(E) \oplus G_1 \oplus G_2 \oplus \cdots$. Next, since $1/\tilde{k}$ is a temperate weight function on $\mathbb{R}^n$ such that $1/\tilde{k}^{p'} \in A_{p'}^*$ and $E' \in \mathit{UMD}$ (see [39]), we see that $\mathcal{B}_{p',1/\tilde{k}}^{\mathrm{c}}(\Omega, E') \simeq \bigoplus_{j=0}^\infty B_j$ where $B_0 \simeq l_{p'}(E')$ and $B_j < l_{p'}(E')$ for $j = 1, 2, \ldots$. Therefore, by Theorem 3.2 of [31] (see [16] also), we get $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \simeq (\mathcal{B}_{p',1/\tilde{k}}^{\mathrm{c}}(\Omega, E'))' \simeq (\bigoplus_{j=0}^\infty B_j)' \simeq \prod_{j=0}^\infty B_j' = \prod_{j=0}^\infty H_j$ (here $H_j = B_j'$) where $H_0 \simeq l_p(E)$ and $H_j < l_p(E)$ for $j = 1, 2, \ldots,$ and the proof is complete. □

Remark. One can improve Lemma 3.2 by using [45]. Indeed, using the arguments of [45] it can be shown that $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \simeq (\mathcal{B}_{p,k}(E) \cap \mathcal{E}'(Q, E))^{\mathbb{N}}$ where $Q = [0, 1]^n$. Then, reasoning as in the lemma, we obtain the isomorphism $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \simeq (l_p(E))^{\mathbb{N}}$.

We now present the main result of this section, an embedding (and sequence space representation) theorem for vector-valued Hörmander–Beurling spaces (see also Remark 3.1). We also pose a related question (Remark 3.1.3): Is $\mathcal{B}_{\infty,k}^{\mathrm{loc}}(\Omega, l_\infty)$ isomorphic to a complemented subspace of $l_\infty^{\mathbb{N}}$? We will use the Fréchet spaces $l_{q^+} = \bigcap_{p>q} l_p$ and $L_{q^-} = \bigcap_{p<q} L_p([0, 1])$ (these spaces have an interest in the structure theory of Fréchet spaces and are primary and have all nuclear $\Lambda_1(\alpha)$-spaces as complemented subspaces, see [27,3]).

Theorem 3.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \leqslant p, q \leqslant \infty$, and let E be a Fréchet space.*

1. *If $p < \infty$ and E is separable then $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ is isomorphic to a subspace of $(C([0, 1]))^{\mathbb{N}}$ and this space does not contain any complemented copy of $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$.*
2. *If E is separable and infinite-dimensional and $E \not\simeq \mathbb{C}^{\mathbb{N}}$ then $\mathcal{B}_{\infty,k}^{\mathrm{loc}}(\Omega, E)$ is isomorphic to a subspace of $l_\infty^{\mathbb{N}}$ but this space does not contain any complemented copy of $\mathcal{B}_{\infty,k}^{\mathrm{loc}}(\Omega, E)$. If $E \simeq \mathbb{C}^{\mathbb{N}}$ then $\mathcal{B}_{\infty,k}^{\mathrm{loc}}(\Omega, E)$ is isomorphic to $l_\infty^{\mathbb{N}}$.*
3. *Suppose $E \subset F^{\mathbb{N}}$ (resp. $< F^{\mathbb{N}}$) where F is a Banach space. Then $l_1^{\mathbb{N}} < \mathcal{B}_{1,k}^{\mathrm{loc}}(\Omega, E) \subset (l_1(F))^{\mathbb{N}}$ (resp. $< (l_1(F))^{\mathbb{N}}$). If F is a dual space and has the Radon–Nikodým property, then $l_\infty^{\mathbb{N}} < \mathcal{B}_{\infty,k}^{\mathrm{loc}}(\Omega, E) \subset (l_\infty(F))^{\mathbb{N}}$ (resp. $< (l_\infty(F))^{\mathbb{N}}$). If F has the UMD-property then $l_p^{\mathbb{N}} < \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \subset (l_p(F))^{\mathbb{N}}$ (resp. $< (l_p(F))^{\mathbb{N}}$) provided that $1 < p < \infty$ and k is a temperate weight with $k^p \in A_p^*$; in particular, $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, l_p^{\mathbb{N}})$ is isomorphic to $l_p^{\mathbb{N}}$.*
4. *Suppose $1 < p < \infty$ and that k is a temperate weight with $k^p \in A_p^*$, and let $E = l_{q^+}$ with $q < \infty$ (resp. $L_{q^-}([0, 1])$ with $1 < q$). Let $(q_j)_1^\infty$ be any sequence such that $q_j \searrow q$ (resp. $q_j \nearrow q$). Then $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ is isomorphic to a subspace of $G := (\prod_{j=1}^\infty l_p(l_{q_j}))^{\mathbb{N}}$ (resp. $H := (\prod_{j=1}^\infty l_p(L_{q_j}([0, 1])))^{\mathbb{N}}$) but G (resp. H) does not contain any complemented copy of $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$.*

5. *Let p, k, q and $(q_j)_1^\infty$ be as in 4. Let X be a Banach subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_{q^+})$ (resp. $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, L_{q^-}([0,1]))$). Then X is isomorphic to a subspace of $l_p(l_{q_1} \oplus \cdots \oplus l_{q_m})$ (resp. $l_p(L_{q_1}([0,1]) \oplus \cdots \oplus L_{q_m}([0,1]))$) for some integer m.*

Proof. 1. The first claim is a consequence from the fact that every separable Fréchet space is isomorphic to a subspace of $(C([0,1]))^{\mathbb{N}}$ (see e.g. [1, p. 51]). Now suppose that $(C([0,1]))^{\mathbb{N}}$ contains a complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$. Then $(C([0,1]))^{\mathbb{N}}$ also contains a complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ since this space is clearly isomorphic to a complemented subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$. Hence it follows, if $p = 1$, that $(C([0,1]))^{\mathbb{N}}$ contains a complemented copy of $l_1^{\mathbb{N}}$ (the proof given in [45] of the isomorphism $\mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega) \simeq l_1^{\mathbb{N}}$ is also valid for weights $k \in \mathcal{K}_\omega$). Then l_1 becomes isomorphic to a complemented subspace of $C([0,1])$ (see e.g. [6]) which contradicts Corollary 2 in [33]. In case $p > 1$ we can apply Proposition 3.7 in [26] and obtain the isomorphism $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \simeq \mathbb{C}^{\mathbb{N}}$. This contradicts the fact that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ is a non-Montel Fréchet space (see [15, Theorem 2.3.9] and [16]). Consequently, $(C([0,1]))^{\mathbb{N}}$ does not contain any complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$.

2. We know that $E \subset l_\infty^{\mathbb{N}}$ [1, p. 51], that $L_\infty \simeq l_\infty$ [23] and that $L_\infty(L_\infty) \subset (L_1(L_1))' \simeq L_1' \simeq L_\infty$ (but $L_\infty(L_\infty) \not\simeq L_\infty$, see [4]). Hence and from Lemma 3.1 it follows that $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E) \subset \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, L_\infty^{\mathbb{N}}) \simeq (\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, L_\infty))^{\mathbb{N}} \subset ((L_\infty(L_\infty))^{\mathbb{N}})^{\mathbb{N}} \simeq (L_\infty(L_\infty))^{\mathbb{N}} \subset L_\infty^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}}$. However, if $E \not\simeq \mathbb{C}^{\mathbb{N}}$, the space $l_\infty^{\mathbb{N}}$ cannot contain any complemented copy of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ by virtue of Proposition 3.12 in [26] (recall that $E < \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$). On the other hand, if $E \simeq \mathbb{C}^{\mathbb{N}}$ then $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E) \simeq (\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega))^{\mathbb{N}} \simeq (l_\infty^{\mathbb{N}})^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}}$ by Lemma 3.1 and [31, Theorem 4.2(3)].

3. By Lemma 3.1 and by [45] and [31, Theorem 4.2(2)], we have $l_1^{\mathbb{N}} \simeq \mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega) < \mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega, E) \subset$ (resp. $<$) $\mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega, F^{\mathbb{N}}) \simeq (\mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega, F))^{\mathbb{N}} \simeq (l_1(F)^{\mathbb{N}})^{\mathbb{N}} \simeq (l_1(F))^{\mathbb{N}}$. If F is a dual space and has the Radon–Nikodým property then $l_\infty^{\mathbb{N}} \simeq \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega) < \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E) \subset$ (resp. $<$) $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, F^{\mathbb{N}}) \simeq (\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, F))^{\mathbb{N}} \simeq (l_\infty(F)^{\mathbb{N}})^{\mathbb{N}} \simeq (l_\infty(F))^{\mathbb{N}}$ by virtue of Lemma 3.1 and [31, Theorem 4.2(3)].

Suppose now that F has the UMD-property, $1 < p < \infty$ and $k^p \in A_p^*$. By using [31, Remark 4.7(1)] (see also [14]), Lemma 3.1 and Lemma 3.2, we get $l_p^{\mathbb{N}} \simeq \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) < \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \subset$ (resp. $<$) $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, F^{\mathbb{N}}) \simeq (\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, F))^{\mathbb{N}} < ((l_p(F))^{\mathbb{N}})^{\mathbb{N}} \simeq (l_p(F))^{\mathbb{N}}$. Hence and from [42, (1), p. 331] it follows that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p^{\mathbb{N}}) \simeq l_p^{\mathbb{N}}$ (see also [31, Remark 4.7(1)] or [14]).

4. Since the proofs of both claims are similar, we shall only proceed with the proof of the second one.

Put $E = L_{q^-}([0,1])$ and let (q_j) be a sequence such that $q_j \nearrow q$. Then, taking into account Lemma 3.1 and Lemma 3.2 (the spaces $L_{q_j}([0,1])$ have the UMD-property, see e.g. [39]), we have

$$\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \subset \mathcal{B}^{\mathrm{loc}}_{p,k}\left(\Omega, \prod_{j=1}^{\infty} L_{q_j}\big([0,1]\big)\right) \simeq \prod_{j=1}^{\infty} \mathcal{B}^{\mathrm{loc}}_{p,k}\big(\Omega, L_{q_j}\big([0,1]\big)\big) < \prod_{j=1}^{\infty} \big(l_p\big(L_{q_j}\big([0,1]\big)\big)\big)^{\mathbb{N}}$$
$$\simeq \left(\prod_{j=1}^{\infty} l_p\big(L_{q_j}\big([0,1]\big)\big)\right)^{\mathbb{N}} = H.$$

Furthermore, since all complemented subspace of a quojection is a quojection (see [28]), H is a quojection (actually $H \simeq \prod_{r=1}^{\infty} X_r$ where each X_r coincides with some $l_p(L_{q_j}([0,1]))$), $E < \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ and E is not a quojection (see [3]), it follows that H does not contain any complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$.

5. Let X be a Banach subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_{q^+})$ (resp. $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, L_{q^-}([0,1]))$). By using 4 we see that X is isomorphic to a subspace of $\prod_{r=1}^{\infty} Y_r$ (resp. $\prod_{r=1}^{\infty} X_r$) where each Y_r (resp. X_r) coincides with some $l_p(l_{q_j})$ (resp. $l_p(L_{q_j}([0,1]))$), thus [6] X becomes isomorphic to a subspace of $l_p(l_{q_1} \oplus \cdots \oplus l_{q_m})$ (resp. $l_p(L_{q_1}([0,1]) \oplus \cdots \oplus L_{q_m}([0,1]))$) for some integer m. □

Remark 3.1. 1. In [38] Rosenthal showed that if (Ω, Σ, μ) is a finite measure space then every weakly compact subset of $L_\infty(\mu)$ is norm separable. By using this result it is easy to show that if $E \subset l_\infty^{\mathbb{N}}$ then every weakly compact subset of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ (and hence every WCG subspace of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$) is separable. In fact, let K be a weakly compact subset of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$. Then K becomes a weakly compact subset of $(L_\infty([0,1]))^{\mathbb{N}}$ (see the proof of Theorem 3.1(2) and recall that $l_\infty \simeq L_\infty([0,1])$). Now the weak topology

$$\sigma\big(\big(L_\infty\big([0,1]\big)\big)^{\mathbb{N}}, \big(\big(L_\infty\big([0,1]\big)\big)^{\mathbb{N}}\big)'\big)$$

is the product of the weak topologies (see, e.g. [17, p. 167]). Consequently the projection of K on every factor $L_\infty([0,1])$ is weakly compact and, by the Rosenthal's result, is norm separable. Hence it follows that K is separable in $(L_\infty([0,1]))^{\mathbb{N}}$ and so is separable in $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$.

2. Evidently it is possible to replace $C([0,1])$ by l_∞ in Theorem 3.1(1). In the non-separable case we have the following extension: "Let $p < \infty$. Let E be a non-separable Fréchet space and let I be a set such that $\operatorname{card} I = \operatorname{dens} E$. Then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \subset (l_\infty(I))^{\mathbb{N}}$ and this space does not contain any complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$." In fact, let $(E_j)_{j=1}^{\infty}$ be a sequence of Banach spaces, with $\operatorname{dens} E_j \leqslant \operatorname{dens} E$ for all j, such that E is isomorphic to a subspace of $\prod_{j=1}^{\infty} E_j$ (see, e.g. [1, p. 34]). Since $\operatorname{dens} L_p(E_j) \leqslant \operatorname{card} I$, we get $L_p(E_j) \subset l_\infty(I)$ [1, p. 50] and

$$\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \subset \mathcal{B}^{\mathrm{loc}}_{p,k}\left(\Omega, \prod_{j=1}^{\infty} E_j\right) \simeq \prod_{j=1}^{\infty} \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j) \subset \prod_{j=1}^{\infty} \big(L_p(E_j)\big)^{\mathbb{N}}$$
$$\subset \prod_{j=1}^{\infty} \big(l_\infty(I)\big)^{\mathbb{N}} \simeq \big(l_\infty(I)\big)^{\mathbb{N}}.$$

Finally, since $l_\infty(I) = C(\beta I)$ (βI is the Stone–Čech compactification of I regarded in its discrete topology) and βI is extremally disconnected, we apply [26, Proposition 3.12].

3. We finish this note by posing the following question: Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$ and $k \in \mathcal{K}_\omega$. Is $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty)$ isomorphic to a complemented subspace of $l_\infty^{\mathbb{N}}$? (If the answer to this question were yes, $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty)$ would be isomorphic to $l_\infty^{\mathbb{N}}$ since $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega) \simeq l_\infty^{\mathbb{N}} < \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty) < l_\infty^{\mathbb{N}}$ implies $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty) \simeq l_\infty^{\mathbb{N}}$ in virtue of [42, (1), p. 331].)

4. On sequence space representations of Hörmander–Beurling spaces and applications

In this section a number of results on sequence space representations of vector-valued Hörmander–Beurling spaces are given (Theorem 4.1; see also Lemma 3.2 and [30,31]). As a consequence, and using sharp results of Meise, Taylor and Vogt [24], a result of Kaballo (see [19]) on short sequences and hypoelliptic differential operators is extended to ω-hypoelliptic differential operators and to the vector-valued setting.

Lemma 4.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \leqslant p < \infty$. Let E be a Fréchet space. Then the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ on $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$ is intercalated between the ε and π topologies.*

Proof. Taking into account the corresponding fundamental systems of seminorms the proof is immediate since, for every $\varphi \in D_\omega(\Omega)$ and every $\|\cdot\| \in \mathrm{cs}(E)$, we have

$$\|T\|_{p,k,\varphi} \leqslant \inf\left\{ \sum_1^m \|u_j\|_{p,k,\varphi}\|e_j\|\colon\ T = \sum_1^m u_j \otimes e_j \right\}$$

for all $T \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$, and, for every neighborhood U of 0 in $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ and every $\|\cdot\| \in \mathrm{cs}(E)$, we have

$$\sup_{(\xi,e')\in U^0\times V^0} \left| \sum_1^m \langle u_j, \xi\rangle \langle e_j, e'\rangle \right| \leqslant \max_{1\leqslant i\leqslant r} \|T\|_{p,k,\varphi_i}$$

(here $\varphi_1, \ldots, \varphi_r \in D_\omega(\Omega)$ generate U and $V = \{e \in E\colon\ \|e\| \leqslant 1\}$) for all $T = \sum_1^m u_j \otimes e_j \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$. $\square$

Remark 4.1. 1. Note that, in general, the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ on $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$ is strictly finer than the ε topology and strictly coarser than the π topology: In fact let $1 < p < \infty$, let k be a temperate weight function on $\mathbb{R}^n$ with $k^p \in A_p^*$ and assume that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p)$ contains a complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon l_p$. Then, by [31, Remark 4.7(1)] (see also Theorem 3.1(3)) and [22, (5), p. 282], we get $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon l_p \simeq l_p^{\mathbb{N}} \hat{\otimes}_\varepsilon l_p \simeq (l_p \hat{\otimes}_\varepsilon l_p)^{\mathbb{N}} < \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p) \simeq l_p^{\mathbb{N}}$. Hence and from [6] it follows that $l_p \hat{\otimes}_\varepsilon l_p < l_p$, that is to say (since l_p is prime [23, Theorem 2.4.3]), that $l_p \hat{\otimes}_\varepsilon l_p \simeq l_p$. But this is false since $l_p \hat{\otimes}_\varepsilon l_p$ fails to have the uniform approximation property (UAP, for short; see [34, p. 350]) whereas $l_p \in$ UAP by [35]. Therefore, $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon l_p$ cannot be isomorphic to a complemented subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p)$. In particular, since $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes l_p$ is dense in $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p)$, the ε topology is strictly coarser than the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p)$. (A different proof, for the case $2 \leqslant p < \infty$, is given in [31, Remark 4.7(2)].) In a similar way it can be shown that the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p)$ on $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes l_p$ is strictly coarser than the π topology (recall that $l_p \hat{\otimes}_\pi l_p \notin$ UAP [34, p. 350]).

2. If $p = 1$ and k is any weight in $\mathcal{K}_\omega$ one can argue as in 1 (by using [31, Theorem 4.2(2)] and the well-known fact that $l_1 \hat{\otimes}_\varepsilon l_1$ is not isomorphic to l_1 [7, Chapter VIII]) and show that the topology induced by $\mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega, l_1)$ on $\mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega) \otimes l_1$ is strictly finer than the ε topology.

3. The assertions in the above notes continue to hold when one replaces l_p by $l_p^{\mathbb{N}}$ in 1 and l_1 by $l_1^{\mathbb{N}}$ in 2.

4. Notice also that if the answer to the posed question in Remark 3.1.3 were affirmative, then $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega) \hat{\otimes}_\varepsilon l_\infty$ would not be isomorphic to $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty)$ for any $k \in \mathcal{K}_\omega$. In fact, if these spaces were isomorphic then, by [31, Theorem 4.2(3)], [22, (5), p. 282], [22, (2), p. 287] and a result of Cembranos and Freniche [4, Theorem 3.2.1], we would have $l_\infty^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}} \hat{\otimes}_\varepsilon l_\infty \simeq (l_\infty \hat{\otimes}_\varepsilon l_\infty)^{\mathbb{N}} \simeq (C(\beta\mathbb{N}) \hat{\otimes}_\varepsilon l_\infty)^{\mathbb{N}} \simeq (C(\beta\mathbb{N}, l_\infty))^{\mathbb{N}} > c_0^{\mathbb{N}}$. Therefore c_0 would become a complemented subspace of l_∞ which contradicts a classical result of Phillips (see e.g. [4, Corollary 1.3.2]).

Theorem 4.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \leqslant p < \infty$. Let E be a nuclear Fréchet space. Then*

(a) $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) = \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon E$;

(b) *if $p = 1$, or, $1 < p < \infty$ and k is a temperate weight with $k^p \in A^*_p$, then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \simeq (l_p(E))^{\mathbb{N}}$;*

(c) *if $p = 1$, or, $1 < p < \infty$ and k is a temperate weight with $k^p \in A^*_p$, and $E \simeq s$ or $s^{\mathbb{N}}$, then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$;*

(d) *if E is infinite-dimensional and $E \not\simeq \mathbb{C}^{\mathbb{N}}$, then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is isomorphic to a (non-complemented) subspace of $(L_p([0,1]))^{\mathbb{N}}$;*

(e) *if E is a power series space of finite type, then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is isomorphic to a complemented subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_{q^+})$ (resp. $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, L_{q^-}([0,1])))$ for any $q \in [1, \infty[$ (resp. $q \in]1, \infty]$);*

(f) *if X is a Banach subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$, then X is isomorphic to a subspace of $L_p([0,1])$;*

(g) *if $p = 1$, or, $1 < p < \infty$ and k is a temperate weight with $k^p \in A^*_p$, and X is a Banach subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$, then X is isomorphic to a subspace of l_p;*

(h) *if $1 < p_1, p_2 < \infty$, and k_1, k_2 are temperate weights such that $k_1^{p_1} \in A^*_{p_1}$, $k_2^{p_2} \in A^*_{p_2}$, then $\mathcal{B}^{\mathrm{loc}}_{p_1,k_1}(\Omega, E) \simeq \mathcal{B}^{\mathrm{loc}}_{p_2,k_2}(\Omega, E)$ if and only if $p_1 = p_2$;*

(i) *$\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is quasinormable, and if $p > 1$ every quotient of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ by a closed subspace is reflexive*;

(j) *every exact sequence $0 \to \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \to G \to E \to 0$ where G is a Fréchet space, $1 < p < \infty$ and k is a temperate weight with $k^p \in A^*_p$, splits.*

Proof. (a) This is an immediate consequence of Lemma 4.1, the nuclearity of E, the denseness of $\mathcal{D}_\omega(\Omega) \otimes E$ in $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ (use [36, Proposition 3.4]) and the completeness of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$.

(b) By using (a), [31, Theorem 4.2], [31, Remark 4.7(1)], [22, (5), p. 282], [22, (5), p. 198] and [22, (5), p. 291], we get $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) = \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon E \simeq l_p^{\mathbb{N}} \hat{\otimes}_\varepsilon E \simeq (l_p \hat{\otimes}_\varepsilon E)^{\mathbb{N}} \simeq (l_p(E))^{\mathbb{N}}$.

(c) By Valdivia [43] and Vogt [45], we know that $\mathcal{D}_{L^p}$ is isomorphic to $l_p \hat{\otimes}_\varepsilon s$. Hence and from (b) and [22, (5), p. 282] it follows that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, s) \simeq (l_p \hat{\otimes}_\varepsilon s)^{\mathbb{N}} \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$ and $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, s^{\mathbb{N}}) \simeq (l_p \hat{\otimes}_\varepsilon s^{\mathbb{N}})^{\mathbb{N}} \simeq ((l_p \hat{\otimes}_\varepsilon s)^{\mathbb{N}})^{\mathbb{N}} \simeq (l_p \hat{\otimes}_\varepsilon s)^{\mathbb{N}} \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$.

(d) The space E is isomorphic to a subspace of $(L_p([0,1]))^{\mathbb{N}}$ (see e.g. [17, p. 483]). Hence and from Lemma 3.1 it follows that

$$\begin{aligned}\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) &\subset \mathcal{B}^{\mathrm{loc}}_{p,k}\big(\Omega, \big(L_p\big([0,1]\big)\big)^{\mathbb{N}}\big) \simeq \big(\mathcal{B}^{\mathrm{loc}}_{p,k}\big(\Omega, L_p\big([0,1]\big)\big)\big)^{\mathbb{N}} \\ &\subset \big(\big(L_p\big(L_p\big([0,1]\big)\big)\big)^{\mathbb{N}}\big)^{\mathbb{N}} \simeq \big(\big(L_p\big([0,1]\big)\big)^{\mathbb{N}}\big)^{\mathbb{N}} \simeq \big(L_p\big([0,1]\big)\big)^{\mathbb{N}}.\end{aligned}$$

Now we prove that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ cannot be isomorphic to a complemented subspace of $(L_p([0,1]))^{\mathbb{N}}$. If this were not the case, E would also be isomorphic to a complemented subspace of $(L_p([0,1]))^{\mathbb{N}}$. Then E would become a quojection (see e.g. [26]) and thus $E \simeq \mathbb{C}^{\mathbb{N}}$ (see again [26]), a contradiction.

(e) We know that all nuclear $\Lambda_1(\alpha)$-spaces are complemented subspaces of l_{q^+} when $1 \leqslant q < \infty$ [27] and of $L_{q^-}([0,1])$ when $1 < q \leqslant \infty$ [3]. Thus, if $E = \Lambda_1(\alpha)$, we have $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, \Lambda_1(\alpha)) < \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_{q^+})$ (resp. $< \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, L_{q^-}([0,1])))$.

(f) By (d) X is isomorphic to a subspace of $(L_p([0,1]))^{\mathbb{N}}$ and thus (see [6]) isomorphic to a subspace of $L_p([0,1])$.

(g) Since E is isomorphic to a subspace of $l_p^{\mathbb{N}}$ [17, p. 483], we may apply Theorem 3.1(3) and conclude that X is also isomorphic to a subspace of $l_p^{\mathbb{N}}$. Thus [6] X becomes isomorphic to a subspace of l_p.

(h) ($\Rightarrow$) From [31, Remark 4.7(1)], the hypothesis and (g) it follows that $l_{p_1} \subset l_{p_2}$ (and $l_{p_2} \subset l_{p_1}$). As is well known this implies $p_1 = p_2$. ($\Leftarrow$) It suffices to apply (b).

(i) Taking into account (b) and recalling that the product of a family of quasinormable spaces is quasinormable [11, p. 107] and that the tensor product $\hat{\otimes}_\varepsilon$ of a Banach space and a nuclear space is also quasinormable [12, Chapter II, Proposition 13, p. 76], we see that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ becomes a quasinormable space. Finally, since $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \subset (L_p([0,1]))^{\mathbb{N}}$ (see the proof of (d)), we conclude the proof by virtue of [11, Corollary, p. 101].

(j) Since the Fréchet space $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ is a quojection (we know that this space is isomorphic to $l_p^{\mathbb{N}}$, see [31] or [14]) it suffices to apply [46, Theorems 5.2 and 1.8]. □

Remark 4.2. 1. Concerning Theorem 4.1(c) let us recall that a large number of standard spaces of test functions are isomorphic to s or $s^{\mathbb{N}}$. For example, $\mathcal{S}(\mathbb{R}^n) \simeq s$ [42,25], $\mathcal{D}(K) \simeq s$ (K is a compact set in $\mathbb{R}^n$ such that $\mathring{K} \neq \emptyset$; see [42,45]), $C^\infty(\Omega) \simeq s^{\mathbb{N}}$ (Ω is an open set in $\mathbb{R}^n$; see [42,45]), $C^\infty(V) \simeq s$ (V is an n-dimensional compact C^∞-differentiable manifold; see [42]), $C^\infty(W) \simeq s^{\mathbb{N}}$ (W is an n-dimensional C^∞-differentiable manifold not compact and countable at infinity; see [42]).

2. It is well known (see [25]) that the space $A(\mathbb{C}^d)$ of all entire analytic functions cannot be isomorphic to either s or $s^{\mathbb{N}}$ but it is isomorphic to a complemented subspace of s. However, if p and k are as in Theorem 4.1(c), $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, A(\mathbb{C}^d))$ and $(\mathcal{D}_{L^p})^{\mathbb{N}}$ are isomorphic. In fact, we know that

$$\mathcal{B}_{p,k}^{\mathrm{loc}}\bigl(\Omega, A(\mathbb{C}^d)\bigr) \simeq \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega) \hat{\otimes}_\varepsilon A(\mathbb{C}^d) \simeq l_p^{\mathbb{N}} \hat{\otimes}_\varepsilon A(\mathbb{C}^d) \simeq \bigl(l_p \hat{\otimes}_\varepsilon A(\mathbb{C}^d)\bigr)^{\mathbb{N}}$$

and that $A(\mathbb{C}^d) \simeq \Lambda_\infty(\alpha)$ with $\alpha_n = n^{1/\alpha}$. But, by [47, 1.1 Proposition] (the proof given there works for any $p \geqslant 1$) we have $l_p \hat{\otimes}_\varepsilon A(\mathbb{C}^d) \simeq l_p \hat{\otimes}_\varepsilon s$, therefore $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, A(\mathbb{C}^d)) \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$.

In [19] Kaballo showed that the short sequence $0 \to N(P(D)) \to \mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega) \to \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega) \to 0$ is an (ϵL)-triple when the differential operator $P(D)$ is hypoelliptic and it does not split when $P(D)$ is elliptic (recall that a short exact sequence of locally convex spaces $0 \to E \to F \xrightarrow{q} G \to 0$ is called an (ϵL)-triple, if for every Banach space X the mapping $q \hat{\otimes}_\epsilon \mathrm{id} : F \hat{\otimes}_\epsilon X \to G \hat{\otimes}_\epsilon X$ is surjective). In the next theorem this result is extended to ω-hypoelliptic differential operators and to the vector-valued setting. The extension is essentially a consequence of results of Meise, Taylor and Vogt [24, Theorem 2.10, Corollary 2.16] (see also Vogt [46]) and Theorem 4.1. We will consider weights in the class $\mathcal{M}^*$ ($\omega \in \mathcal{M}^*$ if $\omega(x) = \sigma(|x|) \in \mathcal{M}$ and σ is as in [24, Definition 1.1]). For example, the weight $\omega(x) = |x|^\beta$ belongs to $\mathcal{M}^*$ when $0 < \beta < 1$. On the other hand, if $P(x) = \sum_{|\alpha| \leqslant m} a_\alpha x^\alpha$ is a complex polynomial in n variables then $P'(x)$ denotes the function $x \to (\sum_{|\alpha| \geqslant 0} |\partial^\alpha P(x)|^2)^{1/2}$. An open set $\Omega \subset \mathbb{R}^n$ is called P-convex (P-convex for supports in [16, Definition 10.6.1]) if to every compact set $K \subset \Omega$ there exists another compact set $K' \subset \Omega$ such that $\phi \in \mathcal{D}(\Omega)$ and $\operatorname{supp} P(-D)\phi \subset K$ implies $\operatorname{supp} \phi \subset K'$. Finally we refer the reader to [2,15,16] for the theory of linear partial differential operators.

Theorem 4.2. *Let $P(D)$ be a linear partial differential operator with constant coefficients in $\mathbb{R}^n$ ($n \geqslant 2$), Ω an open subset of $\mathbb{R}^n$, $\omega \in \mathcal{M}^*$, $k \in \mathcal{K}_\omega$ and $1 \leqslant p < \infty$.*

1. *If $P(D)$ is ω-hypoelliptic and Ω is P-convex, then the short sequence*

$$0 \to N\bigl(P(D)\bigr) \to \mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega) \xrightarrow{P(D)} \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega) \to 0$$

is exact, it does not split and it is an (ϵL)-triple (here $N(D)$ is the kernel of $P(D)$). The dual sequence

$$0 \to \bigl(\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)\bigr)' \xrightarrow{{}^tP(D)} \bigl(\mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega)\bigr)' \to \bigl(N\bigl(P(D)\bigr)\bigr)' \to 0$$

is topologically exact and it does not split either.

2. *If $P(D)$ is ω-hypoelliptic, Ω is $\widetilde{P}$-convex and $1 < p < \infty$, there exists a short sequence*

$$0 \to \mathcal{B}_{p,k}^{\mathrm{c}}(\Omega) \to \mathcal{B}_{p,k/P'}^{\mathrm{c}}(\Omega) \to \bigl(N\bigl(P(-D)\bigr)\bigr)' \to 0$$

which is topologically exact and it does not split.

3. *If $P(D)$ is ω-hypoelliptic, Ω is P-convex and E is a nuclear Fréchet space, the short sequence*

$$0 \to N\bigl(P_E(D)\bigr) \to \mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega, E) \xrightarrow{P_E(D)} \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \to 0$$

is exact and an (ϵL)-triple (here $P_E(D) : \mathcal{D}'_\omega(\Omega, E) \to \mathcal{D}'_\omega(\Omega, E)$ is defined by $\langle \varphi, P_E(D)T \rangle = \langle P(-D)\varphi, T \rangle$ for all $\varphi \in \mathcal{D}_\omega(\Omega)$ and all $T \in \mathcal{D}'_\omega(\Omega, E)$).

Proof. 1. It follows from the hypothesis and [2, Theorem 3.3.3] that $P(D)$ is a continuous linear operator of $\mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega)$ (resp. $\mathcal{E}_\omega(\Omega)$) onto $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)$ (resp. $\mathcal{E}_\omega(\Omega)$). Furthermore $N(P(D))$ coincides, algebraic and topologically, with the subspace $\{f \in \mathcal{E}_\omega(\Omega)\colon\ P(D)f = 0\}$ of $\mathcal{E}_\omega(\Omega)$ in virtue of [2, Theorem 4.1.1], the embedding $\mathcal{E}_\omega(\Omega) \hookrightarrow \mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega)$ [2, Theorem 2.3.5] and the closed graph theorem; thus $N(P(D))$ is a nuclear Fréchet space ($\mathcal{E}_\omega(\Omega)$ is nuclear by [45]). It is then clear that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N(P(D)) & \longrightarrow & \mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega) & \xrightarrow{P(D)} & \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N(P(D)) & \longrightarrow & \mathcal{E}_\omega(\Omega) & \xrightarrow{P(D)} & \mathcal{E}_\omega(\Omega) & \longrightarrow & 0
\end{array}$$

is commutative. Since, by the Meise–Taylor–Vogt theorem [24, Theorem 2.10, Corollary 2.16], the second row of this diagram does not split, it follows that the first row does not split either (see [32]). The first row is an (ϵL)-triple by the nuclearity of $N(P(D))$ and [19, Theorem 2.9]. Next consider the dual diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega))' & \xrightarrow{{}^tP(D)} & (\mathcal{B}_{p,kP'}^{\mathrm{loc}}(\Omega))' & \longrightarrow & (N(P(D)))' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathcal{E}'_\omega(\Omega) & \xrightarrow{{}^tP(D)} & \mathcal{E}'_\omega(\Omega) & \longrightarrow & (N(P(D)))' & \longrightarrow & 0.
\end{array}$$

This diagram is also commutative and since $N(P(D))$ is quasinormable (see e.g. [25, Corollary 28.5]) its rows are topologically exact sequences (use [25, Proposition 26.18]). Its second row does not split because the second row of the previous diagram does not split either and the space $\mathcal{E}_\omega(\Omega)$ is reflexive (see [32]). Hence it follows that the first row does not split either.

2. Since $\widetilde{P}(D) = P(-D)$ and Ω is $\widetilde{P}$-convex, it follows from 1 that the short sequence $0 \to (\mathcal{B}^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega))' \xrightarrow{{}^tP(D)} (\mathcal{B}^{\mathrm{loc}}_{p',\frac{1}{\tilde{k}}\widetilde{P}'}(\Omega))' \to (N(P(-D)))' \to 0$ is topologically exact and it does not split. Using the isomorphisms [31, Theorem 3.2] $(\mathcal{B}^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega))' \simeq \mathcal{B}^{\mathrm{c}}_{p,k}(\Omega)$, $(\mathcal{B}^{\mathrm{loc}}_{p',\frac{1}{\tilde{k}}\widetilde{P}'}(\Omega))' \simeq \mathcal{B}^{\mathrm{c}}_{p,k/P'}(\Omega)$ one easily concludes the proof.

3. According to 1 we have the exact sequence $0 \to N(P(D)) \to \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega) \xrightarrow{P(D)} \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \to 0$ then also $0 \to N(P(D)) \hat{\otimes}_\varepsilon E \to \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega) \hat{\otimes}_\varepsilon E \xrightarrow{P(D) \hat{\otimes}_\varepsilon \mathrm{id}} \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon E \to 0$ is exact (the second arrow is injective by [22, Proposition 5, p. 277] and $P(D) \hat{\otimes}_\varepsilon \mathrm{id}$ is surjective by the nuclearity of E and [22, Proposition 7, p. 189]). On the other hand from [22, Proposition 7, p. 189] and [22, Proposition 7, p. 174] it follows that $N(P_E(D)) = N(P(D) \hat{\otimes}_\varepsilon \mathrm{id}) = \overline{N(P(D)) \otimes E}^{\mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega) \hat{\otimes}_\varepsilon E} = N(P(D)) \hat{\otimes}_\varepsilon E$. Furthermore, by virtue of Theorem 4.1(a), we have $\mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega) \hat{\otimes}_\varepsilon E = \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega, E)$ and $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \hat{\otimes}_\varepsilon E = \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$. Therefore we have the exact sequence $0 \to N(P_E(D)) \to \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega, E) \xrightarrow{P_E(D)} \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \to 0$. Finally the nuclearity of $N(P_E(D))$ and Theorem 2.9 in [19] show that this sequence is also an (ϵL)-triple. □

Remark. For results on the splitting of partial differential operators between $\mathcal{B}^{\mathrm{loc}}_{p,k}$-spaces in the temperate case see also [14].

Acknowledgments

The authors wish to thank the referees for the bibliographic references [14,47], and for their comments and suggestions leading to the improvement of this paper. In particular, credit must be given to one of them for the remark immediately following Lemma 3.2 and for providing us with [47, 1.1 Proposition] which allowed us to prove Remark 4.2.2.

It is also a pleasure for us to thank O. Blasco and J.M.F. Castillo for several very helpful discussions about Remark 4.1.

References

[1] C. Bessaga, A. Pelczyński, Selected Topics in Infinity-Dimensional Topology, Monogr. Math., vol. 58, PWN, 1975.
[2] G. Björck, Linear partial differential operators and generalized distributions, Ark. Mat. 6 (1966) 351–407.
[3] J.M.F. Castillo, J.C. Díaz, J. Motos, On the Fréchet space L_{p^-}, Manuscripta Math. 96 (1998) 219–230.
[4] P. Cembranos, J. Mendoza, Banach Spaces of Vector-Valued Functions, Lecture Notes in Math., vol. 1676, Springer-Verlag, 1997.
[5] P. Cembranos, J. Mendoza, $l_\infty(l_1)$ and $l_1(l_\infty)$ are not isomorphic, J. Math. Anal. Appl. 341 (2008) 295–297, doi:10.1016/j.jmaa.2007.10.027.
[6] J.C. Díaz, A note on isomorphisms between powers of Banach spaces, Collect. Math. 38 (1987) 137–140.
[7] J. Diestel, J.J. Uhl, Vector Measures, Math. Surveys Monogr., vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[8] A. Favini, Su una estensione del metodo d'interpolazione complesso, Rend. Sem. Mat. Univ. Padova 50 (1972) 223–249.
[9] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Math. Stud., vol. 116, North-Holland, Amsterdam, 1985.
[10] H.G. Garnir, M. De Wilde, J. Schmets, Analyse Fonctionnelle, vols. II, III, Birkhäuser, Basel/Stuttgart, 1972/1973.
[11] A. Grothendieck, Sur les spaces (F) et (DF), Summa Bras. Math. 3 (1954) 57–123.
[12] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc., vol. 16, Amer. Math. Soc., Providence, RI, 1955.
[13] O. Grudzinski, Temperierte Beurling-Distributionen, Math. Nachr. 91 (1979) 297–320.
[14] V. Hermanns, Zur Existenz von Rechtsinversen linearer partieller Differentialoperatoren mit konstanten Koeffizienten auf $\mathcal{B}^{\mathrm{loc}}_{p,k}$-Räumen, Dissertation, Wuppertal, 2005.
[15] L. Hörmander, Linear Partial Differential Operators, Grundlehren, vol. 116, Springer-Verlag, Berlin, 1963.
[16] L. Hörmander, The Analysis of Linear Partial Differential Operators II, Grundlehren, vol. 257, Springer-Verlag, Berlin, 1983.
[17] H. Jarchow, Locally Convex Spaces, Teubner-Verlag, Stuttgart, 1981.
[18] D. Jornet, A. Oliaro, Functional composition in $B_{p,k}$ spaces and applications, Math. Scand. 99 (2) (2006) 175–203.
[19] W. Kaballo, Lifting theorems for vector-valued functions and the ϵ-tensor product, in: K.D. Bierstedt, B. Fuchssteiner (Eds.), Functional Analysis: Surveys and Recent Results, North-Holland, 1977, pp. 149–166.
[20] H. Komatsu, Ultradistributions I. Structure theorems and a characterization, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973) 25–105.
[21] H. Komatsu, Ultradistributions III. Vector-valued ultradistributions and the theory of kernels, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982) 653–718.
[22] G. Köthe, Topological Vector Spaces, II, Springer-Verlag, Berlin, 1979.
[23] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I. Sequence Spaces, Springer-Verlag, Berlin, 1977.
[24] R. Meise, B.A. Taylor, D. Vogt, Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions, Math. Nachr. 180 (1996) 213–242.
[25] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxf. Grad. Texts Math., vol. 2, Clarendon Press, Oxford, 1997.
[26] G. Metafune, V.B. Moscatelli, Complemented subspaces of sums and products of Banach spaces, Ann. Mat. Pura Appl. 153 (1988) 175–190.
[27] G. Metafune, V.B. Moscatelli, On the space $l_{p^+} = \bigcap_{q>p} l_q$, Math. Nachr. 147 (1990) 7–12.
[28] G. Metafune, V.B. Moscatelli, Quojection and prequojections, in: T. Terzioglu (Ed.), Advances in the Theory of Fréchet Spaces, in: NATO ASI Ser. C, vol. 287, Kluwer Academic Publishers, 1989, pp. 235–254.
[29] J. Motos, M.J. Planells, C. Talavera, On weighted L_p-spaces of vector-valued entire analytic functions, Math. Z. 260 (2008) 451–472, doi:10.1007/s00209-007-0283-4.
[30] J. Motos, M.J. Planells, C. Talavera, On some iterated weighted spaces, J. Math. Anal. Appl. 338 (2008) 162–174, doi:10.1016/j.jmaa.2007.05.009.
[31] J. Motos, M.J. Planells, On sequence space representations of Hörmander–Beurling spaces, J. Math. Anal. Appl. 348 (2008) 395–403, doi:10.1016/j.jmaa.2008.07.031.

[32] V.B. Palamodov, Homological methods in the theory of locally convex spaces, Russian Math. Surveys 26 (1) (1971) 1–64.
[33] A. Pelczyński, Projections in certain Banach spaces, Studia Math. 19 (1960) 209–228.
[34] A. Pelczyński, Selected problems on the structure of complemented subspaces of Banach spaces, in: J.M.F. Castillo, W.B. Johnson (Eds.), Methods in Banach Space Theory, Proc. of the V Conference on Banach Spaces, Cáceres, Spain, September 2004, in: London Math. Soc. Lecture Note Ser., vol. 337, Cambridge Univ. Press, 2006, pp. 341–354.
[35] A. Pelczyński, H. Rosenthal, Localization techniques in L^p spaces, Studia Math. 52 (1975) 263–289.
[36] M.J. Planells, J. Villegas, On Hörmander–Beurling spaces $\mathcal{B}^c_{p,k}(\Omega, E)$, J. Appl. Anal. 13 (1) (2007) 97–116.
[37] M.J. Planells, J. Villegas, A note on traces of Hörmander spaces, Bol. Soc. Mat. Mexicana 14 (1) (2008), in press.
[38] H. Rosenthal, On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measure μ, Acta Math. 124 (1970) 205–248.
[39] J.L. Rubio de Francia, Martingale and integral transforms of Banach space valued functions, in: J. Bastero, M. San Miguel (Eds.), Probability and Banach Spaces, in: Lecture Notes in Math., vol. 1221, Springer, Berlin, 1986, pp. 195–222.
[40] M. Schechter, Complex interpolation, Compos. Math. 18 (1967) 117–147.
[41] H.I. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, Wiley, Chichester, 1987.
[42] M. Valdivia, Topics in Locally Convex Spaces, North-Holland Math. Stud., vol. 67, North-Holland, Amsterdam, 1982.
[43] M. Valdivia, On the space $\mathcal{D}_{L^p}$, in: Mathematical Analysis and Applications, Part B, in: Adv. Math. Suppl. Stud., vol. 7B, Acad. Press, 1981, pp. 759–767.
[44] J. Villegas, On vector-valued Hörmander–Beurling spaces, Extracta Math. 18 (2003) 91–106.
[45] D. Vogt, Sequence space representations of spaces of test functions and distributions, in: G.I. Zapata (Ed.), Functional Analysis, Holomorphy and Approximations Theory, in: Lect. Notes Pure Appl. Math., vol. 83, 1983, pp. 405–443.
[46] D. Vogt, Some results on continuous linear maps between Fréchet spaces, in: K.D. Bierstedt, B. Fuchssteiner (Eds.), Functional Analysis: Surveys and Recents Results III, in: North-Holland Math. Stud., vol. 90, 1984, pp. 349–381.
[47] D. Vogt, On two classes of (F)-spaces, Arch. Math. 45 (1985) 255–266.