On the Maximal Operator Ideal Associated with a Tensor Norm Defined by Interpolation Spaces

M. E. Puerta and G. Loaiza

Abstract. The classical approach to studying operator ideals using tensor norms mainly focuses on those tensor norms and operator ideals defined by means of $\ell_p$ spaces. In a previous paper, an interpolation space, defined via the real method and using $\ell_p$ spaces, was used to define a tensor norm, and the associated minimal operator ideals were characterized. In this paper, the next natural step is taken, that is, the corresponding maximal operator ideals are characterized. As an application, necessary and sufficient conditions for the coincidence of the maximal and minimal ideals are given. Finally, the previous results are used in order to find some new metric properties of the mentioned tensor norm.

1 Introduction

In [17, 18] Matter introduced the notion of $(p, \sigma)$-absolutely continuous operators ($1 \leq p < \infty$, $0 < \sigma < 1$) between Banach spaces. For each choice of $p$ and $\sigma$, these operators form an ideal. One of the most significant features of these ideals is their close relationship with typical calculation procedures in interpolation space theory. This is the case in [17], where interpolation spaces are defined by the real method and are used as essential tools in the characterization of some of the operators Matter introduced. Even though the ideals treated by Matter are all maximal, there is no attempt in [17, 18] to exploit the important relation between these ideals and tensor norms. This relation was used in [16] for the case $1 < p < \infty$, where the ideals of nuclear and integral operators corresponding to a tensor norm were characterized by means of factorizations through interpolation spaces. The case $p = 1$ was studied in [2]. These works put in evidence the role interpolation spaces may play in defining tensor norms that are more general than the classical ones. Each new tensor norm poses the problem of characterizing the associated (in the sense of [7]) nuclear and integral operator ideals. In [19], the authors introduced a tensor norm $g_\xi$ by means of an interpolation space $\xi$ and characterized the associated minimal operator ideals. The aim of the present paper is to give a characterization of the maximal operator ideal associated to this tensor norm, apply this and the characterization obtained in [19] to the study of the coincidence between components of the maximal and minimal operator ideals, and use these coincidence results to prove some metric properties of $g_\xi$ and its dual.

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Notation is standard. We will always consider Banach spaces over the real field, since we shall use results in the theory of Banach lattices. The canonical inclusion map from Banach space $E$ into the bidual $E''$ will be denoted by $I_E$. In general, if $E$ is a subspace of $F$, the inclusion of $E$ into $F$ is denoted by $I_{E,F}$. The set of finite dimensional subspaces of a normed space $E$ will be denoted by $FIN(E)$.

We recall the more relevant aspects on Banach lattices (we refer the reader to [1]). A Banach lattice $E$ is order complete or Dedekind complete if every order bounded set in $E$ has a least upper bound in $E$, and it is order continuous if every order convergent filter is norm convergent. Every dual Banach sequence lattice $E'$ is order complete and all reflexive spaces are even order continuous. A linear map $T$ between Banach lattices $E$ and $F$ is said to be positive if $T(x) \geq 0$ in $F$ for every $x \in E, x \geq 0$. $T$ is called order bounded if $T(A)$ is order bounded in $F$ for every order bounded set $A$ in $E$.

Let $\omega$ be the vector space of all scalar sequences and $\varphi$ its subspace of the sequences with finitely many nonzero coordinates. A sequence space $\xi$ is a linear subspace of $\omega$ containing $\varphi$ provided with a topology finer than the topology of coordinatewise convergence. A Banach sequence space will be a sequence space $\xi$ provided with a norm that makes it a Banach lattice and an ideal in $\omega$, i.e., such that if $|x| \leq |y|$ with $x \in \omega$ and $y \in \xi$, then $x \in \xi$ and $\|x\| \leq \|y\|_\xi$. A sectional subspace $S_k(\xi), k \in \mathbb{N}$, is the topological subspace of $\xi$ of those sequences $(\alpha_i)$ such that $\alpha_i = 0$ for every $i \geq k$. Clearly $S_k(\xi)$ is 1-complemented in $\xi$. A Banach sequence space $\xi$ will be called regular whenever the sequence $\{e_i\}_{i=1}^\infty$, where $e_i := (\delta_{ij})_j$ (Kronecker’s delta) is a Schauder base in $\xi$.

Following [19], we establish some notation. Given a Banach space $E$ and the compatible couple $\lambda = (\ell_{p_0}, \ell_{p_1})$ with $1 < p_0 \leq p_1 < \infty$ and its dual couple $\lambda' = (\ell_{p_0}', \ell_{p_1}')$, we will denote by $\lambda_{q,\beta}, \lambda_{q,\infty}$ the interpolation spaces obtained using methods $\beta$ and $\infty$, respectively, from the couple $\lambda$ (see [3]). Analogously, we denote by $\lambda'_{q,\beta}, \lambda'_{q,\infty}$ the interpolation spaces obtained from the couple $\lambda'$. With this notation, if $1 \leq q < \infty$ and $0 < \theta < 1$, dual spaces $(\lambda_{q,\beta})' = ((\ell_{p_0}, \ell_{p_1}), \lambda_{q,\beta})' = (\ell_{p_0}', \ell_{p_1}')_{\lambda_{q,\beta}} = \lambda'_{q,\infty}$. We can see that $(\lambda_{q,\beta})' \neq \lambda'_{q,\beta}$. Recalling the duality for method $\infty$, since $\lambda_{q,\beta}$ is reflexive for $1 \leq q < \infty$, we get the isometry

$$(\lambda'_{q,\infty})' = \lambda_{q,\beta}.$$

We say that a sequence $(x_n)_{n=1}^\infty \in E$ is strongly $\lambda_{q,\beta}$-summing if $(\|x_n\|) \in \lambda_{q,\beta}$, and we write $\pi_{\lambda_{q,\beta}}((x_n)) := \|(\|x_n\|)\|_{\lambda_{q,\beta}}$. It is said to be weakly $\lambda_{q,\beta}$-summing if $\varepsilon_{\lambda_{q,\beta}}((x_n)) := \sup_{\|\alpha\| \leq 1} \|(\langle\alpha, x_n\rangle)\|_{\lambda_{q,\beta}}$. We denote by $\lambda_{q,\beta}(E)$ (resp. $\lambda_{q,\beta}(E)$) the space of all strongly (resp. weakly) $\lambda_{q,\beta}$-summing in $E$ with the norm $\pi_{\lambda_{q,\beta}}(\cdot)$ (resp. $\varepsilon_{\lambda_{q,\beta}}(\cdot)$).

Let $(\Omega, \Sigma, \mu)$ be a measure space. We denote by $L_0(\mu)$ the space of equivalence classes, modulo equality $\mu$-almost everywhere, of $\mu$-measurable real-valued functions, endowed with the topology of local convergence in measure, and the space of all equivalence classes of $\mu$-measurable $X$-valued functions is denoted by $L_0(\mu, X)$. By a Köthe function space $K(\mu)$ on $(\Omega, \Sigma, \mu)$, we shall mean an order dense ideal of $L_0(\mu)$, which is equipped with a norm $\|\cdot\|_{K(\mu)}$ that makes it a Banach lattice (if $f \in L_0(\mu)$ and $g \in K(\mu)$, then $f \in K(\mu)$ with $\|f\|_{K(\mu)} \leq |g|$, then $f \in K(\mu)$ with $\|f\|_{K(\mu)} \leq |g|_{K(\mu)}$).
Similarly, \( \mathcal{K}(\mu, X) = \{ f \in L_0(\mu, X) : \| f(\cdot) \|_X \in \mathcal{K}(\mu) \} \), endowed with the norm \( \| f \|_{\mathcal{K}(\mu, X)} = \| f(\cdot) \|_X \|_{\mathcal{K}(\mu)} \).

For the theory of operator ideals and tensor norms, we refer to the books [21] and [7] of Pietsch and Defant and Floret respectively.

If \( E \) and \( F \) are Banach spaces and \( \alpha \) is a tensor norm, then \( E \otimes_\alpha F \) represents the space \( E \otimes F \) endowed with the \( \alpha \)-normed topology. The completion of \( E \otimes_\alpha F \) is denoted by \( E \hat{\otimes}_\alpha F \), and the norm of \( z \) in \( E \hat{\otimes}_\alpha F \) by \( \alpha(z; E \otimes F) \). If there is no risk of confusion, we write \( \alpha(z) \) instead of \( \alpha(z; E \otimes F) \).

## 2 The Tensor Norm \( g_{\lambda,q} \) and Associated \( \lambda_{q,\beta} \)-Nuclear Operators

The more natural approach to defining a tensor norm analogous to Saphar’s tensor norm is as follows. Let \( E \) and \( F \) be Banach spaces and \( z \in E \otimes F \), we define

\[
g_{\lambda,q} (z; E \otimes F) := \inf \pi_{\lambda,q} ((x_n)) \varepsilon_{\lambda',\infty} ((y_n))
\]

taking the infimum over all representations of \( z \) as \( \sum_{n=1}^m x_n \otimes y_n \). We will write \( g_{\lambda,q} (z) \) instead of \( g_{\lambda,q} (z; E \otimes F) \) if there is no possibility of confusion.

It is possible that for some interpolation space \( \Lambda_{q,\beta} \), the functional \( g_{\lambda,q} \) does not satisfy the triangle inequality, but it is always a reasonable quasi norm on \( E \otimes F \), see [6, 9]. We denote by \( E \otimes_{\lambda,q} F \) the quasi normed space and by \( E \hat{\otimes}_{\lambda,q} F \) the corresponding quasi Banach space.

To obtain a tensor norm \( g_{\lambda,q}^\ell \) in [19], we took the Minkowski functional, denoted \( g_{\lambda,q}^\ell (z; E \otimes F) \), of the absolutely convex hull of the unit closed ball \( B_{\lambda,q} := \{ z \in E \otimes F / g_{\lambda,q} (z) \leq 1 \} \) of the quasi norm \( g_{\lambda,q} \) in \( E \otimes F \), such that

\[
g_{\lambda,q}^\ell (z; E \otimes F) := \inf \sum_{i=1}^n \pi_{\lambda,q}((x_{ij})) \varepsilon_{\lambda',\infty} ((y_{ij}))
\]

taking the infimum over all representations of \( z \) as \( \sum_{i=1}^n \sum_{j=1}^m x_{ij} \otimes y_{ij} \). Again, we will write \( g_{\lambda,q} \) instead of \( g_{\lambda,q}^\ell (z; E \otimes F) \) if there is no possibility of confusion.

It is easy to see that \( g_{\lambda,q}^\ell \) is a tensor norm on the class of all Banach spaces, using [7, Criterion 12.2] and bearing in mind that if \( 1 \leq q \leq \infty \), then

\[
\| (e_i) \|_{\lambda,q} = c_q^{-1} y \| (e_i) \|_{\lambda,q} = c_q,
\]

where

\[
c_q = \left( \int_0^\infty \left( \min \{1,t\} \frac{\varepsilon_{\lambda',\infty} ((y_i))}{t} \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q}}
\]

for every \( i \in \mathbb{N} \), in relation to method \( \mathcal{K} \) and that for all \( z \in E \otimes F, g_{\lambda,q}^\ell (z; E \otimes F) \leq g_{\lambda,q} (z; E \otimes F) \). We denote by \( E \hat{\otimes}_{\lambda,q} F \) the corresponding Banach space.

Proceeding as in [6, 23], it is easy to see that if \( z \in E \hat{\otimes}_{\lambda,q} F \), there are \( (x_i)_{i=1}^\infty \in \lambda_{q,\beta} [E] \) and \( (y_i)_{i=1}^\infty \in \lambda'_{q,\infty} (F) \) such that \( \pi_{\lambda,q} ((x_i)) \varepsilon_{\lambda',\infty} ((y_i)) < \infty \) and \( z = \sum_{i=1}^\infty x_i \otimes y_i \).

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\[ \sum_{i=1}^{\infty} x_i \otimes y_i. \] Moreover, the quasi norm of \( z \) in \( E \hat{\otimes}_{\lambda_{q,\beta}} F \) (again denoted by \( g_{\lambda_{q,\beta}}(z) \)) is given by

\[ g_{\lambda_{q,\beta}}(z) = \inf \pi_{\lambda_{q,\beta}}((x_i)) \varepsilon_{\lambda'_{q,\infty}}((y_i)) \]

taking the infimum over all such representations of \( z \) as \( \sum_{n=1}^{m} x_n \otimes y_n \). Similarly, if \( z \in E \hat{\otimes}_{\lambda_{q,\beta}} F \) then \( z \) can be represented as \( z = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \), where \( (x_{ij})_{i=1}^{\infty} \in \lambda_{q,\beta}[E] \) for each \( j \in \mathbb{N}, (y_{ij})_{i=1}^{\infty} \in \lambda'_{q,\infty}(F) \) for each \( j \in \mathbb{N} \) and

\[ \sum_{j=1}^{\infty} \pi_{\lambda_{q,\beta}}((x_{ij})) \varepsilon_{\lambda'_{q,\infty}}((y_{ij})) < \infty. \]

Moreover, the norm of \( z \) in \( E \hat{\otimes}_{\lambda_{q,\beta}} F \) is

\[ g'_{\lambda_{q,\beta}}(z) = \inf \sum_{j=1}^{\infty} \pi_{\lambda_{q,\beta}}((x_{ij})) \varepsilon_{\lambda'_{q,\infty}}((y_{ij})), \]

taking the infimum over all representations of \( z \) as \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \).

The topology defined by the quasi norm \( g_{\lambda_{q,\beta}} \) on \( E \otimes F \) is normable with norm equivalent to \( g'_{\lambda_{q,\beta}} \). In fact, being \( \lambda_{q,\beta} \) a reflexive interpolation space for \( 1 \leq q < \infty \) and following the arguments of [6, Proposition 16], we consider the bilinear onto map \( R: \lambda_{q,\beta}[E] \times \lambda'_{q,\infty}(F) \to E \hat{\otimes}_{\lambda_{q,\beta}} F \), such that \( R((x_i),(y_i)) = \sum_{i=1}^{\infty} x_i \otimes y_i \). \( R \) is continuous with quasi norm less than or equal to one. Then there exists a unique linear and continuous map \( \lambda_{q,\beta}[E] \otimes_{\pi} \lambda'_{q,\infty}(F) \to E \hat{\otimes}_{\lambda_{q,\beta}} F \), see [25]. This map can be extended to a continuous linear and onto map \( \lambda_{q,\beta}[E] \hat{\otimes}_{\pi} \lambda'_{q,\infty}(F) \to E \hat{\otimes}_{\lambda_{q,\beta}} F \) which is open by the open mapping theorem. Then \( E \hat{\otimes}_{\lambda_{q,\beta}} F \) is isomorphic to a quotient of a Banach space and so it is a Banach space itself. In this way, there is a norm \( w_{\lambda_{q,\beta}} (\cdot; E \otimes F) \) equivalent to the quasi norm \( g_{\lambda_{q,\beta}} (\cdot; E \otimes F) \); furthermore, it is easy to see that \( w_{\lambda_{q,\beta}} (\cdot; E \otimes F) \), \( g_{\lambda_{q,\beta}} (\cdot; E \otimes F) \) and \( g'_{\lambda_{q,\beta}} (\cdot; E \otimes F) \) are equivalent with \( g'_{\lambda_{q,\beta}} (\cdot; E \otimes F) \leq w_{\lambda_{q,\beta}} (\cdot; E \otimes F) \). Given the last equivalence, \( g_{\lambda_{q,\beta}} \) seems appropriate for our purposes, but we need \( g'_{\lambda_{q,\beta}} \) for our main results.

To introduce \( \lambda_{q,\beta} \)-nuclear operators, bearing in mind that every representation of \( z \in E' \hat{\otimes}_{\lambda'_{q,\beta}} F \) as \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \), defines a map \( T_z \in \mathcal{L}(E,F) \) such that for \( x \in E \),

\[ T_z(x) := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (x_{ij},x)y_{ij}, \]

Furthermore, \( T_z \) is well defined and independent of the chosen representation for \( z \). Let \( \Phi_{EF}: E' \hat{\otimes}_{\lambda'_{q,\beta}} F \to \mathcal{L}(E,F) \) be defined by \( \Phi_{EF}(z) := T_z \).

**Definition 2.1** An operator between Banach spaces \( T: E \to F \) is said to be \( \lambda_{q,\beta} \)-nuclear if \( T = \Phi_{EF}(z) \) for some \( z \in E' \hat{\otimes}_{\lambda'_{q,\beta}} F \).

Given any pair of Banach spaces \( E \) and \( F \), the space of the \( \lambda_{q,\beta} \)-nuclear operators \( T: E \to F \) endowed with the topology of the norm

\[ N_{\lambda_{q,\beta}}(T) := \inf \{ g_{\lambda_{q,\beta}}(z)/\Phi_{EF}(z) = T \}, \]
or with the equivalent quasi-norm $N_{\lambda, q}^\epsilon(T) := \inf \{ g_{\lambda, q}^\epsilon(z) / \Phi_{EF}(z) = T \}$, is denoted by $N_{\lambda, q}(E, F)$. Also $(N_{\lambda, q}(E, F), N_{\lambda, q}^\epsilon)$ denotes a component of the minimal Banach operator ideal $(N_{\lambda, q}, N_{\lambda, q}^\epsilon)$ associated to the tensor norm $g_{\lambda, q}^\epsilon$. As in [19], we obtain the following result.

**Theorem 2.2** Let $E, F$ be any pair of Banach spaces and an operator $T \in \mathcal{L}(E, F)$. Then the following are equivalent:

(i) $T$ is $\lambda_{q, \beta}$-nuclear.

(ii) $T$ factors continuously in the following way:

```
\begin{array}{c}
\ell_1[\lambda_{q, \beta}] \\
B \\
A \\
E \\
\end{array}
\begin{array}{c}
T \\
C \\
F \\
\end{array}
```

where $B$ is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \lambda_{q, \beta}$.

Furthermore, $N_{\lambda, q}(T) = \inf \{ \| C \| \| B \| \| A \| \}$, infimum taken over all such factors.

(iii) $T$ factors continuously in the following way:

```
\begin{array}{c}
\ell_1[\lambda_{q, \beta}] \\
B \\
A \\
E \\
\end{array}
\begin{array}{c}
T \\
C \\
F \\
\end{array}
```

where $B$ is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_1[\lambda_{q, \beta}]$.

Furthermore, $N_{\lambda, q}^\epsilon(T) = \inf \{ \| C \| \| B \| \| A \| \}$, infimum taken over all such factors.

There are other important operator ideals associated with $g_{\lambda, q}$ and $g_{\lambda, q}^\epsilon$.

**Definition 2.3** Let $T \in \mathcal{L}(E, F)$. We say that $T$ is $\lambda_{q, \beta}$-absolutely summing if a real number $C > 0$ exists such that for all sequences $(x_i)$ in $E$, with $\varepsilon_{\lambda_{q, \beta}}((x_i)) < \infty$, the following is satisfied:

\begin{equation}
\| (T(x_i)) \|_{\lambda_{q, \beta}} \leq C \varepsilon_{\lambda_{q, \beta}}((x_i)).
\end{equation}

For $\mathcal{P}_{\lambda_{q, \beta}}(E, F)$, we denote the Banach ideal of the $\lambda_{q, \beta}$-absolutely summing operators $T : E \rightarrow F$ endowed with the topology of the norm $\mathcal{P}_{\lambda_{q, \beta}}(T) := \inf C$, taking the infimum over all $C$ that satisfies (2.1).

**Theorem 2.4** Let $E$ and $F$ be Banach spaces. Then $(E \otimes g_{\lambda_{q, \beta}}^\epsilon F)' = \mathcal{P}_{\lambda_{q, \beta}}'(F, E')$ isometrically.
3 \( \lambda_{q,d} \)-Integral Operators

In this paper, we give a characterization of the maximal operator ideal by considering the structure of finite dimensional subspaces of the interpolation spaces involved. The behavior of the interpolation sequences spaces under ultraproducts is also crucial, see [14, 15].

For ultraproducts of Banach spaces, we refer the reader to [11]. We only set the notation we will use. Let \( D \) be a nonempty index set and \( \mathcal{U} \) a non-trivial ultrafilter in \( D \). Given a family \( \{X_d, d \in D\} \) of Banach spaces, \( (X_d)_\mathcal{U} \) denotes the corresponding ultraproduct Banach space. If every \( X_d, d \in D \), coincides with a fixed Banach space \( X \), the corresponding ultraproduct is called an ultrapower of \( X \) and is denoted by \( (X)_\mathcal{U} \). Recall that if every \( X_d, d \in D \) is a Banach lattice, \( (X_d)_\mathcal{U} \) has a canonical order that makes it a Banach lattice. If we have another family of Banach spaces \( \{Y_d, d \in D\} \) and a family of operators \( \{T_d \in \mathcal{L}(X_d, Y_d), d \in D\} \) such that \( \sup_{d \in D} \|T_d\| < \infty \), then \( (T_d)_\mathcal{U} \in \mathcal{L}((X_d)_\mathcal{U}, (Y_d)_\mathcal{U}) \) denotes the canonical ultraprodct operator.

We now give a local definition that was inspired by Gordon and Lewis’ definition of local unconditional structure.

**Definition 3.1** Given a sequence \( \xi \), we say that a Banach space \( X \) has an \( S_k(\xi) \)-local unconditional structure if there exists a real constant \( c > 0 \) such that for every finite dimensional subspace \( F \) of \( X \), there is a section \( S_n(\xi) \) of \( \xi \) and linear operators \( u: F \to S_n(\xi) \) and \( v: S_n(\xi) \to X \) such that \( \|u\|\|v\| \leq c \) and \( vu = I_F \).

The constant \( c \) that appears in above definition is called \( S_k(\xi) \)-local unconditional structure constant of \( X \), and in this case we say that \( X \) has \( c\)-\( S_k(\xi) \)-local unconditional structure. If a Banach space \( X \) has \( c\)-\( S_k(\xi) \)-local unconditional structure for every \( c > C \), we say that it has \( C^+\)-\( S_k(\xi) \)-local unconditional structure.

The following definition was introduced by Pelczyński and Rosenthal in 1975 [20].

**Definition 3.2** A Banach space \( X \) has the uniform projection property if there is a \( b > 0 \) such that, for each natural number \( n \), there is a natural number \( m(n) \) such that, for every \( n \)-dimensional subspace \( M \subset X \), there exists a \( k \)-dimensional and \( b \)-complemented subspace \( Z \) of \( X \) containing \( M \) with \( k \leq m(n) \).

The constant \( b \) of the above definition is called a uniform projection property constant of \( X \), and in this case we say that \( X \) has the \( b \)-uniform projection property. If \( X \) has the \( b \)-uniform projection property for every \( b > B \), we say that \( X \) has the \( B^+ \)-uniform projection property.

Now, the uniform projection property is satisfied by \( L_p \) spaces for \( 1 < p < \infty \) [20], and since \( \lambda_{q,d} \) is an interpolation space defined by means of \( \ell_{p_1} \) and \( \ell_{p_2} \), with \( 1 < p_0 \leq p_1 < \infty \), with continuous inclusions \( \ell_{p_1} \subset \lambda_{q,d} \subset \ell_{p_2} \), the uniform projection property is satisfied by \( \lambda_{q,d} \).

Furthermore, if \( 1 \leq p \leq \infty \), then the Bochner space \( L_p(\mu, E) \) and \( \ell_p(E) \) has the \( b \)-uniform projection property if \( E \) does, see [11]. We highlight that the uniform projection property is stable under ultrapowers, see [11]. Moreover, from [22], since \( \lambda_{q,d} \) is reflexive, then every ultrapower of \( \ell_1[\lambda_{q,d}] \) (of \( \lambda_{q,d} \)) has \( 1^+\)-\( S_k(\ell_1)[\lambda_{q,d}] \)-local unconditional structure (resp. \( 1^+\)-\( S_k(\lambda_{q,d}) \)-local unconditional structure).
According to the general theory of tensor norms and operator ideals, the normed ideal of $\lambda_{q,\beta}$-integral operators $(J_{\lambda_{q,\beta}}, I_{\lambda_{q,\beta}})$ is the maximal operator ideal associated with the tensor norm $g_{\lambda_{q,\beta}}^*$ in the sense of Defant and Floret [7], or in an equivalent way, the maximal normed operator ideal associated with the normed ideal of $\lambda_{q,\beta}$-nuclear operators in the sense of Pietsch [21]. From [7], for every pair of Banach spaces $E$ and $F$, an operator $T: E \to F$ is $\lambda_{q,\beta}$-integral if and only if $J_{\beta}T \in (E \otimes (g_{\lambda_{q,\beta}}^*)'F)'$.

For every pair of Banach spaces $E, F$ we define the finitely generated tensor norm $g_{\lambda_{q,\beta}}^*$ such that if $M \in FIN(E)$ and $N \in FIN(F)$, for every $z \in M \otimes N$,

$$g_{\lambda_{q,\beta}}^*(z; M \otimes N) := \sup \{ |\langle z, w \rangle|/g_{\lambda_{q,\beta}}(w; M' \otimes N') \leq 1 \}.$$ 

Clearly, $g_{\lambda_{q,\beta}}^* = (g_{\lambda_{q,\beta}}^*)'$, since the unit ball in $M' \otimes (g_{\lambda_{q,\beta}}^*)'N'$ is the convex hull of the unit ball of $M' \otimes (J_{\lambda_{q,\beta}}(N'))'$. But we remark that $E' \otimes (J_{\lambda_{q,\beta}}(N'))'$ is an isometric subspace of $(E \otimes (g_{\lambda_{q,\beta}}^*)'F)'$ because $g_{\lambda_{q,\beta}}^*$ is finitely generated, see [7, 15.3].

In this case, we define $I_{\lambda_{q,\beta}}(T)$ to be the norm of $J_{\beta}T$ considered as an element of the topological dual of the Banach space $E \otimes (g_{\lambda_{q,\beta}}^*)'F'$. Remark that $I_{\lambda_{q,\beta}}(T) = I_{\lambda_{q,\beta}}(J_{\beta}T)$ as a consequence of $F'$ being canonically complemented in $F'''$.

First we give a non trivial example of $\lambda_{q,\beta}$-integral operators.

**Theorem 3.3** Let $(\Omega, \Sigma, \mu)$ be a measure space and let $\lambda_{q,\beta}$ be an interpolation space. Then every order bounded operator $S: L_{\infty}(\mu) \to \lambda_{q,\beta}$ and every order bounded operator $S: L_{\infty}(\mu) \to \ell_p(\lambda_{q,\beta})$ are $\lambda_{q,\beta}$-integral with $I_{\lambda_{q,\beta}}(S) = \|S\|$. 

**Proof** We will only give the proof if $S: L_{\infty}(\mu) \to \lambda_{q,\beta}$ is an order bounded operator, since the proof in the other case is similar.

The predual space of $\lambda_{q,\beta}$ is $\lambda_{q,\beta}'$, which is a regular space for $1 \leq q < \infty$, because $\lambda_{q,\beta}'$ is an interpolation space between regular spaces $\ell_{p_0}$ and $\ell_{p_1}$ with $1 < p_0 \leq p_1 < \infty$, where $\ell_{p_i}$ is dense in $\lambda_{q,\beta}'$ and $\lambda_{q,\beta}'$ is dense in $\ell_{p_i}'$. Then, the linear span $\mathcal{T}$ of the set $\{e_i, i \in \mathbb{N}\}$ is dense in $\lambda_{q,\beta}'$, and by the representation theorem of maximal operator ideals (see [7, 17.5]) and the density lemma ([7, Theorem 13.4]) we only have to see that $S \in (L_{\infty}(\mu) \otimes \lambda_{q,\beta})'$. Given $z \in L_{\infty}(\mu) \otimes \lambda_{q,\beta}$ and $\varepsilon > 0$, let $X$ and $Y$ be finite dimensional subspaces of $L_{\infty}(\mu)$ and $\mathcal{T}$ respectively such that $z \in X \otimes Y$ and

$$g_{\lambda_{q,\beta}}^*(z; X \otimes Y) \leq g_{\lambda_{q,\beta}}^*(z; L_{\infty}(\mu) \otimes \mathcal{T}) + \varepsilon.$$  

Let $\{g_i\}_{i=1}^m$ be a basis for $Y$ and let $k \in \mathbb{N}$ be such that $\forall 1 \leq s \leq m g_i = \sum_{j=1}^k c_{ij} e_j$. Then $\forall f \in X, \forall 1 \leq s \leq m$

$$\langle f, \otimes g_i \rangle = \langle f, S'(g_i) \rangle = \left\langle f, \left( \sum_{i=1}^k c_{ij} S'(e_i) \right) \right\rangle = \left\langle f \otimes \sum_{j=1}^k c_{ij} e_j, S'(e_i) \otimes e_i \right\rangle.$$  

Then if $U$ denotes the tensor $U := \sum_{i=1}^k S'(e_i) \otimes e_i \in L_{\infty}(\mu)' \otimes \lambda_{q,\beta}$, by bilinearity we get $\forall z \in X \otimes Y(z, S) = \langle U, z \rangle$.
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Given \( \nu > 0 \), for every \( 1 \leq i \leq k \) there is \( f_i \in L_\infty(\mu) \) such that \( \|f_i\| \leq 1 \) and \( \|S'(e_i)\| \leq |(S'(e_i), f_i)| + \nu \). Then \( f := \sup_{1 \leq i \leq k} f_i \) lies in the closed unit ball of \( L_\infty(\mu) \). On the other hand, \( \lambda_{q,3} \) is a dual lattice and hence it is order complete. By the Riesz–Kantorovich theorem (see [1, Theorem 1.13], for instance), the modulus \( |S| \) of the operator \( S \) exists in \( \mathcal{L}(L_\infty(\mu), \lambda_{q,3}) \). By the lattice properties of \( \lambda_{q,3} \), we have

\[
\pi_{\lambda_{q,3}}((S'(e_i))) = \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} \|S'(e_i)\| e_i \right) \\
\leq \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} |(S'(e_i), f_i)| e_i \right) + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right) \\
\leq \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} |S(f_i), e_i| e_i \right) + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right) \\
\leq \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} |S(f_i)|, e_i \right) + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right) \\
\leq \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} |S(f_i)|, e_i \right) + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right) \\
= \pi_{\lambda_{q,3}}\left( |S(f)| \right) + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right) \leq \|S\| + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right).
\]

Moreover, \( \varepsilon|_{\lambda_{q,3},\infty}((e_i)_{i=1}^{k}) \leq 1 \). Hence, denoting by \( I_X \) and \( I_Y \) the corresponding inclusion maps into \( L_\infty(\mu) \) and \( \lambda_{q,3} \) respectively, we have

\[
|\langle S, z \rangle| = |\langle U, z \rangle| = |\langle U, ((I_X)' \otimes (I_Y)')(z) \rangle| \\
\leq g_{\lambda_{q,3}}(U; X \otimes Y)g_{\lambda_{q,3}}(\mathcal{L}(X)' \otimes (I_Y)'(z); (I_X)' \otimes Y') \\
\leq g_{\lambda_{q,3}}(U; X \otimes Y)g_{\lambda_{q,3}}(\mathcal{L}(X)' \otimes (I_Y)'(z); X' \otimes Y') \\
\leq (g_{\lambda_{q,3}}(U; L_\infty \otimes \lambda_{q,3}) + \varepsilon)g_{\lambda_{q,3}}(z; L_\infty(\mu) \otimes \lambda_{q,3}'(\mathcal{X})) \\
\leq g_{\lambda_{q,3}}(z; L_\infty(\mu) \otimes \lambda_{q,3}'(\mathcal{X}))(\pi_{\lambda_{q,3}}((S'(e_i))\varepsilon|_{\lambda_{q,3},\infty}(\mathcal{X}))) + \varepsilon) \\
\leq g_{\lambda_{q,3}}(z; L_\infty(\mu) \otimes \lambda_{q,3}'(\mathcal{X}))(\|S\| + \nu \pi_{\lambda_{q,3}}\left( \sum_{i=1}^{k} e_i \right) + \varepsilon),
\]

and, \( \nu \) being arbitrary, \( |\langle S, z \rangle| \leq g_{\lambda_{q,3}}(z; L_\infty(\mu) \otimes \lambda_{q,3}'(\mathcal{X}))(\|S\| + \varepsilon). \) Finally, since \( \varepsilon \) is arbitrary, we get \( |\langle S, z \rangle| \leq g_{\lambda_{q,3}}(z; L_\infty(\mu) \otimes \lambda_{q,3}'(\mathcal{X}))(\|S\|). \) But from [1, Theorem 1.10], \( |\langle S, \chi_{\Omega} \rangle| = \sup\{|\langle S(f), f \rangle|; f \leq \chi_{\Omega}\} \) and as \( \lambda_{q,3} \) is order continuous,

\[
\|S\| = \|\langle S, \chi_{\Omega} \rangle\| = \sup\{|\langle S(f), f \rangle|; \|f\| \leq 1\} = |\langle S, z \rangle|.
\]
Then \( S \) is \( \lambda_{q,\beta} \)-integral with \( I_{\lambda_{q,\beta}}(S) \leq \| S \| \). But as \((\mathcal{J}_{\lambda_{q,\beta}}, I_{\lambda_{q,\beta}})\) is a Banach operators ideal, \( \| S \| \leq I_{\lambda_{q,\beta}}(S) \), hence \( I_{\lambda_{q,\beta}}(S) = \| S \| \).

**Corollary 3.4** Let \((\Omega, \Sigma, \mu)\) be a measure space and \( n, k \in \mathbb{N} \). Then every operator \( T : L_\infty(\mu) \to S_k(\lambda_{q,\beta}) \) and every operator \( T : L_\infty(\mu) \to S_n(\ell_1)[S_k(\lambda_{q,\beta})] \) satisfy \( I_{\lambda_{q,\beta}}(T) = \| T \| \).

**Proof** The result follows easily from Theorem 3.3 since every operator \( T : L_\infty(\mu) \to S_k(\lambda_{q,\beta}) \) \((T : L_\infty(\mu) \to S_n(\ell_1)[S_k(\lambda_{q,\beta})]\) in the other case) is order bounded and \( S_k(\lambda_{q,\beta}) \) (resp. \( S_n(\ell_1)[S_k(\lambda_{q,\beta})]\)) is reflexive, hence order continuous.

For our next theorem we need a very deep technical result of Lindenstrauss and Tzafriri [13] that gives us a kind of “uniform approximation” of finite dimensional subspaces by finite dimensional sublattices in Banach lattices.

**Lemma 3.5** ([13]) Let \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) be fixed. There is a natural number \( h(n, \varepsilon) \) such that for every Banach lattice \( X \) and every subspace \( F \subset X \) of dimension \( \dim(F) = n \), there are \( h(n, \varepsilon) \) disjoints elements \( \{ z_i, 1 \leq i \leq h(n, \varepsilon) \} \) and an operator \( A \) from \( F \) into the linear span \( G \) of \( \{ z_i, 1 \leq i \leq h(n, \varepsilon) \} \) such that

\[
\forall x \in F \| A(x) - x \| \leq \varepsilon \| x \| .
\]

**Theorem 3.6** Let \( \lambda_{q,\beta} \) be a interpolation space, regular for \( 1 \leq q < \infty \), \( G \) an abstract \( M \)-space, and \( X \) a Banach space with \( c-S_k(\lambda_{q,\beta}) \) or \( c-S_k(\ell_1)[S_k(\lambda_{q,\beta})] \)-local uniform structure. Then every operator \( T : G \to X \) is \( \lambda_{q,\beta} \)-integral and \( I_{\lambda_{q,\beta}}(T) \leq c \| T \| \).

**Proof** We will prove the case where \( X \) has \( c-S_k(\lambda_{q,\beta}) \)-local unconditional structure since the other case is similar. By the representation theorem of maximal operator ideals [see [7, 17.7]], we only need to show that \( I_{\lambda}T \in (G \otimes X')' \).

Given \( z \in G \otimes X' \) and \( \varepsilon > 0 \), let \( P \subset G \) and \( Q \subset X' \) be finite dimensional subspaces and let \( z = \sum_{i=1}^n f_i \otimes x_i' \) be a fixed representation of \( z \) with \( f_i \in P \) and \( x_i' \in Q \), \( i = 1, 2, \ldots, n \) such that

\[
g_{\lambda_{q,\beta}}(z; G \otimes X') \leq g_{\lambda_{q,\beta}}(z; P \otimes Q) \leq g_{\lambda_{q,\beta}}(z; G \otimes X') + \varepsilon.
\]

From Lemma 3.5 we have a finite dimensional sublattice \( P_1 \) of \( G \) and an operator \( A : P \to P_1 \) so that \( \forall f \in P, \| A(f) - f \| \leq \varepsilon \| f \| \). Then, if \( id_G \) denotes the identity map on \( G \), we have

\[
\| (I_{\lambda}T, z) \| = \left\| \sum_{i=1}^n \langle T(f_i), x_i' \rangle \right\| \leq \left\| \sum_{i=1}^n \langle T(id_G - A)(f_i), x_i' \rangle \right\| + \left\| \sum_{i=1}^n \langle TA(f_i), x_i' \rangle \right\|
\]

\[
\leq \varepsilon \| T \| \sum_{i=1}^n \| f_i \| \| x_i' \| + \left\| \sum_{i=1}^n \langle TA(f_i), x_i' \rangle \right\|.
\]

Let \( X_1 := T(P_1) \). As \( X \) has \( S_k(\lambda_{q,\beta}) \)-local unconditional structure, there are \( k \in \mathbb{N}, u : X_1 \to S_k(\lambda_{q,\beta}) \) and \( v : S_k(\lambda_{q,\beta}) \to X \) such that \( I_{\lambda, X} = vu \) and \( \| u \| \| v \| \leq c \).
Let $X_2 := vu(X_1)$, which is a finite dimensional subspace of $X$ containing $X_1$ and $I_{X_1} = vu$. Put $K_2: X''' 	o X' = X'''/X_2''$ be the canonical quotient map. Then

\[
\sum_{i=1}^{n} \langle T(A(f_i)), x'_i \rangle = \sum_{i=1}^{n} \langle I_{X_1} x_i, T(A(f_i)), K_2(x'_i) \rangle = \sum_{i=1}^{n} \langle vuT(A(f_i)), K_2(x'_i) \rangle
\]

\[
= \sum_{i=1}^{n} \langle uT(A(f_i)), v'K_2(x'_i) \rangle = \langle uT; \sum_{i=1}^{n} A(f_i) \otimes v'K_2(x'_i) \rangle
\]

with $\sum_{i=1}^{n} A(f_i) \otimes v'K_2(x'_i) \in P_1 \otimes (S_k(\lambda_{q,3}))'$ and $uT: P_1 \to S_k(\lambda_{q,3})$. Since $P_1$ is a reflexive abstract $M$-space, it is lattice isometric to some $L_\infty(\mu)$ space, hence, by Corollary 3.4, this map is $\lambda_{q,3}$-integral with $I_{\lambda_{q,3}}(uT) \leq \|u\||T||. Then

\[
\left| \sum_{i=1}^{n} \langle T(A(f_i)), x'_i \rangle \right| = \left| \langle uT; \sum_{i=1}^{n} A(f_i) \otimes v'K_2(x'_i) \rangle \right|
\]

\[
\leq I_{\lambda_{q,3}}(uT)g'_{\lambda_{q,3}} \left( \sum_{i=1}^{n} A(f_i) \otimes v'K_2(x'_i); P_1 \otimes S_k(\lambda_{q,3}) \right)
\]

\[
\leq \|u\||T\|\|A\||\|v'\||\|K_2\|g'_{\lambda_{q,3}}(z; P \otimes Q)
\]

\[
\leq (1 + \varepsilon)c\|T\|g'_{\lambda_{q,3}}(z; G \otimes X') + \varepsilon,
\]

and since $\varepsilon$ is arbitrary, we obtain $\langle J_{k} T, z \rangle \leq c\|T\|g'_{\lambda_{q,3}}(z; G \otimes X')$. 

Concerning the characterization theorem of $\lambda_{q,3}$-integral operators, we have the following.

**Theorem 3.7** Let $\lambda_{q,3}$ be a regular interpolation space for $1 \leq q < \infty$ and let $E$ and $F$ be Banach spaces. The following statements are equivalent:

(i) $T \in \mathfrak{I}_{\lambda_{q,3}}(E, F)$.

(ii) $J_{k} T$ factors continuously in the following way:

\[
\begin{array}{ccc}
E & \xrightarrow{J_{k} T} & F'' \\
A \downarrow & & \uparrow C \\
L_\infty(\mu) & \xrightarrow{B} & X
\end{array}
\]

where $X$ is an ultrapower of $\ell_1[\lambda_{q,3}]$ and $B$ is a lattice homomorphism. Furthermore, $I_{\lambda_{q,3}}(T)$ is equivalent to $\inf\{\|D\|\|B\|\|A\|\}$, taking it over all such factors.
Then from Theorem 2.2 of characterization of \( \lambda \)-subspaces of \( E \), for every \((P, Q) \) \( \in \) FIS\( (E), Q \in \text{FIN}(F') \), where FIS\( (Y) \) is the set of finite dimensional subspaces of a Banach space \( Y \), endowed with the natural inclusion order

\[
(P_1, Q_1) \leq (P_2, Q_2) \iff P_1 \subset P_2, \quad Q_1 \subset Q_2.
\]

For every \((P_0, Q_0) \in D, \quad R(P_0, Q_0) := \{(P, Q) \in D : (P_0, Q_0) \subset (P, Q)\}\) and \( \mathcal{R} \) is filter basis in \( D \), and according to Zorn’s lemma, let \( \mathcal{D} \) be an ultrafilter on \( D \) containing \( \mathcal{R} \). If \( d \in D \), \( P_d \) and \( Q_d \) denote the finite dimensional subspaces of \( E \) and \( F' \) respectively so that \( d = (P_d, Q_d) \). For every \( d \in D \), if

\[
z \in P_d \otimes Q_d, \quad J_{F'}T_{|P_d \otimes Q_d} \in (P_d \otimes_{\kappa_{q,d}} Q_d)' = P'_d \otimes_{\kappa_{q,d}} Q'_d = N_{\lambda_{q,d}}(P_d, Q_d').
\]

Then from Theorem 2.2 of characterization of \( \lambda_{q,d} \)-nuclear operators, \( J_{F'}T_{|P_d \otimes Q_d} \) factors as

\[
\begin{array}{ccc}
P_d & \xrightarrow{J_{F'}T_{|P_d \otimes Q_d}} & Q'_d \\
A_d \downarrow & & \downarrow C_d \\
\ell_\infty \otimes \ell_\infty & \xrightarrow{\ell_1[\lambda_{q,d}]} & \ell_1[\lambda_{q,d}]
\end{array}
\]

where \( B_d \) is a positive diagonal operator and

\[
\|A_d\|\|B_d\|\|C_d\| \leq N_{\lambda_{q,d}}(T_{|P_d \otimes Q_d}) + \varepsilon = I_{\lambda_{q,d}}(T_{|P_d \otimes Q_d}) + \varepsilon.
\]

Then

\[
\|A_d\|\|B_d\|\|C_d\| \leq I_{\lambda_{q,d}}(T_{|P_d \otimes Q_d}) + \varepsilon \leq I_{\lambda_{q,d}}(T) + \varepsilon
\]

Without loss of generality, we can suppose that \( \|A_d\| = \|C_d\| = 1 \). We define \( W_{F'} : E \to (P_d)_D \) such that \( W_{F'}(x) = (x_d)_D \), so that \( x_d = x \) if \( x \in P_d \), and \( x_d = 0 \) if \( x \notin P_d \). In the same way, we define \( W_{F'} : F' \to (Q_d)_D \) such that \( W_{F'}(a) = (a_d)_D \), so that \( a_d = a \) if \( a \in Q_d \), and \( a_d = 0 \) if \( a \notin Q_d \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{W_{F'}} & F' \\
\downarrow & & \downarrow \\
(P_d)_D & \xrightarrow{(J_{F'}T_{|P_d \otimes Q_d})_D} & ((Q_d)_D)' \\
(A_d)_D \downarrow & & \downarrow \\
(\ell_\infty[\ell_\infty])_D & \xrightarrow{(C_d)_D} & (\ell_1[\lambda_{q,d}])_D
\end{array}
\]

where \( I \) is the canonical inclusion map. As in [13] \( ((\ell_1[\lambda_{q,d}])_D)' \) is a 1-complemented subspace of some ultrapower \( ((\ell_1[\lambda_{q,d}])_D)_U \), which, from [24], is another
ultralinear \((\ell_1[\lambda_{q,\beta}])_\mathcal{U}\) with projection \(Q\). The result follows with \(A = (A_d)_\mathcal{D}, B = ((B_d)_\mathcal{D})''\), which is a lattice homomorphism, \(C = P_{F''}(W'_p, I(C_d)_\mathcal{D})''_Q\), where \(P_{F''}\) is the projection of \(F''\) in \(F''\), and \(X = (\ell_1[\lambda])_\mathcal{U}\), having in mind that as \((\ell_1(\ell_1)_\mathcal{D})_\mathcal{D}\) is an abstract space, there is a measure space such that \(L_\infty(\mu) = ((\ell_1(\ell_1)_\mathcal{D})_\mathcal{D})''\), where equality means that the spaces are lattice isometric.

(iii) \(\Rightarrow\) (i) As \((\lambda_{q,\beta}, I_{\lambda_{q,\beta}})\) is an operator ideal, it follows easily from Theorem 3.6 and considering that every ultralinear \(\ell_1[\lambda_{q,\beta}]\) has \(1^+\)-\(S_\mathcal{T}(\ell_1)[S_\mathcal{K}(\lambda_{q,\beta})]\)-local unconditional structure.

![Diagram](https://doi.org/10.4153/CMB-2010-075-6)
4 On Equality Between $\lambda_{q,3}$-Nuclear and $\lambda_{q,3}$-Integral Operators

Finally, using the preceding characterization theorems, we give some properties of $\lambda_{q,3}$-nuclear and $\lambda_{q,3}$-integral operators. Let us now establish a necessary condition for equality between components of $\lambda_{q,3}$-nuclear and $\lambda_{q,3}$-integral operator ideals. First, we introduce a new operator ideal, which is contained in the ideal of the $\lambda_{q,3}$-integral operators.

**Definition 4.1** Given $E$ and $F$ Banach spaces, let $\lambda_{q,3}$ be an interpolation space. We say that $T \in \mathcal{L}(E, F)$ is strictly $\lambda_{q,3}$-integral if there exists a $\sigma$-finite measure space $(\Omega, \mathcal{B}, \mu)$ and a Köthe function space $\mathcal{K}(\mu)$ which is complemented in a space with $S_k(\ell_1)[S_k(\lambda_{q,3})]$-local unconditional structure, such that $T$ factors continuously in the following way:

![Diagram](image)

where $B$ is a multiplication operator for a positive function of $\mathcal{K}(\mu)$, endowed with the topology of the norm $S(\lambda_{q,3})(T) = I_{\lambda_{q,3}}(T)$.

Obviously, if $F$ is a dual space, or it is complemented in its bidual space, then $S(\lambda_{q,3})(E, F) = \mathcal{N}(\lambda_{q,3})(E, F)$.

**Theorem 4.2** Let $\ell_{\lambda_{q,3}}$ be a interpolation space, and let $E$ and $F$ be Banach spaces, such that $E'$ satisfies the Radon–Nikodým property, then $\mathcal{N}(\lambda_{q,3})(E, F) = S(\lambda_{q,3})(E, F)$.

**Proof** Let $T \in S(\lambda_{q,3})(E, F)$, where $E'$ has the Radon–Nikodým property.

(a) First, we suppose that $B$ is a multiplication operator for a function $g \in \mathcal{K}(\mu)$ with finite measure support $D$. We denote by $v_D$ the restriction of $\mu$ to $D$.

As $(\chi_D A) : E \to L_\infty(v_D)$, then $(\chi_D A)' : (L_\infty(v_D))' \to E'$ and the restriction of $(\chi_D A)' |_{L_1(v_D)} : L_1(v_D) \to E'$, thus, for every $x \in E$ and $f \in L_1(v_D),$

$$\langle x, (\chi_D A)'(f) \rangle = \langle \chi_D A(x), f \rangle = \int_D \chi_D A(x) f d(v_D).$$

As $E'$ has the Radon–Nikodým property, by [8, III(5)], we have that $(\chi_D A)'$ has a Riesz representation, therefore a function $\phi \in L_\infty(v_D, E')$ exists such that for every $f \in L_1(v_D)$

$$(\chi_D A)'(f) = \int_D f \phi d(v_D).$$

Then, for every $x \in E$, we have that $\chi_D A(x) = \langle \phi(t), x \rangle$, $v_D$-almost everywhere in $D$, and then $B(\chi_D A)(x) = \langle \phi(t) \cdot, x \rangle_{\nu_D}$, $v_D$-almost everywhere in $D$. Let $g \phi$ be this last operator, and we can consider it as a $\mathcal{K}(\nu_D, E')$ element. As the simple functions are dense in $\mathcal{K}(\nu_D, E')$, $g \phi$ can be approximated by a sequence of simple functions $(\langle S_k \rangle)_{k=1}^\infty$. 

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Maximal Operator Ideal Associated with a Tensor Norm Defined by Interpolation Spaces

We suppose $S_k = \sum_{j=1}^{m_k} x_j^k \chi_{A_j}$, where $\{A_i : i = 1, \ldots, m\}$ is a family of $\nu$-measure set of $\Omega$ pairwise disjoint. For each $k \in \mathbb{N}$, we can interpret $S_k$ as a map $S_k : E \to \mathcal{K}(\nu)$ such that $S_k(x) = \sum_{j=1}^{m_k} x_j^k, x > \chi_{A_j}$ with norm less than or equal to the norm of $S_k$ in $\mathcal{K}(\nu, E')$.

Clearly for all $k \in \mathbb{N}$, $S_k$ is $\lambda_{q,3}$-nuclear since it has finite range, but we need to evaluate its $\lambda_{q,3}$-nuclear norm coinciding with its $\lambda_{q,3}$-integral norm. Let $S_k : E \to L_\infty(\nu)$ be such that $S_k(x) = \sum_{j=1}^{m_k} \langle x_j^k, x \rangle \chi_{A_j}$, and let $S_k^2 : L_\infty(\nu) \to \mathcal{K}(\nu)$ be such that $S_k^2(f) = \sum_{j=1}^{m_k} \|x_j^k\| \|f\chi_{A_j}\|$. We suppose $S_k(x) \leq x_0 \chi_{A_0}$.

Then $\|S_k\| \leq 1$, $\|S_k\| \leq \|S_k\|_{\mathcal{K}(\nu, E')}$ and $S_k = S_k^2 k$. But as $\mathcal{K}(\nu)$ is a complemented subspace of space with $S_k(\ell_1)(S_0(\lambda_{q,3}))$-local unconditional structure, from Theorem 3.6 there is $K > 0$ such that $I_{\lambda_{q,3}}(S_k^2) \leq K\|S_k\| \leq K\|S_k\|_{\mathcal{K}(\nu, E')}$, hence $\mathcal{N}_{\lambda_{q,3}}(S_k^2) \leq K\|S_k\|_{\mathcal{K}(\nu, E')}$, hence $\mathcal{N}_{\lambda_{q,3}}(S_k) \leq K\|S_k\|_{\mathcal{K}(\nu, E')}$. Then, as $(S_k)_{k=1}^\infty$ converges in the $\mathcal{K}(\nu, E')$ space, it is a Cauchy sequence in $\mathcal{N}_{\lambda_{q,3}}(E, \mathcal{K}(\nu, E'))$, and since this is complete, $(S_k)_{k=1}^\infty$ converges to $g\phi$, that is to say, $g\phi \in \mathcal{N}_{\lambda_{q,3}}(E, \mathcal{K}(\nu, E'))$. Therefore, $g\phi = B_{12}A$ is $\lambda_{q,3}$-nuclear and so $T$ is also $\lambda_{q,3}$-nuclear.

(b) Now, if $g$ is any element of $\mathcal{K}(\nu)$, it can be approximated in norm by means of a sequence $(t_n)_{n=1}^\infty$ of simple functions with finite measure support, and therefore by (a), the sequence $T_n = CB_{12}A$ is a Cauchy sequence in $\mathcal{N}_{\lambda_{q,3}}(E, F)$ converging to $T$ in $\mathcal{L}(E, F)$, and then $T \in \mathcal{N}_{\lambda_{q,3}}(E, F)$. □

As consequence of the former result and of the factorization Theorems 3.8 and 272 we obtain the following metric properties of $g^c_{\lambda_{q,3}}$ and $(g^c_{\lambda_{q,3}})'$.

**Theorem 4.3** $(g^c_{\lambda_{q,3}})'$ is a totally accessible tensor norm.

**Proof** Since $(g^c_{\lambda_{q,3}})'$ is finitely generated, it is sufficient to prove that the map $F \otimes (g^c_{\lambda_{q,3}})' \to \mathcal{P}_{\lambda_{q,3}}(E', F'')$ is isometric.

Let

$$z = \sum_{i=1}^n \sum_{j=1}^l y_{ij} \otimes x_{ij} \in F \otimes (g^c_{\lambda_{q,3}})' ,$$

and let $H_z \in \mathcal{P}_{\lambda_{q,3}}(E', F'')$ be the canonical map associated with $z$, that is to say,

$$H_z(x') = \sum_{i=1}^n \sum_{j=1}^l \langle x_{ij}, x' \rangle y_{ij}$$

for all $x' \in E'$, with $H_z \in \mathcal{L}(E', F) \subseteq \mathcal{L}(E', F'')$.
Applying [7, Theorem 15.5] for \( \alpha = (g_{\lambda,\beta}^c)' \), the theorem and the equality \((g_{\lambda,\beta}^c)' = g_{\lambda,\beta}^c \), since \( g_{\lambda,\beta}^c \) is finitely generated, we have that inclusion
\[
F \otimes (g_{\lambda,\beta}^c)' \rightarrow (F' \otimes g_{\lambda,\beta}^c) \rightarrow \mathcal{P}_{\lambda',\infty} (E', E'')
\]
is an isometry, and therefore by [7, Proposition 12.4], we obtain
\[
\mathbf{P}_{\lambda',\infty} (H_2) = (g_{\lambda,\beta}^c)' (z; F \otimes E) \leq (g_{\lambda,\beta}^c)' (z; F \otimes E).
\]

Now, given \( N \), a finite dimensional subspace of \( F \) such that \( z \in N \otimes (g_{\lambda,\beta}^c)' \), there exists
\[
V \in (N \otimes (g_{\lambda,\beta}^c)') E' = J(N, E')
\]
such that \( I_{\lambda,\beta} (V) \leq 1 \) and \((g_{\lambda,\beta}^c)' (z; N \otimes E) = \langle z, V \rangle \). Clearly enough
\[
V \in \mathcal{S} J_{\lambda,\beta} (N, E') = J_{\lambda,\beta} (N, E')
\]
because \( E' \) is a dual space, and \( N' \), being finite dimensional, has the Radon–Nikodým property. Therefore, by Theorem 4.2 \( V \in \mathcal{N}_{\lambda,\beta} (N, E') \) and by Theorem 2.2, given \( \epsilon > 0 \), there is a factorization \( V \) in the way
\[
\begin{array}{c c c c}
\ell_\infty & \rightarrow & T & \rightarrow & E' \\
A & \downarrow & \rightarrow & \rightarrow & \downarrow \\
\ell_\infty [\ell_\infty] & \rightarrow & B & \rightarrow & \ell_1 [\lambda,\beta] \\
\end{array}
\]
such that \( ||C|| ||B|| ||A|| \leq N_{\lambda,\beta}^c (V) + \epsilon = I_{\lambda,\beta} (V) + \epsilon \leq 1 + \epsilon. \)

As \( \ell_\infty [\ell_\infty] \) has the extension metric property, (see [21, Proposition 1, C.3.2]), \( A \) can be extended to a continuous map \( \tilde{A} \in \mathcal{L}(F, \ell_\infty [\ell_\infty]) \) such that \( ||\tilde{A}|| = ||A||. \)

By Theorem 2.2 again, \( W := C \tilde{A} \) is in \( \mathcal{N}_{\lambda,\beta} (F, E') \), so there is a representation \( w := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y_{ij}' \otimes x_{ij}' \in F' \otimes (g_{\lambda,\beta}^c)' E' \) of \( W \) verifying
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \pi_{\lambda,\beta} ((y_{ij}') \otimes (x_{ij}')) \leq N_{\lambda,\beta}^c (W) + \epsilon \leq ||C|| ||B|| ||\tilde{A}|| + \epsilon \leq 1 + 2 \epsilon.
\]

Then, \((g_{\lambda,\beta}^c)' (z; F \otimes E) \leq (g_{\lambda,\beta}^c)' (z; N \otimes E) = \langle z, V \rangle = \langle z, W \rangle \). It follows that
\[
(g_{\lambda,\beta}^c)' (z; F \otimes E) \leq g_{\lambda,\beta}^c (w) \mathbf{P}_{\lambda',\infty} (H_2) \leq (1 + 2 \epsilon) \mathbf{P}_{\lambda',\infty} (H_2),
\]
whence \((g_{\lambda,\beta}^c)' (z; F \otimes E) \leq \mathbf{P}_{\lambda',\infty} (H_2) \), and the equality is obvious.

Finally, as a consequence of the former theorem and of [7, Proposition 15.6], we have the following.

**Corollary 4.4** \( g_{\lambda,\beta}^c \) is an accessible tensor norm.
Maximal Operator Ideal Associated with a Tensor Norm Defined by Interpolation Spaces

References


Universidad EAFIT, Departamento de Ciencias Básicas, Medellín, Colombia
e-mail: mpuerta@eafit.edu.co
gloaiza@eafit.edu.co

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