Multivariate Outlier Detection and Robust Estimation Using Skewness and Projections

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November 12, 2019

Abstract

This work introduces a new statistical technique for outlier detection in multivariate data. Our methodology is based on finding projection that maximizes the univariate skewness coefficient. The performance of this procedure is analyzed and compared with common outlier detection methods described in the literature. Simulations with different kind of multivariate distributions and contamination structures are made, and some well-known datasets are used to check out performance. We present an empirical evidence that both, the correct detection rate and false detection rate show the advantages of the presented proposal. Finally, a new robust version of covariance matrix is also introduced.

Keywords: covariance matrix, outlier detection, projection pursuit, robust statistics, skewness coefficient.

1 Introduction

Multivariate outlier detection is recognized as an interesting issue in statistical data analysis; being very useful in topics such as quality control (Rocke, 1989; Davis and Adams, 2005), time series (Galeano et al., 2006), medical image analysis (Magnotti and Billor, 2014), chemistry (Hubert and Van der Veeken, 2008) and so on. Datasets frequently contain one or more outliers; that is, observations that are far away from the bulk of the data, or deviated from the general pattern (Maronna et al., 2006). Most multivariate techniques require the use of estimations for both location and shape parameters (Maronna and Zamar, 2002), nevertheless the presence of outliers and higher dimensions may affect these estimators (Rocke and Woodruff, 1996; Peña and Prieto, 2001). Thus, reasonable location and scale estimations follow being a challenging in statistics.

Methods have been reported in the literature for identifying multivariate outliers. Some of them are based on a robustification of the Mahalanobis distance, through the improvement of both location and scale parameters, such as Fast–MCD, proposed by Rousseeuw and Van Driessen (1999), the Orthogonalized Gnanadesikan–Kettenring proposal by Maronna and Zamar (2002) or the Comedian approach technique, proposed by Sajesh and Srinivasan (2012). Other are based on data projections, looking for less complexity in low dimensions; the Stahel–Donoho outlyingness proposal by Stahel (1981) and Donoho (1982); projections based on the optimization of the kurtosis coefficient by Peña and Prieto (2001, 2007) and a proposal for multivariate skewed distributions based on a modified Stahel–Donoho outlyingness by Hubert and Van der Veeken (2008).

The main contribution of this work is to introduce a new outlier detection procedure for numerical data, that aims to study multivariate data in an univariate way, based on the search and analysis of directions that maximizes the univariate skewness coefficient of the projected data. The proposed method is inspired in that introduced by Peña and Prieto (2001), but we use the third moment instead. Our method is tested in both simulated and real data and we present quite evidence that the multivariate outlier detection method introduced in this paper has outstanding performance than other techniques already studied in the literature. The paper is organized as follows, Section 2 describes the proposed method showing in a set of steps the data projection and the outlier detection decision, also an introduction of robust location and scale estimators and an illustrative example of the method is presented. Section 3 presents a simulation study with different contaminated multivariate distributions to check performance against other well-known methods in the literature. Finally, in Section 4 are given some conclusions of this work.

2 Description of Proposed Method

Let $(x_1, x_2, ..., x_n)$ be a given sample of a p–dimensional random vector X that follows a contaminated multivariate distribution of the form $(1-\alpha)\mathcal{F}_1 + \alpha\mathcal{F}_2$, given a contamination percentage $\alpha < 0.5$, i.e., $X_1 \sim \mathcal{F}_1$, $X_2 \sim \mathcal{F}_2$ and where $X = [X_1, X_2]$. The proposed method is based on the projection of each observation onto one vector that maximizes the univariate skewness. This direction is obtained as the solutions of an optimization procedure, as follows:

1. Data are transformed through a standardizing. Let $\bar{x} \in \mathbb{R}^p$ denote the sample mean and $H_{\hat{\sigma}} \in \mathbb{R}^{p \times p}$ a diagonal matrix whose entries are the margin standard deviations $\hat{\sigma}_1, \hat{\sigma}_2, \cdots, \hat{\sigma}_p$ then the points are centered and rescaled using

$$z_i = H_{\hat{\sigma}}^{-1}(x_i - \bar{x}) \quad \forall i = 1, \cdots, n. \tag{1}$$

2. Compute the unit vector that maximizes the skewness coefficient of the projection. This direction is obtained as the solution of the optimization problem

$$\begin{aligned} \underset{\mathbf{r}}{\text{argmax}} \quad & \frac{1}{n}\sum_{i=1}^{n} (\mathbf{r}' z_i)^3 \\ \text{subjected to} \quad & \mathbf{r}'\mathbf{r} = 1 \end{aligned} \tag{2}$$

3. Once multivariate observations are projected onto vector $\mathbf{r}$, compute an estimated probability density function $\hat{f}_h(y)$ of the data $y_i = \mathbf{r}' z_i$, fitting a kernel density estimation. The aim in this step is find the center point $\text{CP} = \{y_k \mid y_k \in (y_1, \cdots, y_n) \wedge \forall y_l \in (y_1, \cdots, y_n) : \hat{f}_h(y_l) \leq \hat{f}_h(y_k)\}$ of the data as the global maximal of $\hat{f}_h(y)$.

4. To determine if any y_i is an outlier, compute an univariate cutoff level $\beta^* = \text{CP} + \psi\text{MAD}^*(y)\ \forall\psi > 0$ where $\text{MAD}^*(y) = \text{median}\left(|y_i - \text{meda}(y)|\right)$ and $\text{meda}(y) = \text{median}\left(y_i - \text{median}(y)\right)$. Keeping in mind that positive skewness is maximized and projected outliers are positioned right side the main bulk of the univariate data. The value β^* is used to identify if a given observation can be considered as outlier, if $y_i > \beta^*$, then observation i is labeled as an outlier.

   This cutoff is chosen in each iteration and depends on the contamination level. β^* could be considered as a modified version of the "empirical three-sigma rule" for right tail and a robust cutoff following the notion of outlyingness (Stahel, 1981; Donoho, 1982). Thus, to ensure a reasonable level of type I error, it is suggested in first iteration a value $\psi = 2$ and then for rest iterations a value $\psi \geq 3$.

5. If condition in step 4 is satisfied for some i, a new multivariate sample is composed of all observations i such that $y_i \leq \beta^*$. Then, a Mahalanobis distance MD^* is computed for all non-labeled as outliers using the remaining observations $z^* \in \{z_i : y_i \leq \beta^*\}$. It computes

$$\begin{gathered} MD^* = (z^* - \mu^*)'\Sigma^{*-1}(z^* - \mu^*) \\ \text{where} \quad \mu^* = \text{meda}(z^*) \quad \text{and} \quad \Sigma^* = \text{cov}(z^*) \end{gathered} \tag{3}$$

All this process is repeated until no more outliers are detected, it is $MD^* < 1.4826 * \chi^2_{0.975,p}$ (Sajesh and Srinivasan, 2012) for all points or when the number of remaining observations are greater or equal than $\lfloor (n+p+1)/2 \rfloor$ (Rousseeuw and Leroy, 1987).

2.1 Computation of the Projections Algorithm

The essence of the preceding procedure is associated with the computation of the solution for Equation (2). This computation is conducted obtaining the solution directly from the first-order optimality conditions. This is

$$
\begin{aligned}
3\sum_{i=1}^{n}\left(\mathbf{r}'z_i\right)^2 z_i &= 2\lambda\mathbf{r} \\
3\sum_{i=1}^{n}\left(\mathbf{r}'z_i\right)^2 \mathbf{r}'z_i &= 2\lambda
\end{aligned}
\tag{4}
$$

Replacing the value 2λ in (4) we obtain the following resulting condition

$$
\begin{aligned}
\sum_{i=1}^{n}\left(\mathbf{r}'z_i\right)^2 z_i &= \sum_{i=1}^{n}\left(\mathbf{r}'z_i\right)^2 \mathbf{r}'z_i\mathbf{r} \\
\sum_{i=1}^{n}\left(\mathbf{r}'z_i\right) \mathbf{r}'z_i z_i &= \sum_{i=1}^{n}\left(\mathbf{r}'z_i\right)^3 \mathbf{r} \\
\left(\sum_{i=1}^{n}\left(\mathbf{r}'z_i\right) z_i z_i'\right)\mathbf{r} &= \left(\sum_{i=1}^{n}\left(\mathbf{r}'z_i\right)^3\right)\mathbf{r}
\end{aligned}
\tag{5}
$$

Equation (5) indicates that the optimal $\mathbf{r}$, in each iteration, is a unit eigenvector of the matrix $\mathbb{A}(\mathbf{r}) = \sum_{i=1}^{n}\left(\mathbf{r}'z_i\right) z_i z_i'$. Since the eigenvalue of the solutions correspond to the value of the third moment of the projected data y_i. The interest here is computing the eigenvector corresponding to the largest eigenvalue. An iterative procedure to compute the vector $\mathbf{r}$ proceeds following these steps:

1. Set iteration index $g = 0$ and compute an initializer unit vector $\mathbf{r_g}$. For our case, we choose $\mathbf{r_0}$ corresponding to the largest principal component of the normalized observations $z_i/\|z_i\|$ as Peña and Prieto (2001) proposed.

2. In iteration $g+1$ compute the spectral decomposition of $\mathbb{A}(\mathbf{r_g})$, find the unit eigenvector corresponding to the largest eigenvalue and then set it as $\mathbf{r_{g+1}}$.

3. Finish when $\|\mathbf{r_{g+1}} - \mathbf{r_g}\| < \epsilon$ and set $\mathbf{r} = \mathbf{r_{g+1}}$.

Sometimes this iteratively procedure may not converge, hence the algorithm will stay in an infinite loop. Thus, it is necessary add a stop criterion in this case, Cuesta-Albertos and Nieto-Reyes (2008) established that less than 100 projections are required to reach reasonable stability. Thus, we set a level of 200 iterations as a sufficient iterated projections search to find a good direction. According to the explained above, in each iteration we save the vectors $\mathbf{r_{min}}$ and $\mathbf{r_{max}}$, the directions (eigenvectors) with minimum and maximum eigenvalue respectively.

A relevant issue is how to select the best direction, $\mathbf{r_{min}}$ or $\mathbf{r_{max}}$, when convergence in the computation of the projection is not reached. Our choice has been to compare these directions through the skewness coefficient (b_1) and the robust measure of skewness medcouple (MC) (Brys et al., 2004). If $b_1(\mathbf{r'_{min}}z_i) > b_1(\mathbf{r'_{max}}z_i)$ and $MC(\mathbf{r'_{min}}z_i) > MC(\mathbf{r'_{max}}z_i)$, then the best direction is $\mathbf{r_{min}}$. Otherwise If $b_1(\mathbf{r'_{min}}z_i) < b_1(\mathbf{r'_{max}}z_i)$ and $MC(\mathbf{r'_{min}}z_i) < MC(\mathbf{r'_{max}}z_i)$, then the best direction is $\mathbf{r_{max}}$.

2.2 Robust Location and Scale Estimators

Once we have labeled some observations of z_i as outliers, following the procedure described previously, it is possible to generate robust estimates for the mean and covariance matrix considering the non-outliers remaining observations. Let Q be the set of all observations not labeled as outliers, the cardinality of Q is denoted by $|Q|$. The proposed estimators of location and shape are, respectively:

$$\begin{aligned} \hat{\mu}_R &= \frac{1}{|Q|}\sum_{x_i \in Q} x_i \\ \hat{S}_R &= \frac{1}{|Q|-1}\sum_{x_i, x_j \in Q} (x_i - \hat{\mu}_R)(x_j - \hat{\mu}_R)' \end{aligned} \tag{6}$$

2.2.1 Limit Error Estimator

Let μ and Σ be the location and scale parameters of a p–dimensional population following a multivariate distribution and let $\hat{\mu} \in \mathbb{R}^p$ and $\hat{\Sigma} \in \mathbb{R}^{p\times p}$ be realizations of the sample estimators, respectively. It would be expected that if these estimations are significant and representative for the population then $\mu \approx \hat{\mu}$ and $\Sigma \approx \hat{\Sigma}$. Remember that by definition, $\|\hat{\mu}\|_2 = 0 \iff \hat{\mu} = \mathbf{0}$ and $\left\|\hat{\Sigma}\right\|_F = 0 \iff \hat{\Sigma} = \mathbf{0}$; here, $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the vectorial entry-wise norms, 2–norm for vectors and Frobenius norm for matrices. Thus, if $\mu \approx \hat{\mu}$ then $\mu - \hat{\mu} \approx \mathbf{0}$ and this implies that $\|\mu - \hat{\mu}\|_2 \approx 0$; and again if $\Sigma \approx \hat{\Sigma}$ then $\Sigma - \hat{\Sigma} \approx 0$ and this implies that $\left\|\Sigma - \hat{\Sigma}\right\|_F \approx 0$.

According to described above, it is possible to set a limit error given an estimation, following these steps:

1. Set a multivariate distribution, values of sample size n, sample-space dimension p, theoretical mean μ and covariance matrix Σ and number of iterations k.

2. In each iteration, generate a p–dimensional random sample of size n following the fixed multivariate distribution. Compute the sample estimators $\hat{\mu}_j$, $\hat{\Sigma}_j$ and the entry-wise norms

$$\begin{aligned} N_\mu(j) &= \|\mu - \hat{\mu}_j\|_2 \\ N_\Sigma(j) &= \left\|\Sigma - \hat{\Sigma}_j\right\|_F \end{aligned} \qquad \forall\, j = 1, \cdots, k \tag{7}$$

3. Once iterations have finished, use bootstrap B times to estimate samples of the maximum statistics

$$\begin{aligned} T_l^{(\mu,k)} &= \max\left(N_\mu(1), \cdots, N_\mu(k)\right) \\ T_l^{(\Sigma,k)} &= \max\left(N_\Sigma(1), \cdots, N_\Sigma(k)\right) \end{aligned} \qquad \forall\, l = 1, \cdots, B \tag{8}$$

4. Finally, set the error limits $\widehat{EL}_{\mu,boot}$ and $\widehat{EL}_{\Sigma,boot}$ for μ and Σ respectively, as the minimum of the maximum bootstrapped entry-wise norms

$$\begin{aligned} \widehat{EL}_{\mu,boot} &= \min\left(T_1^{(\mu,k)}, \cdots, T_B^{(\mu,k)}\right) \\ \widehat{EL}_{\Sigma,boot} &= \min\left(T_1^{(\Sigma,k)}, \cdots, T_B^{(\Sigma,k)}\right) \end{aligned} \tag{9}$$

These values $\widehat{EL}_{\mu,boot}$ and $\widehat{EL}_{\Sigma,boot}$ are actually upper limits, because of lowers are 0 by definition for all sample estimations. Thus, given $\widetilde{\mu}$ and $\widetilde{\Sigma}$ two estimations of a p–dimensional vector random variable X, if $\left\|\Sigma - \tilde{\Sigma}_j\right\|_F \leq \widehat{EL}_{\Sigma,boot}$ and $\|\mu - \tilde{\mu}_j\|_2 \leq \widehat{EL}_{\mu,boot}$ then $\tilde{\mu}$ and $\tilde{\Sigma}$ are both significant and representative estimations of population parameters.

2.3 Examples

To illustrate the exposed procedure and how the projection vectors and the skewness coefficient work, some results from the computation of the method in few simple cases are shown. The first ones are based on generating 500 observations from a model of the form $(1-\alpha)N(\mathbf{0}, I) + \alpha N(5\delta, I)$ in dimension 3, where $\delta = (1,1,1)'$ and $\alpha = 0.1$.

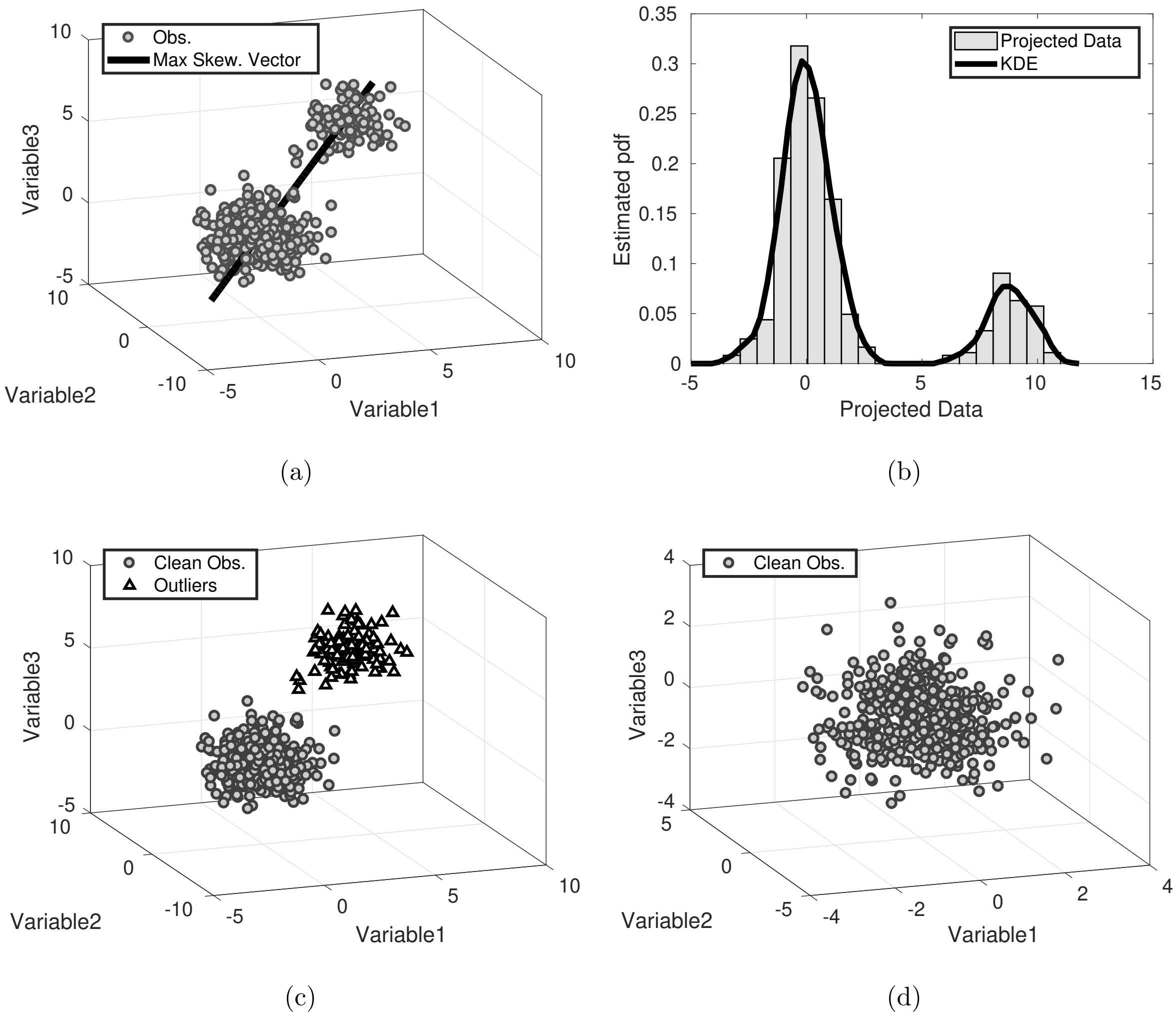


Figure 1: Outlier detection procedure for a 3–dimensional simulated data, in (a) there are the points and the projection vector that maximizes the skewness coefficient of projected data, in (b) there is the histogram of the projected data and its KDE estimation, in (c) there are the outliers detected in the sample, in triangles, and the clean observations in balls and in (d) there are the clean observations.

Consider Figure 1, corresponding to the example explained above; this figure shows the behavior or steps of the proposed method, in Figure 1(a) there is the scatterplot of the observations, 90% in balls, considered clean data and a 10%, in diamonds, the outlier points;

it is easy to see that a contamination is presented. Also there is a line, that goes through the two bulks, it corresponds to the direction that maximize the skewness coefficient of all projected points, allowing the correct identification of the outliers.

In Figure 1(b) there is the histogram of the projected data, i.e. the univariate representation of the multivariate data in that direction; there it can be appreciated the bimodality and the positive skewness, indicating the presence of two populations. Once the histogram shows, in a univariate way, the presence of outliers in the sample, these are eliminated through the computation of β^*, finding points $y_i > \beta^*$. Finally in Figure 1(c) the points not labeled as outliers are presented, corresponding to all observations considered clean at the beginning.

The last example has analyzed the behavior of the outlier detection proposal in a controlled experiment. Now, two well-known datasets were selected to test the performance of our method. These two datasets, "Hawkins-Bradu-Kass" (HBK) and "Rousseeuw Data" (RD) were taken from Rousseeuw and Leroy (1987). HBK was studied by Rousseeuw and Van Driessen (1999), Peña and Prieto (2001) and Peña and Prieto (2007) and RD by Rousseeuw (1984) and Atkinson (1994).

Table 1 gives the corresponding results for the algorithm, indicating the name of the dataset, the number of outliers detected and the specific points labeled as outliers. These results are equivalents to those reported in the literature for other outlier detection techniques, indicating that the presented proposal behaves according to the expected on these kind of datasets.

Table 1: Multivariate outlier detection in some datasets.

Dataset	# Outliers	Outliers
HBK	14	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14
RD	20	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

3 Simulations Results

To explore more extensive the performance, we have carried out a set of simulation experiments for different contamination levels and sample-space dimensions. These experiments compare the performance of the proposed method regarding the outlier detection by the measure of the *average correct detection rates* "c" and the *average false detection rates* "f".

Five well-known methods in the literature are selected to compare the performance with our proposal (denoted Skew): *Minimum Covariance Determinant* (Fast–MCD), method by Rousseeuw and Van Driessen (1999), *Orthogonalized Gnanadesikan-Kettenring* (OGK) by Maronna and Zamar (2002), the *Median Ball Algorithm* proposal (O–H) by Olive (2004), *Kurtosis Projections* (Kur1) refers to Peña and Prieto (2001) approach and the method *Random Directions + Kurtosis Projections* (Kur2) by Peña and Prieto (2007). All simulations were performed in Matlab.

3.1 Normal Distribution

Consider a p–dimensional random variable X following a contaminated multivariate normal distribution, given as a mixture of normal distributions of the form $(1-\alpha)N(\mathbf{0}, I) + \alpha N(\delta\mathbf{e}, \lambda I)$, where e denotes the p–dimensional vector of ones, this kind of experiment is equivalent to those used by Rousseeuw and Van Driessen (1999), Peña and Prieto (2001), Maronna and Zamar (2002), Filzmoser (2005), Peña and Prieto (2007) and Sajesh and Srinivasan (2012).

This experiment has been conducted for different contamination levels $\alpha = 0, 0.1, 0.2, 0.3, 0.4$, different values of sample-space dimension $p = 10, 20, 40, 100, 200$, a distance of outliers $\delta = 3, 7$ and a concentration of the contamination $\lambda = 0.1, 1$. The sample size is fixed in $n = 500$ observations and for each set of experiments 500 random repetitions have been generated.

In Table 2 there are the most representative results, where distance of outliers and outliers concentration are fixed in $\delta = 3$ and $\lambda = 0.1$, respectively. There, it can be appreciated that our proposal gets better c values in high contamination levels ($\alpha > 0.2$) than the others; when $0 < \alpha \leq 0.2$ Kur1 and Kur2 methods achieve good results until a

Table 2: c and f values for multivariate normal distribution, $\lambda = 0.1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.005**	-	0.028	-	0.061	-	0.029	-	0.012	-	0.017
	0.1	**0.998**	**0**	0.998	0.016	1	0.051	0.996	0.019	1	0.012	1	0.016
	0.2	**1**	**0**	0	0.129	0	0.085	0	0.072	1	0.012	0.900	0.019
	0.3	**1**	**0**	0	0.481	0	0.312	0.056	0.131	0.994	0.013	0.327	0.180
	0.4	**1**	**0**	0	0.834	0	0.615	0	0.279	0.962	0.011	0.696	0.199
20	0.0	**-**	**0.009**	-	0.034	-	0.073	-	0.038	-	0.011	-	0.022
	0.1	**0.998**	**0.001**	0	0.083	1	0.062	0	0.059	1	0.012	1	0.024
	0.2	**1**	**0**	0	0.393	0.008	0.111	0	0.093	0.996	0.013	0.943	0.037
	0.3	**1**	**0**	0	0.717	0	0.429	0.002	0.166	0.968	0.023	0.280	0.493
	0.4	**0.998**	**0**	0	0.998	0	0.789	0	0.361	1	0.014	0.984	0.031
40	0.0	**-**	**0.016**	-	0.071	-	0.101	-	0.084	-	0.011	-	0.062
	0.1	**0.998**	**0**	0	0.299	1	0.080	0	0.132	1	0.014	1	0.070
	0.2	**1**	**0**	0	0.576	0.098	0.134	0	0.239	0.896	0.071	0.946	0.110
	0.3	**1**	**0**	0	0.956	0	0.618	0	0.476	0.472	0.346	0.144	0.572
	0.4	**1**	**0**	0	1	0	0.934	0	0.744	1	0.022	1	0.053
100	0.0	**-**	**0.021**	-	0.233	-	0.112	-	0.400	-	0.009	-	0.379
	0.1	**0.924**	**0.002**	0	0.417	1	0.091	0	0.444	0.922	0.023	1	0.319
	0.2	**0.998**	**0**	0	0.649	0.743	0.090	0	0.500	0.306	0.358	0.012	0.497
	0.3	**1**	**0**	0	0.986	0	0.907	0	0.571	0	0.571	0.010	0.567
	0.4	**1**	**0**	0	1	0	0.998	0	0.667	0.506	0.329	0.508	0.328
200	0.0	**-**	**0.014**	-	0.277	-	0.121	-	0.300	-	0.003	-	0.200
	0.1	**0.950**	**0.001**	0	0.336	1	0.101	0	0.333	0.002	0.101	0	0.222
	0.2	**0.998**	**0**	0	0.588	1	0.075	0	0.375	0.002	0.247	0	0.250
	0.3	**1**	**0**	0	0.982	0	0.995	0	0.429	0	0.285	0	0.286
	0.4	**1**	**0**	0	1	0	1	0	0.500	0	0.329	0	0.333

sample-space dimension $p = 40$, further p their c values drop.

Regarding f values, we can appreciate that for all sample-space dimensions and contamination levels the proposed method achieves the lowest values, even smaller than 0.05; when the sample-space dimension increases, f values of the other methods tend to increase. In the Supplementary Material there are Table 10, Table 11 and Table 12, corresponding to remaining simulations. However, these tables show similar outcomes as in Table 2.

3.2 Correlated Normal Distribution

Consider a p–dimensional random variable X following a contaminated multivariate normal distribution, given a dependence structure, as a combinations of correlated normal distributions of the form $(1-\alpha)N(\mathbf{0}, \Sigma_1) + \alpha N(\mathbf{e}^*, \lambda\Sigma_2)$, where $\mathbf{e}^*$ denotes a p–dimensional vector $\in \mathbb{Z}^p \setminus \{\mathbf{0}\}$ with $|e_i^*| \leq \delta \quad \forall\ i = 1, \cdots, p$ and at least some $e_i^* = \delta$. Σ_1 and Σ_2 are two different matrices, each one with a defined correlation structure. These matrices are constructed following the procedure proposed by Agostinelli et al. (2015).

This experiment has been conducted for different contamination levels $\alpha = 0, 0.1, 0.2, 0.3, 0.4$, different values of sample-space dimension $p = 10, 20, 40, 100, 200$, a distance of outliers $\delta = 3, 7$ and a concentration of the contamination $\lambda = 0.1, 1$. The sample size is fixed in $n = 500$ observations and for each set of experiments 500 random repetitions have been generated.

In Table 3 there are the most representative results, when distance of outliers and outliers concentration are fixed in $\delta = 3$ and $\lambda = 0.1$, respectively. There, it can be appreciated that our proposal gets better c values in high contamination levels ($\alpha \geq 0.2$) than the others, but for $\alpha = 0.1$ Kur1 achieves good c values until $p = 20$ and Kur2 achieves proficient c values until $p = 100$. Regarding f values, our proposal achieves the best performance, with all f values smaller than the other methods in all contaminations levels and sample-space dimensions. In the Supplementary Material there are Table 13, Table 14 and Table 15, corresponding to remaining simulations. However, these tables show similar outcomes as in Table 3.

3.3 Uniform Distribution

In order to check the behavior of all these methods when the multivariate distribution is not elliptical, we perform a simulation study considering a p–dimensional random variable X following a contaminated multivariate uniform distribution, given as a mixture of uniform distributions of the form $(1-\alpha)U(\mathbf{0}, \mathbf{e}) + \alpha\lambda U(\delta+\mathbf{0}, \delta+\mathbf{e})$. The notation $\alpha\lambda U(\cdot, \cdot)$ refers to a compression level λ in all dimensions of the multivariate uniform contamination proportion $\alpha U(\cdot, \cdot)$. Here, parameters α, λ, δ, n, p and $\mathbf{e}$ are the same as in the multivariate normal

Table 3: c and f values for multivariate correlated normal distribution, $\lambda = 0.1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.003**	-	0.028	-	0.061	-	0.029	-	0.012	-	0.016
	0.1	**0.807**	**0.002**	0.999	0.015	0.988	0.051	0.999	0.019	0.999	0.011	0.999	0.016
	0.2	**0.990**	**0.001**	0.942	0.014	0.395	0.063	0.941	0.014	0.994	0.012	0.996	0.017
	0.3	**0.967**	**0.001**	0.344	0.275	0.027	0.212	0.337	0.081	0.948	0.013	0.849	0.029
	0.4	**0.838**	**0**	0.020	0.873	0.004	0.546	0.002	0.319	0.906	0.062	0.805	0.135
20	0.0	**-**	**0.005**	-	0.034	-	0.071	-	0.037	-	0.011	-	0.022
	0.1	**0.805**	**0.001**	1	0.024	1	0.062	1	0.029	0.935	0.013	1	0.025
	0.2	**1**	**0**	0.197	0.159	0.604	0.062	0.195	0.074	0.385	0.026	0.963	0.026
	0.3	**1**	**0**	0.014	0.666	0.002	0.319	0.019	0.165	0.548	0.116	0.722	0.151
	0.4	**1**	**0**	0	0.997	0	0.715	0	0.428	0.968	0.035	0.794	0.154
40	0.0	**-**	**0.009**	-	0.072	-	0.102	-	0.084	-	0.011	-	0.062
	0.1	**0.834**	**0.003**	0.441	0.119	1	0.083	0.408	0.100	0.140	0.019	1	0.075
	0.2	**0.996**	**0**	0.038	0.496	0.518	0.078	0.086	0.202	0.008	0.090	0.917	0.104
	0.3	**1**	**0**	0	0.782	0	0.464	0.010	0.428	0.174	0.496	0.438	0.391
	0.4	**1**	**0**	0	1	0	0.879	0	0.731	0.780	0.179	0.642	0.295
100	0.0	**-**	**0.019**	-	0.234	-	0.122	-	0.400	-	0.009	-	0.378
	0.1	**0.856**	**0.003**	0.149	0.339	1	0.102	0.232	0.418	0.029	0.026	0.830	0.343
	0.2	**0.998**	**0**	0	0.510	0.198	0.145	0.114	0.472	0	0.459	0.034	0.491
	0.3	**1**	**0**	0	0.863	0	0.754	0	0.571	0	0.571	0	0.571
	0.4	**1**	**0**	0	1	0	0.995	0	0.667	0.148	0.567	0.104	0.597
200	0.0	**-**	**0.020**	-	0.277	-	0.141	-	0.300	-	0.003	-	0.200
	0.1	**0.920**	**0.002**	0	0.328	1	0.118	0.206	0.310	0	0.015	0.002	0.222
	0.2	**0.980**	**0.001**	0	0.386	0.052	0.267	0.026	0.369	0	0.213	0	0.250
	0.3	**1**	**0**	0	0.800	0	0.926	0	0.429	0	0.274	0	0.286
	0.4	**1**	**0**	0	0.999	0	1	0	0.500	0	0.320	0	0.333

case presented at the beginning; for each set of experiments 500 random repetitions have been generated.

In Table 4 there are the most representative results, when distance of outliers and outliers concentration are fixed in $\delta = 3$ and $\lambda = 0.1$. There, it can be appreciated that our proposal gets better c values again in high contamination levels ($\alpha \geq 0.2$) than the others; even though for $\alpha = 0.1$ our proposal achieves outstanding results in terms of c values. Regarding the other methods, we can appreciate that OGK and O–H methods have

Table 4: c and f values for multivariate uniform distribution, $\lambda = 0.1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.001**	-	0.001	-	0.001	-	0	-	0	-	0
	0.1	**0.963**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.2	**1**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.3	**1**	**0**	1	0	0.013	0	1	0	0.995	0	0.856	0
	0.4	**1**	**0**	0.800	0	0.002	0	1	0	0.985	0	0.869	0
20	0.0	**-**	**0.005**	-	0.001	-	0.004	-	0.001	-	0	-	0
	0.1	**0.990**	**0**	1	0	1	0.004	1	0	1	0	1	0
	0.2	**0.998**	**0**	1	0	1	0.003	1	0	0.923	0	0.996	0
	0.3	**1**	**0**	0.693	0	0.030	0.003	1	0	0.692	0	0.193	0
	0.4	**1**	**0**	0.008	0.001	0.005	0.003	1	0	0.924	0	0.591	0
40	0.0	**-**	**0.014**	-	0.001	-	0.018	-	0.012	-	0	-	0.005
	0.1	**0.998**	**0**	1	0.006	1	0.014	1	0.011	0.754	0	1	0.007
	0.2	**1**	**0**	0.729	0.005	1	0.011	1	0.009	0	0	0.354	0.005
	0.3	**1**	**0**	0.017	0.008	0.498	0.006	1	0.007	0	0	0.010	0.004
	0.4	**1**	**0**	0.012	0.010	0.019	0.011	0.996	0.005	0.112	0	0.058	0.005
100	0.0	**-**	**0.025**	-	0.197	-	0.028	-	0.399	-	0	-	0.330
	0.1	**0.998**	**0**	0.419	0.176	1	0.024	1	0.333	0.004	0	0.958	0.308
	0.2	**1**	**0**	0.248	0.185	1	0.017	1	0.250	0.001	0	0.408	0.318
	0.3	**1**	**0**	0.216	0.188	1	0.009	0.799	0.229	0	0	0.362	0.321
	0.4	**1**	**0**	0.204	0.192	0.022	0.036	0.415	0.389	0	0	0.354	0.334
200	0.0	**-**	**0.018**	-	0.281	-	0.028	-	0.300	-	0	-	0.200
	0.1	**0.998**	**0**	0.321	0.277	1	0.031	0.987	0.224	0.001	0	0.259	0.193
	0.2	**1**	**0**	0.298	0.277	1	0.023	0.395	0.276	0	0	0.223	0.194
	0.3	**1**	**0**	0.286	0.279	1	0.011	0.322	0.291	0	0	0.208	0.197
	0.4	**1**	**0**	0.284	0.279	0.005	0.089	0.307	0.295	0	0	0.204	0.197

proficient c values for all contaminations levels until the sample-space dimension $p = 40$, this because for $p = 100, 200$, in these two methods, the higher c the higher f. It should be noted regarding Fast-MCD, Kur1 and Kur2 methods their unsatisfactory behavior in higher sample-space dimensions, $p > 20$, for this kind of distribution.

In f values, our proposal achieves the best performance for all $\alpha > 0$ experiments, with all f values smaller than the other methods in all sample-space dimensions; despite Kur1 method obtains best f values for all $\alpha = 0$, our proposal achieves small values close to zero.

In the Supplementary Material there are Table 16, Table 17 and Table 18, corresponding to remaining simulations. However, these tables show similar outcomes as in Table 4.

3.4 t_4–Student Distribution

In order to validate the performance of all these methods when the multivariate distribution differs from normality by a heavy-tail appearance, we perform another simulation study considering a p–dimensional random variable X following a contaminated multivariate t_4–Student distribution, given as a mixture of t_4–Student distributions of the form $(1-\alpha)t_4(\mathbf{0}, I) + \alpha t_4(\delta\mathbf{e}, \lambda I)$. The first parameter of the notation $t_4(\cdot,\cdot)$ refers to location parameter, mean, and the other to scale parameter, covariance matrix; similar to the experiments proposed and used by Filzmoser (2005). Parameters α, λ, δ, n, p and $\mathbf{e}$ are the same as in the multivariate normal case presented above; for each set of experiments 500 random repetitions have been generated.

In Table 5 there are the most representative results, when distance of outliers and outliers concentration are fixed in $\delta = 3$ and $\lambda = 0.1$, respectively. Regarding the method Fast–MCD it can be appreciated a good performance both in c and f for sample-space dimension $p = 10$ but for contamination level $\alpha < 0.4$, achieving f values lower or equal than 0.1, but for higher dimensions this method gets non satisfactory results. For methods Kur1 and Kur2 there are good results in c values until dimension $p = 20$, however for f values the results are not suitable, getting high f values.

The OGK and O–H proposals in general are good outlier detection methods in all dimension but only in contamination levels $\alpha = 0.1, 0.2$, but are procedures with high false positives rates, due to f values greater or equal than 0.1 in all set of experiments. Our proposal in general shows a very good performance in all scenarios resulting in c values close to one and f values at least less than 0.1; in global terms, the proposed method achieves the best performance. In the Supplementary Material there are Table 19, Table 20 and Table 21, corresponding to remaining simulations. However, these tables show similar outcomes as in Table 5.

Table 5: c and f values for multivariate t_4–Student distribution, $\lambda = 0.1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.053**	-	0.125	-	0.197	-	0.129	-	0.106	-	0.122
	0.1	**0.998**	**0.046**	1	0.104	1	0.173	1	0.104	1	0.104	1	0.122
	0.2	**1**	**0.042**	1	0.087	1	0.150	1	0.080	1	0.101	0.998	0.125
	0.3	**1**	**0.040**	1	0.071	0.210	0.134	1	0.054	1	0.095	0.993	0.125
	0.4	**1**	**0.042**	0.250	0.099	0.185	0.162	0.310	0.096	0.994	0.074	0.651	0.100
20	0.0	**-**	**0.085**	-	0.163	-	0.223	-	0.158	-	0.127	-	0.171
	0.1	**1**	**0.077**	1	0.144	1	0.195	1	0.133	1	0.128	1	0.177
	0.2	**1**	**0.076**	1	0.125	1	0.168	1	0.105	0.877	0.123	0.962	0.181
	0.3	**1**	**0.072**	0.373	0.134	0.270	0.157	0.965	0.079	0.960	0.117	0.830	0.158
	0.4	**1**	**0.071**	0.159	0.144	0.221	0.184	0.157	0.143	1	0.085	0.999	0.129
40	0.0	**-**	**0.083**	-	0.250	-	0.242	-	0.240	-	0.152	-	0.355
	0.1	**1**	**0.075**	1	0.230	1	0.211	1	0.214	0.634	0.146	1	0.287
	0.2	**1**	**0.068**	0.349	0.227	1	0.178	1	0.178	0.176	0.134	0.576	0.308
	0.3	**1**	**0.064**	0.269	0.230	0.398	0.164	0.331	0.214	0.161	0.136	0.379	0.322
	0.4	**1**	**0.056**	0.252	0.236	0.257	0.198	0.241	0.225	0.808	0.093	0.784	0.177
100	0.0	**-**	**0.013**	-	0.306	-	0.261	-	0.400	-	0.131	-	0.399
	0.1	**0.998**	**0.006**	0.461	0.290	1	0.224	1	0.333	0.157	0.119	0.894	0.345
	0.2	**1**	**0.005**	0.345	0.293	1	0.180	0.770	0.308	0.131	0.117	0.465	0.384
	0.3	**1**	**0.005**	0.325	0.293	0.896	0.140	0.432	0.386	0.124	0.118	0.430	0.387
	0.4	**1**	**0**	0.312	0.297	0.290	0.224	0.413	0.392	0.123	0.120	0.410	0.390
200	0.0	**-**	**0.009**	-	0.272	-	0.274	-	0.300	-	0.039	-	0.200
	0.1	**0.990**	**0**	0.310	0.267	1	0.231	0.749	0.250	0.026	0.036	0.234	0.196
	0.2	**1**	**0**	0.286	0.268	1	0.182	0.327	0.293	0.028	0.036	0.211	0.197
	0.3	**1**	**0**	0.277	0.267	0.998	0.128	0.311	0.300	0.030	0.035	0.203	0.199
	0.4	**1**	**0**	0.273	0.269	0.330	0.213	0.304	0.297	0.033	0.034	0.201	0.199

3.5 Gamma Distribution

In order to validate the performance of all these outlier detection methods when the multivariate distribution is not symmetrical or is multivariate skewed, we perform the last outlier detection simulation study considering a p–dimensional random variable X following a contaminated multivariate Gamma distribution, in notation shape-scale, given as a mixture of Gamma distributions of the form $(1-\alpha)Gamma_{[\mathbf{0},1]}(2,2)+\alpha Gamma_{[\delta+\mathbf{0},\lambda]}(2,2)$. The first parameter of the notation $Gamma_{[\cdot,\cdot]}$ refers to the point $\in \mathbb{R}^p$ where the support of the distribution begins and the second to the factor in which data is compressed or expanded. Parameters α, λ, δ, n and p are the same as in all remaining multivariate simulation cases presented; for each set of experiments 500 random repetitions have been generated.

Table 6: c and f values for multivariate gamma distribution, $\lambda = 0.1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.071**	-	0.128	-	0.198	-	0.132	-	0.096	-	0.115
	0.1	**0.970**	**0.056**	1	0.106	1	0.169	1	0.106	1	0.096	1	0.117
	0.2	**1**	**0.051**	1	0.084	0.999	0.130	1	0.077	1	0.095	0.983	0.116
	0.3	**1**	**0.051**	1	0.064	0.229	0.124	1	0.047	0.998	0.087	0.946	0.115
	0.4	**1**	**0.048**	0.224	0.085	0.186	0.147	0.275	0.082	1	0.067	0.519	0.088
20	0.0	**-**	**0.117**	-	0.153	-	0.209	-	0.148	-	0.102	-	0.148
	0.1	**0.992**	**0.103**	1	0.132	1	0.177	1	0.121	1	0.105	1	0.156
	0.2	**1**	**0.095**	1	0.111	1	0.146	1	0.092	0.544	0.089	0.906	0.154
	0.3	**1**	**0.091**	0.292	0.114	0.255	0.141	0.993	0.061	0.887	0.097	0.754	0.131
	0.4	**1**	**0.081**	0.141	0.121	0.207	0.162	0.136	0.118	1	0.070	0.998	0.122
40	0.0	**-**	**0.100**	-	0.227	-	0.218	-	0.221	-	0.107	-	0.309
	0.1	**0.998**	**0.073**	1	0.204	1	0.182	1	0.193	0.479	0.094	1	0.279
	0.2	**1**	**0.061**	0.321	0.194	1	0.146	1	0.157	0.116	0.086	0.503	0.262
	0.3	**1**	**0.048**	0.234	0.198	0.357	0.131	0.325	0.182	0.101	0.086	0.311	0.266
	0.4	**1**	**0.028**	0.218	0.200	0.241	0.158	0.209	0.191	0.608	0.075	0.608	0.192
100	0.0	**-**	**0.022**	-	0.286	-	0.221	-	0.400	-	0.073	-	0.398
	0.1	**0.996**	**0**	0.414	0.272	1	0.179	1	0.333	0.090	0.063	0.901	0.344
	0.2	**1**	**0**	0.321	0.275	1	0.138	0.910	0.273	0.071	0.063	0.443	0.384
	0.3	**1**	**0**	0.299	0.277	0.868	0.098	0.423	0.390	0.066	0.061	0.418	0.390
	0.4	**1**	**0**	0.289	0.280	0.271	0.167	0.409	0.394	0.064	0.062	0.406	0.392
200	0.0	**-**	**0.017**	-	0.272	-	0.226	-	0.300	-	0.016	-	0.200
	0.1	**0.994**	**0**	0.303	0.267	1	0.183	0.858	0.238	0.014	0.014	0.241	0.195
	0.2	**1**	**0**	0.287	0.268	1	0.137	0.327	0.293	0.014	0.015	0.215	0.196
	0.3	**1**	**0**	0.276	0.270	0.998	0.088	0.312	0.295	0.014	0.014	0.207	0.197
	0.4	**1**	**0**	0.275	0.271	0.366	0.127	0.304	0.297	0.014	0.014	0.203	0.198

In Table 6 there are the most representative results, when distance of outliers and outliers concentration are fixed in $\delta = 3$ and $\lambda = 0.1$, respectively. There we can find that our proposal achieves desired results in both c and f in all sets of experiments, obtaining the best performance in multivariate skewed scenarios.

Regarding Fast–MCD method, it can be appreciated a good performance in c values for low sample-space dimension $p < 40$ but for contamination levels $\alpha < 0.3$, but in terms of f values in those sets of experiments, Fast–MCD gets greater values than our proposal, for higher p this drops also in c and f values. For OGK and O–H procedures, we can find proficient performance for more sample-space dimensions than Fast–MCD but not for high contamination levels, again we note here that these methods achieves non satisfactory results in f values, getting false positives rates over 0.1. Finally, regarding Kur1 and Kur2 methods, these get the most non satisfactory results in all experiments, only achieve moderate c values for $p = 10$, however f values are higher than our proposed method. In the Supplementary Material there are Table 22, Table 23 and Table 24, corresponding to remaining simulations. However, these tables show similar outcomes as in Table 6.

3.6 Computational Affine Equivariance

We investigate the possible lack of equivariance of our method on the correct outlier detection performance. Consider $W = \{\mathbf{w_1}, \cdots, \mathbf{w_n}\}$, m and S a pair of multivariate location and scale estimators, respectively, and $A \in \mathbb{R}^{p\times p}$ a nonsingular matrix and let $W_A = \{\mathbf{Aw_1}, \cdots, \mathbf{Aw_n}\}$ the affine transformation of W. If m and S are affine equivariant, then $m_A = m(W_A) = Am(W)$ and $S_A = S(W_A) = AS(W)A'$. Affine equivariance property implies that estimators transform suitably under any nonsingular parameterization. Thus, for any data rotation, shift or rescale these estimators should hold stable.

Maronna and Zamar (2002) proposed the following procedure to study the lack of equivariance property. Let $A = TD$, where T is a random orthogonal matrix and D is diagonal matrix which its entries $d_1, \cdots, d_p$ are independent and uniformly distributed in $(0, 1)$, then, a simulation is repeated for several sampling situations and the performance of "c" and "f" are evaluated. Consider a p–dimensional random variable X following a contaminated multivariate normal distribution, given as a mixture of normal distributions of the form $(1-\alpha)N(\mathbf{0}, I) + \alpha N(\delta\mathbf{e}, \lambda I)$, where $\mathbf{e}$ denotes the p–dimensional vector of ones, and let $X_A = XA$ the affine transformation of X.

Table 7: c and f values for transformed data with different λ and δ.

p	α	$\lambda = 0.1$, $\delta = 3$		$\lambda = 1$, $\delta = 3$		$\lambda = 0.1$, $\delta = 7$		$\lambda = 1$, $\delta = 7$	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	-	0.008	-	0.008	-	0.009	-	0.006
	0.1	0.939	0.002	0.967	0.001	0.980	0.001	0.972	0.001
	0.2	1	0.003	0.995	0	0.998	0	0.994	0
	0.3	1	0	0.990	0	0.998	0	1	0
	0.4	0.951	0.001	0.992	0	0.998	0	1	0.019
20	0.0	-	0.010	-	0.010	-	0.011	-	0.010
	0.1	0.962	0.001	0.978	0.001	0.990	0	0.984	0
	0.2	0.996	0	0.998	0	0.990	0	0.992	0
	0.3	1	0	1	0	1	0	1	0
	0.4	1	0	1	0	1	0	1	0
40	0.0	-	0.014	-	0.013	-	0.015	-	0.014
	0.1	0.964	0.001	0.972	0.001	0.974	0.001	0.985	0
	0.2	0.998	0	0.992	0	0.994	0	0.992	0
	0.3	1	0	1	0	1	0	1	0
	0.4	1	0	1	0	1	0	1	0
100	0.0	-	0.014	-	0.013	-	0.014	-	0.014
	0.1	0.964	0.001	0.962	0.001	0.979	0.001	0.979	0
	0.2	0.982	0.001	0.984	0	0.998	0	1	0
	0.3	1	0	1	0	1	0	1	0
	0.4	1	0	1	0	1	0	1	0
200	0.0	-	0.009	-	0.011	-	0.009	-	0.010
	0.1	0.950	0.001	0.968	0.001	0.988	0	0.979	0
	0.2	0.992	0	0.994	0	1	0	1	0
	0.3	1	0	1	0	1	0	1	0
	0.4	1	0	1	0	1	0	1	0

This experiment has been conducted for different contamination levels $\alpha = 0, 0.1, 0.2, 0.3, 0.4$, different values of sample-space dimension $p = 10, 20, 40, 100, 200$, a distance of outliers $\delta = 3, 7$ and a concentration of the contamination $\lambda = 0.1, 1$. The sample size is fixed in $n = 500$ observations and for each set of experiments 500 random repetitions have been generated. The proposed simulation sets consist on affinely transform data matrix X in each iteration applying the transformation A to X and our proposed method is then applied to transformed data X_A to detect outliers.

Table 7 presents these simulation results with c and f values, even under affine transformations, our method is able to outlier detection, since c values are close to one, and f values close to zero. However, when contamination level increases ($\alpha > 0.1$) our method shows better performance. Regarding these experiments we can conclude that our method achieves a proficient outlier detection performance in affine transformations.

3.7 Computing Times

To compare the computing times of the different outlier detection methods exposed in previous section, we generated random samples of a p–dimensional random variable X following a contaminated multivariate normal distribution, given as a mixture of normal distributions of the form $(1-\alpha)N(\mathbf{0}, I) + \alpha N(\delta\mathbf{e}, \lambda I)$, where e denotes the p–dimensional vector of ones, with different values of $\alpha = 0.1, 0.2, 0.3, 0.4$, $\delta = 3, 7$, $\lambda = 0.1, 1$ and $p = 10, 20, 40$ analogous to previous experiments, the sample size is fixed in $n = 500$ observations and 500 random repetitions have been generated. We ran the experiments on a PC with a 4-GHz Intel Core i7-6700K processor with 64 Gb RAM.

Table 8 provides the mean running times, measured in seconds, for all outlier detection methods described above in many different experiments sets. From this table, it is clear that O–H method is the fastest of all followed by Kur2 proposal. Although our proposed method is not the best in terms of computing speed, it achieves reasonable mean times that could be consider adequate and competitive for the outlier detection aim, since the maximum mean computing time is less than two seconds, that we contemplate low on average.

3.8 Location and Scale Estimators Performance

3.8.1 $\widehat{\mathbf{EL}}_{\mu,\mathbf{boot}}$ and $\widehat{\mathbf{EL}}_{\mathbf{\Sigma},\mathbf{boot}}$ Estimation

According to steps (7), (8) and (9) introduced in the error limit procedure on Subsection 2.2, it is possible to estimate these limits for both location and scale parameters. Following this, we perform a simulation study in order to compute estimations for $\widehat{EL}_{\mu,boot}$ and $\widehat{EL}_{\Sigma,boot}$

Table 8: Mean computational times in seconds for different λ and δ.

	p	**10**				**20**				**40**			
	α	**0.1**	**0.2**	**0.3**	**0.4**	**0.1**	**0.2**	**0.3**	**0.4**	**0.1**	**0.2**	**0.3**	**0.4**
	Skew	0.25	0.29	0.54	0.82	0.16	0.15	0.13	0.99	0.21	0.12	0.11	1.31
	Fast–MCD	0.18	0.18	0.17	0.17	0.25	0.26	0.25	0.24	0.46	0.47	0.46	0.45
$\lambda = 0.1$	OGK	0.05	0.05	0.05	0.05	0.22	0.22	0.22	0.21	0.89	0.88	0.87	0.86
$\delta = 3$	O–H	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	Kur1	0.46	0.37	0.23	0.14	1.06	0.82	0.52	0.33	2.84	2.49	1.86	1.26
	Kur2	0.04	0.03	0.03	0.02	0.07	0.05	0.05	0.04	0.19	0.11	0.11	0.10
	Skew	0.93	0.81	0.92	0.47	0.61	0.38	0.89	0.97	0.24	0.11	0.26	1.37
	Fast–MCD	0.18	0.18	0.17	0.18	0.26	0.25	0.25	0.25	0.47	0.48	0.47	0.46
$\lambda = 1$	OGK	0.05	0.05	0.05	0.05	0.22	0.22	0.21	0.21	0.89	0.87	0.87	0.86
$\delta = 3$	O–H	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	Kur1	0.51	0.50	0.44	0.39	1.27	1.33	1.14	0.91	3.75	3.74	3.76	3.48
	Kur2	0.05	0.05	0.04	0.03	0.09	0.09	0.10	0.04	0.29	0.30	0.33	0.30
	Skew	1.06	0.99	0.93	0.20	1.12	1.06	1.02	0.27	1.46	1.39	1.36	0.92
	Fast–MCD	0.18	0.17	0.17	0.18	0.25	0.25	0.25	0.25	0.47	0.46	0.46	0.45
$\lambda = 0.1$	OGK	0.05	0.05	0.05	0.05	0.21	0.21	0.21	0.21	0.89	0.86	0.86	0.85
$\delta = 7$	O–H	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	Kur1	0.45	0.35	0.23	0.15	1.02	0.79	0.51	0.33	2.87	2.43	1.81	1.27
	Kur2	0.05	0.03	0.02	0.02	0.07	0.04	0.05	0.04	0.20	0.11	0.11	0.10
	Skew	0.78	0.68	0.35	0.16	1.00	1.07	0.79	0.10	1.45	1.42	1.39	0.13
	Fast–MCD	0.18	0.18	0.18	0.18	0.25	0.25	0.25	0.28	0.47	0.47	0.47	0.46
$\lambda = 1$	OGK	0.05	0.05	0.05	0.05	0.22	0.22	0.21	0.22	0.88	0.87	0.86	0.86
$\delta = 7$	O–H	0.01	0.01	0.01	0.01	0.01	0.01	0.0	0.01	0.01	0.01	0.01	0.01
	Kur1	0.54	0.50	0.45	0.40	1.24	1.33	1.10	0.92	3.75	3.76	3.76	3.47
	Kur2	0.05	0.05	0.04	0.02	0.09	0.08	0.09	0.04	0.30	0.27	0.32	0.31

given a multivariate distribution and theoretical parameters. We generated random samples of a p–dimensional random variable X following a multivariate normal distribution $N(\mathbf{0}, I)$, with different values of $n = 50, 100, 200, 300, 350, 400, 450, 500, 600, \cdots, 5000$ and $p = 10, 20, 40, 100, 200$, theoretical parameter are set to $\mu = \mathbf{0}$ and $\Sigma = I^{p\times p}$, finally $B = 10000$ bootstrap repetitions and $k = 10000$ iterations have been generated.

Consider Figure 2, corresponding to the error limits estimations for μ and Σ of a multivariate $N(\mathbf{0}, I)$ distribution. Figure 2(a) shows different curves of $\widehat{EL}_{\Sigma,boot}$ as a function of sample size n when the sample-space dimension p is set, similar occurs in

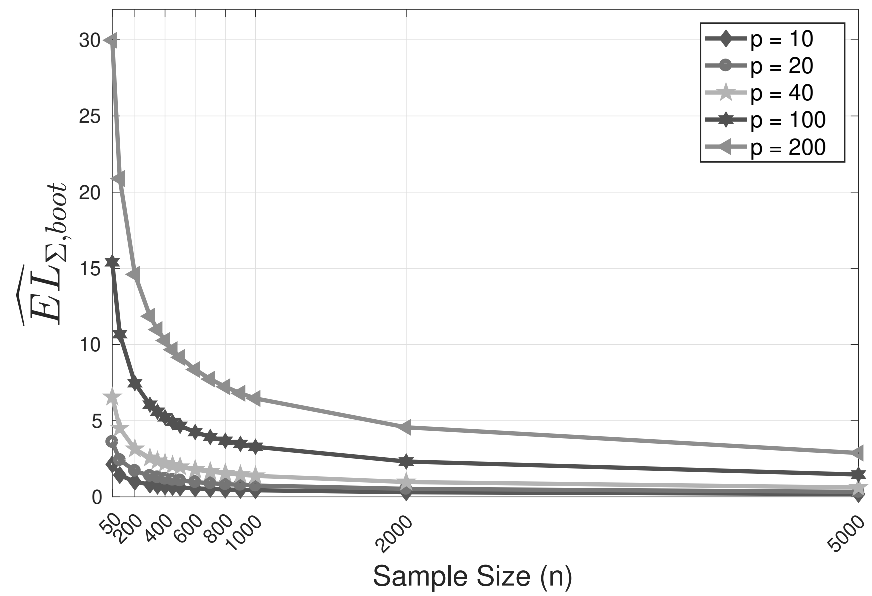
Figure 2(b) for $\widehat{EL}_{\mu,boot}$. In both (a) and (b) plots, the curves are ordered from highest sample space-dimension to smallest and all of them decreases, in an exponentially shape, when sample size increases. This has a lot of sense, since the smaller p and the higher n the more accurate and consistent turns out to be the estimations $\hat{\Sigma}$ and $\hat{\mu}$ from theoretical Σ and μ, respectively. In Table 25 of the Supplementary Material there are the corresponding numerical values of $\widehat{EL}_{\mu,boot}$ and $\widehat{EL}_{\Sigma,boot}$ of a multivariate $N(\mathbf{0}, I)$ distribution showed in Figure 2.

3.8.2 $\hat{\mu}_{\mathbf{R}}$ and $\hat{\mathbf{S}}_{\mathbf{R}}$ Performance

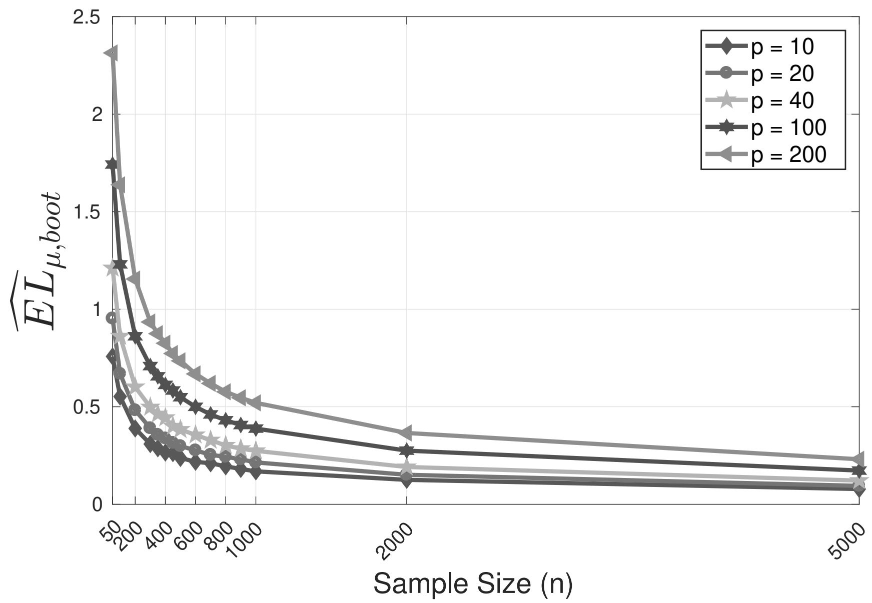
Once $\widehat{EL}_{\Sigma,boot}$ and $\widehat{EL}_{\mu,boot}$ estimations have been computed, for a multivariate $N(\mathbf{0}, I)$ distribution, due to the procedure explained above for different n and p, we can test performance of the proposed robust location and scale estimators $\hat{\mu}_R$ and $\hat{S}_R$ presented in Equation (6) by a simulation study. Consider a p–dimensional random variable X following a contaminated multivariate normal distribution, given as a mixture of normal distributions of the form $(1-\alpha)N(\mathbf{0}, I) + \alpha N(\delta\mathbf{e}, \lambda I)$, where e denotes the p–dimensional vector of ones.

This experiment has been conducted for different contamination levels $\alpha = 0, 0.1, 0.2, 0.3, 0.4$, different values of sample-space dimension $p = 10, 20, 40, 100, 200$, a distance of outliers $\delta = 3, 7$ and a concentration of the contamination $\lambda = 0.1, 1$. The sample size is fixed in $n = 500$ observations and for each set of experiments $k = 500$ random repetitions have been generated. In the j-th iteration, given the parameters set, we run our proposed outlier detection method in order to get the estimations of $\hat{\mu}_R(j)$ and $\hat{S}_R(j)$.

Followed this, we compute the statistics $N_{\hat{\mu}_R}(j) = \|\mu - \hat{\mu}_R(j)\|_2$, $N_{\hat{S}_R}(j) = \left\|\Sigma - \hat{S}_R(j)\right\|_F$ and set the random variables $c_{\hat{\mu}_R}(j) = \mathbb{I}_{\left(N_{\hat{\mu}_R}(j) \leq \widehat{EL}_{\mu,boot}(n_1,p)\right)}$, $c_{\hat{S}_R}(j) = \mathbb{I}_{\left(N_{\hat{S}_R}(j) \leq \widehat{EL}_{\Sigma,boot}(n_1,p)\right)}$ $\forall\ j = 1, \cdots, k$ as indicator functions; here n_1 denotes the sample size of non-outliers points. Thus, the performance of estimators $\hat{\mu}_R$ and $\hat{S}_R$ are measured by the expected values $\overline{c}_{\hat{\mu}_R}$ and $\overline{c}_{\hat{S}_R}$, where, if these averages are close to one, it can concluded that the estimations are significant and representative. Table 9 presents the results of this simulation study, there we can find that both $\overline{c}_{\hat{\mu}_R}$ and $\overline{c}_{\hat{S}_R}$ are close to one, achieving values greater or equal



(a)



(b)

Figure 2: Error limit estimation by simulation varying sample size n and sample-space dimension p of a multivariate $N(\mathbf{0}, I)$, in a) $\widehat{EL}_{\Sigma,boot}$ estimation for Σ and in b) $\widehat{EL}_{\mu,boot}$ estimation for μ.

than 0.9 in all simulation scenarios, regarding these experiments we can conclude that our proposed location and scale estimators, for normal distribution in this case, can be consider adequate for robust estimation.

Table 9: $\overline{c}_{\hat{\mu}_R}$ and $\overline{c}_{\hat{S}_R}$ values for multivariate normal distribution with different λ and δ.

p	α	$\lambda=0.1$, $\delta=3$		$\lambda=1$, $\delta=3$		$\lambda=0.1$, $\delta=7$		$\lambda=1$, $\delta=7$	
		$\overline{c}_{\hat{\mu}_R}$	$\overline{c}_{\hat{S}_R}$	$\overline{c}_{\hat{\mu}_R}$	$\overline{c}_{\hat{S}_R}$	$\overline{c}_{\hat{\mu}_R}$	$\overline{c}_{\hat{S}_R}$	$\overline{c}_{\hat{\mu}_R}$	$\overline{c}_{\hat{S}_R}$
10	0.0	1	1	0.994	1	0.990	0.998	1	0.998
	0.1	0.996	0.996	0.978	0.974	0.976	0.978	0.982	0.980
	0.2	0	1	0.994	0.996	0.998	0.998	0.986	0.984
	0.3	1	1	0.998	0.998	1	0.998	1	0.996
	0.4	0.998	0.998	0.998	0.990	1	1	1	1
20	0.0	0.998	1	0.998	0.998	0.996	0.998	0.998	0.996
	0.1	1	1	0.992	0.988	0.988	0.988	0.972	0.970
	0.2	1	0.998	1	1	1	1	0.984	0.986
	0.3	1	0.996	0.998	0.998	0.998	1	1	1
	0.4	1	1	1	1	1	0.998	1	1
40	0.0	0.992	0.998	0.994	0.998	1	1	1	1
	0.1	1	0.998	0.996	0.994	0.980	0.982	0.978	0.982
	0.2	1	0.998	1	1	0.994	0.994	0.994	0.994
	0.3	1	1	0.998	1	1	0.998	1	0.998
	0.4	1	1	1	1	1	0.996	1	0.998
100	0.0	0.990	0.974	0.996	0.994	0.998	0.982	0.996	0.988
	0.1	0.916	0.916	1	0.998	0.968	0.964	0.952	0.952
	0.2	1	0.998	1	1	0.986	0.984	0.994	0.994
	0.3	1	1	0.998	1	1	0.998	0.998	0.996
	0.4	0.998	1	1	1	0.998	1	1	0.998
200	0.0	0.998	0.944	1	0.942	0.996	0.928	0.998	0.938
	0.1	0.964	0.966	1	0.998	0.978	0.978	0.984	0.986
	0.2	0.998	1	0.998	0.996	0.988	0.988	0.976	0.976
	0.3	0.998	0.998	1	1	1	1	0.998	0.998
	0.4	1	0.998	0.996	1	1	0.992	0.998	1

4 Conclusion

In this work we have developed an iteratively method for multivariate outlier detection, based on the analysis of univariate projections onto the direction that maximizes the skewness coefficient, as a tool to studying multivariate data in an univariate case. Also we have introduced an empirical relation between the presence of outlier points and the increasing of univariate skewness of projected data.

In the same way, we have introduced a computational nonparametric procedure to test significance and representativeness of sample estimations, setting limit errors for both location and scale parameters given a multivariate distribution and a correlation structure, all of this based on the computation of entry-wise norms, 2–norm for location parameter μ and Frobenius norm for scale parameter Σ. According to simulations experiments, we have showed that the proposed robust estimators $\hat{\mu}_R$ and $\hat{\Sigma}_R$ are proficient for multivariate normal distribution and notice that the scale estimator does not require a reweighting parameter in its formula, as other proposals (Rousseeuw and Van Driessen, 1999; Peña and Prieto, 2001, 2007) does to achieve a reasonable estimations; this results in a less complexity estimation, since the reweighting parameter is not required to obtain a competent covariance matrix estimation.

The exposed outlier detection method shows a satisfactory performance in various simulation scenarios, especially for high contamination levels, large sample-space dimensions and high concentrated contaminations, compared with five well-known procedures for outlier detection and robust estimation of the literature. Also, it is important to highlight the proficient performance of our proposal for many different multivariate distribution, each one with special features like multivariate skew and heavy-tail or non-elliptically distributed, emphasizing the outstanding performance in outlier detection, robust estimation, computational affine equivariance and computing times. Followed this, two real dataset examples were also studied, in which the results bear out expected performance. Finally, the results presented in this article show the advantages of our method using projections that maximizes the univariate skewness in the search of multivariate outliers.

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SUPPLEMENTARY MATERIAL

Table 10: c and f values for multivariate normal distribution, $\lambda = 1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.005**	-	0.027	-	0.059	-	0.029	-	0.012	-	0.016
	0.1	**0.976**	**0.001**	1	0.016	1	0.053	1	0.019	1	0.012	1	0.016
	0.2	**0.999**	**0**	1	0.009	1	0.041	1	0.011	0.999	0.012	1	0.017
	0.3	**1**	**0.001**	0	0.005	0.086	0.036	1	0.005	0.998	0.011	0.966	0.017
	0.4	**0.999**	**0**	0.724	0.008	0.058	0.045	0.990	0.001	0.995	0.010	0.892	0.017
20	0.0	**-**	**0.009**	-	0.034	-	0.072	-	0.038	-	0.012	-	0.023
	0.1	**0.990**	**0.001**	1	0.024	1	0.062	1	0.028	1	0.013	1	0.024
	0.2	**1**	**0**	1	0.015	1	0.048	1	0.018	0.328	0.010	0.948	0.025
	0.3	**1**	**0**	0.299	0.023	0.116	0.042	1	0.010	0.636	0.012	0.471	0.022
	0.4	**1**	**0**	0.041	0.029	0.074	0.056	0.607	0.017	0.990	0.013	0.987	0.027
40	0.0	**-**	**0.016**	-	0.070	-	0.100	-	0.084	-	0.011	-	0.061
	0.1	**1**	**0.001**	1	0.056	1	0.079	1	0.071	0.535	0.011	1	0.073
	0.2	**1**	**0**	0.261	0.058	1	0.057	1	0.054	0.019	0.009	0.329	0.059
	0.3	**1**	**0**	0.087	0.062	0.322	0.038	0.989	0.039	0.017	0.009	0.073	0.053
	0.4	**1**	**0**	0.076	0.064	0.107	0.066	0.093	0.077	0.224	0.011	0.201	0.055
100	0.0	**-**	**0.020**	-	0.233	-	0.112	-	0.400	-	0.009	-	0.369
	0.1	**1**	**0**	0.401	0.217	1	0.086	1	0.333	0.025	0.008	0.944	0.325
	0.2	**1**	**0**	0.273	0.221	1	0.061	0.999	0.250	0.012	0.008	0.439	0.367
	0.3	**1**	**0**	0.252	0.225	0.989	0.036	0.448	0.379	0.010	0.008	0.398	0.364
	0.4	**1**	**0**	0.239	0.227	0.122	0.089	0.413	0.392	0.009	0.008	0.388	0.373
200	0.0	**-**	**0.015**	-	0.276	-	0.122	-	0.300	-	0.003	-	0.200
	0.1	**1**	**0**	0.312	0.273	1	0.093	0.978	0.225	0.005	0.002	0.253	0.194
	0.2	**1**	**0**	0.287	0.274	1	0.065	0.345	0.289	0.003	0.003	0.221	0.195
	0.3	**1**	**0**	0.281	0.276	1	0.037	0.317	0.293	0.003	0.002	0.201	0.197
	0.4	**1**	**0**	0.280	0.275	0.108	0.097	0.306	0.296	0.003	0.003	0.206	0.196

Table 11: c and f values for multivariate normal distribution, $\lambda = 0.1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.005**	-	0.028	-	0.061	-	0.029	-	0.011	-	0.016
	0.1	**0.974**	**0.001**	1	0.016	1	0.053	1	0.019	1	0.012	1	0.017
	0.2	**0.994**	**0**	0	0.126	1	0.043	1	0.011	1	0.012	1	0.017
	0.3	**1**	**0**	0	0.480	0.009	0.122	1	0.005	0.994	0.012	0.996	0.019
	0.4	**1**	**0**	0	0.834	0	0.658	1	0.001	1	0.010	1	0.018
20	0.0	**-**	**0.009**	-	0.034	-	0.073	-	0.038	-	0.012	-	0.022
	0.1	**0.982**	**0.001**	0.992	0.024	1	0.061	1	0.028	1	0.013	1	0.025
	0.2	**0.994**	**0.001**	0	0.389	1	0.047	1	0.018	0.998	0.014	1	0.026
	0.3	**1**	**0**	0	0.718	0.014	0.193	1	0.001	0.968	0.021	1	0.028
	0.4	**1**	**0**	0	0.998	0	0.816	1	0.004	1	0.013	1	0.022
40	0.0	**-**	**0.016**	-	0.071	-	0.101	-	0.084	-	0.012	-	0.061
	0.1	**0.990**	**0**	0	0.303	1	0.080	1	0.071	1	0.014	1	0.069
	0.2	**0.996**	**0.001**	0	0.577	1	0.062	1	0.054	0.922	0.056	1	0.092
	0.3	**1**	**0**	0	0.956	0.088	0.286	1	0.039	0.604	0.262	1	0.085
	0.4	**1**	**0**	0	1	0	0.948	1	0.021	1	0.022	1	0.055
100	0.0	**-**	**0.021**	-	0.233	-	0.112	-	0.400	-	0.008	-	0.373
	0.1	**0.962**	**0**	0	0.412	1	0.095	0.958	0.338	0.992	0.016	1	0.314
	0.2	**0.986**	**0**	0	0.651	1	0.071	1	0.250	0.388	0.316	0.314	0.421
	0.3	**1**	**0**	0	0.986	0.459	0.275	1	0.143	0	0.571	0.012	0.566
	0.4	**1**	**0**	0	1	0	0.999	1	0	0.602	0.265	0.592	0.272
200	0.0	**-**	**0.014**	-	0.277	-	0.122	-	0.300	-	0.003	-	0.200
	0.1	**0.990**	**0**	0	0.336	1	0.105	0	0.333	0	0.099	0	0.222
	0.2	**0.992**	**0**	0	0.589	1	0.079	1	0.0125	0.008	0.246	0	0.250
	0.3	**1**	**0**	0	0.982	0.818	0.134	1	0	0	0.285	0	0.286
	0.4	**1**	**0**	0	1	0	1	0	0.500	0	0.330	0	0.333

Table 12: c and f values for multivariate normal distribution, $\lambda = 1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.004**	-	0.028	-	0.061	-	0.029	-	0.012	-	0.016
	0.1	**0.988**	**0.001**	1	0.016	1	0.053	1	0.019	1	0.012	1	0.017
	0.2	**0.992**	**0.001**	1	0.009	1	0.043	1	0.012	1	0.013	1	0.018
	0.3	**1**	**0**	1	0.005	1	0.034	1	0.005	1	0.012	1	0.018
	0.4	**1**	**0**	0.809	0.007	0.058	0.041	1	0.001	1	0.011	1	0.018
20	0.0	**-**	**0.008**	-	0.034	-	0.072	-	0.038	-	0.012	-	0.022
	0.1	**0.970**	**0.001**	1	0.023	1	0.061	1	0.028	1	0.012	1	0.024
	0.2	**0.992**	**0.001**	1	0.015	1	0.047	1	0.018	0.409	0.009	1	0.025
	0.3	**1**	**0**	0.409	0.022	1	0.037	1	0.010	0.728	0.013	0.915	0.029
	0.4	**1**	**0**	0.053	0.030	0.077	0.044	1	0.004	0.998	0.014	0.994	0.028
40	0.0	**-**	**0.016**	-	0.071	-	0.100	-	0.083	-	0.011	-	0.061
	0.1	**0.986**	**0**	1	0.057	1	0.089	1	0.071	0.614	0.011	1	0.075
	0.2	**0.992**	**0**	0.331	0.057	1	0.058	1	0.053	0.022	0.009	0.814	0.076
	0.3	**1**	**0**	0.085	0.061	1	0.037	1	0.038	0.016	0.009	0.082	0.053
	0.4	**1**	**0**	0.073	0.065	0.149	0.034	1	0.021	0.267	0.012	0.209	0.054
100	0.0	**-**	**0.019**	-	0.232	-	0.113	-	0.400	-	0.009	-	0.369
	0.1	**0.966**	**0**	0.418	0.216	1	0.088	1	0.333	0.025	0.007	0.999	0.325
	0.2	**0.992**	**0**	0.276	0.222	1	0.064	1	0.250	0.013	0.008	0.422	0.351
	0.3	**1**	**0**	0.249	0.227	0.459	0.038	1	0.143	0.012	0.008	0.389	0.358
	0.4	**1**	**0**	0.240	0.226	0.657	0.022	0.998	0.001	0.009	0.008	0.382	0.365
200	0.0	**-**	**0.015**	-	0.278	-	0.121	-	0.300	-	0.003	-	0.200
	0.1	**0.998**	**0**	0.317	0.273	1	0.094	1	0.222	0.005	0.002	0.259	0.193
	0.2	**0.988**	**0**	0.288	0.274	1	0.068	0.749	0.188	0.003	0.003	0.247	0.196
	0.3	**0.998**	**0**	0.281	0.275	1	0.039	1	0	0.003	0.003	0.207	0.197
	0.4	**1**	**0**	0.282	0.273	0.969	0.015	0.749	0	0.003	0.002	0.204	0.198

Table 13: c and f values for multivariate correlated normal distribution, $\lambda = 1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.003**	-	0.027	-	0.060	-	0.029	-	0.011	-	0.015
	0.1	**0.933**	**0.001**	0.999	0.016	1	0.052	0.999	0.019	0.999	0.012	0.999	0.017
	0.2	**0.975**	**0**	0.999	0.009	0.998	0.039	0.999	0.011	0.999	0.012	0.999	0.018
	0.3	**0.942**	**0**	0.999	0.005	0.806	0.026	0.999	0.005	0.999	0.013	0.997	0.019
	0.4	**0.885**	**0**	0.999	0.003	0.331	0.045	0.999	0.001	0.990	0.010	0.891	0.014
20	0.0	**-**	**0.004**	-	0.035	-	0.073	-	0.038	-	0.012	-	0.023
	0.1	**0.963**	**0.001**	1	0.023	1	0.061	1	0.027	1	0.012	1	0.023
	0.2	**0.997**	**0.001**	1	0.016	1	0.051	1	0.019	1	0.014	1	0.027
	0.3	**1**	**0.001**	1	0.011	0.932	0.029	1	0.010	1	0.014	1	0.029
	0.4	**0.992**	**0.001**	1	0.006	0.323	0.051	1	0.004	0.994	0.013	0.930	0.022
40	0.0	**-**	**0.009**	-	0.071	-	0.101	-	0.084	-	0.011	-	0.062
	0.1	**0.945**	**0.001**	1	0.056	1	0.080	1	0,068	1	0.013	1	0.068
	0.2	**0.988**	**0**	1	0.047	1	0.065	1	0,055	1	0.017	1	0.093
	0.3	**1**	**0.001**	1	0.034	0.995	0.040	1	0.039	0.994	0.018	0.996	0.075
	0.4	**1**	**0.001**	1	0.024	0.357	0.059	1	0.022	0.803	0.011	0.838	0.041
100	0.0	**-**	**0.020**	-	0.232	-	0.122	-	0.399	-	0.009	-	0.376
	0.1	**0.920**	**0.002**	1	0.189	1	0.097	1	0.333	1	0.013	1	0.320
	0.2	**0.996**	**0**	1	0.137	1	0.073	1	0.250	0.982	0.016	1	0.248
	0.3	**1**	**0**	1	0.107	1	0.047	1	0.143	0.807	0.008	1	0.143
	0.4	**1**	**0**	0.976	0.027	0.457	0.051	1	0	0.386	0.006	0.964	0.024
200	0.0	**-**	**0.019**	-	0.278	-	0.143	-	0.300	-	0.003	-	0.200
	0.1	**0.964**	**0.001**	1	0.203	1	0.109	1	0.222	0.908	0.006	1	0.111
	0.2	**0.994**	**0**	0.951	0.159	1	0.079	1	0.125	0.480	0.001	0.965	0.009
	0.3	**1**	**0**	0	0.800	0	0.926	0	0.429	0	0.274	0	0.286
	0.4	**1**	**0**	0.625	0.149	1	0.048	1	0	0.263	0.001	0.665	0.001

Table 14: c and f values for multivariate correlated normal distribution, $\lambda = 0.1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.003**	-	0.028	-	0.060	-	0.029	-	0.012	-	0.016
	0.1	**0.947**	**0.001**	1	0.016	1	0.054	1	0.020	1	0.012	1	0.017
	0.2	**1**	**0**	1	0.009	0.998	0.046	1	0.011	1	0.012	1	0.017
	0.3	**1**	**0**	0.560	0.186	0.094	0.150	1	0.005	0.990	0.012	0.997	0.017
	0.4	**0.998**	**0**	0.106	0.799	0.003	0.564	1	0.001	1	0.010	0.992	0.020
20	0.0	**-**	**0.005**	-	0.034	-	0.073	-	0.038	-	0.012	-	0.022
	0.1	**0.944**	**0.001**	1	0.023	1	0.059	1	0.027	0.939	0.012	1	0.023
	0.2	**0.996**	**0**	0.498	0.113	1	0.049	0.998	0.018	0.391	0.026	1	0.025
	0.3	**1**	**0**	0.026	0.659	0.007	0.274	1	0.009	0.596	0.098	0.994	0.033
	0.4	**1**	**0**	0	0.997	0	0.723	1	0.004	0.974	0.034	0.994	0.026
40	0.0	**-**	**0.009**	-	0.071	-	0.102	-	0.084	-	0.011	-	0.062
	0.1	**0.966**	**0**	0.976	0.058	1	0.081	0.966	0.069	0.168	0.019	1	0.074
	0.2	**0.994**	**0**	0.056	0.488	1	0.065	0.998	0.053	0.008	0.088	0.998	0.072
	0.3	**1**	**0**	0	0.782	0	0.483	1	0.038	0.218	0.447	0.899	0.153
	0.4	**1**	**0**	0	1	0	0.890	1	0.022	0.788	0.178	0.778	0.207
100	0.0	**-**	**0.019**	-	0.232	-	0.124	-	0.399	-	0.009	-	0.371
	0.1	**0.960**	**0**	0.141	0.342	1	0.105	0.234	0.418	0.032	0.027	0.966	0.325
	0.2	**0.970**	**0**	0	0.511	1	0.085	0.994	0.252	0	0.462	0.026	0.493
	0.3	**1**	**0**	0	0.860	0	0.758	1	0.143	0	0.571	0.002	0.571
	0.4	**1**	**0**	0	1	0	0.997	1	0	0.160	0.559	0.114	0.591
200	0.0	**-**	**0.019**	-	0.278	-	0.141	-	0.300	-	0.003	-	0.200
	0.1	**0.980**	**0.001**	0	0.328	1	0.121	0.232	0.308	0	0.015	0.003	0.222
	0.2	**0.980**	**0.001**	0	0.385	1	0.096	0.054	0.362	0	0.214	0	0.250
	0.3	**1**	**0**	0	0.804	0	0.938	1	0	0	0.275	0	0.286
	0.4	**1**	**0**	0	0.999	0	1	0.006	0.483	0	0.321	0	0.333

Table 15: c and f values for multivariate correlated normal distribution, $\lambda = 1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.003**	-	0.028	-	0.060	-	0.029	-	0.012	-	0.016
	0.1	**0.976**	**0.001**	1	0.016	1	0.054	1	0.019	1	0.012	1	0.017
	0.2	**0.992**	**0**	1	0.009	1	0.046	1	0.011	1	0.012	1	0.017
	0.3	**1**	**0**	1	0.005	0.969	0.029	1	0.005	1	0.012	0.999	0.018
	0.4	**1**	**0**	1	0.003	0.308	0.042	1	0.001	1	0.010	0.999	0.014
20	0.0	**-**	**0.005**	-	0.035	-	0.073	-	0.038	-	0.012	-	0.023
	0.1	**0.966**	**0.001**	1	0.023	1	0.061	1	0.028	1	0.013	1	0.024
	0.2	**0.982**	**0**	1	0.016	1	0.050	1	0.018	1	0.014	1	0.026
	0.3	**1**	**0**	1	0.009	0.999	0.037	1	0.009	1	0.013	1	0.029
	0.4	**1**	**0**	1	0.006	0.328	0.041	1	0.004	1	0.013	0.999	0.026
40	0.0	**-**	**0.009**	-	0.071	-	0.101	-	0.083	-	0.011	-	0.062
	0.1	**0.986**	**0**	1	0.058	1	0.082	1	0.069	1	0.014	1	0.070
	0.2	**0.984**	**0.001**	1	0.046	1	0.065	1	0.056	1	0.017	1	0.094
	0.3	**1**	**0.001**	1	0.035	1	0.046	1	0.040	0.995	0.018	0.998	0.071
	0.4	**1**	**0.001**	1	0.024	0.456	0.039	1	0.023	0.817	0.012	0.857	0.041
100	0.0	**-**	**0.019**	-	0.232	-	0.123	-	0.399	-	0.009	-	0.376
	0.1	**0.970**	**0**	1	0.189	1	0.102	1	0.333	1	0.013	1	0.313
	0.2	**0.990**	**0**	1	0.138	1	0.076	1	0.250	0.985	0.017	1	0.247
	0.3	**1**	**0**	1	0.107	1	0.051	1	0.143	0.806	0.008	1	0.143
	0.4	**1**	**0**	0.977	0.027	0.541	0.052	1	0	0.372	0.006	0.964	0.024
200	0.0	**-**	**0.019**	-	0.277	-	0.141	-	0.300	-	0.003	-	0.200
	0.1	**0.963**	**0.001**	1	0.203	1	0.111	1	0.222	0.900	0.006	1	0.111
	0.2	**0.974**	**0**	0.952	0.160	1	0.082	1	0.125	0.474	0.001	0.964	0.009
	0.3	**0.998**	**0**	0.627	0.148	1	0.053	1	0	0.274	0.001	0.664	0.001
	0.4	**1**	**0**	0.410	0.173	0.629	0.048	0.999	0	0.036	0.001	0.417	0.055

Table 16: c and f values for multivariate uniform distribution, $\lambda = 1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.002**	-	0	-	0.001	-	0	-	0	-	0
	0.1	**0.971**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.2	**0.998**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.3	**1**	**0**	1	0.012	0.013	0.001	1	0	0.998	0	0.842	0
	0.4	**1**	**0**	0.794	0	0.003	0.001	1	0	0.991	0	0.897	0
20	0.0	**-**	**0.004**	-	0.002	-	0.004	-	0.002	-	0	-	0
	0.1	**0.992**	**0**	1	0	1	0.004	1	0.001	1	0	1	0
	0.2	**1**	**0**	1	0	1	0.003	1	0.001	0.896	0	0.998	0.001
	0.3	**1**	**0**	0.675	0	0.031	0.003	1	0	0.658	0	0.239	0
	0.4	**1**	**0**	0.006	0.001	0.005	0.003	1	0.001	0.938	0	0.631	0.001
40	0.0	**-**	**0.015**	-	0.010	-	0.017	-	0.012	-	0	-	0.005
	0.1	**0.994**	**0.001**	1	0.006	1	0.014	1	0.011	0.748	0	1	0.006
	0.2	**1**	**0**	0.715	0.005	1	0.010	1	0.009	0.001	0	0.351	0.005
	0.3	**1**	**0**	0.016	0.008	0.523	0.006	1	0.007	0	0	0.008	0.004
	0.4	**1**	**0**	0.011	0.008	0.019	0.010	0.990	0.005	0.100	0	0.046	0.004
100	0.0	**-**	**0.025**	-	0.197	-	0.027	-	0.400	-	0.001	-	0.336
	0.1	**1**	**0**	0.406	0.177	1	0.024	1	0.333	0.004	0	0.966	0.310
	0.2	**1**	**0**	0.246	0.184	1	0.018	1	0.250	0.001	0	0.407	0.321
	0.3	**1**	**0**	0.219	0.188	1	0.009	0.798	0.229	0.001	0.001	0.358	0.316
	0.4	**1**	**0**	0.205	0.193	0.022	0.037	0.415	0.389	0	0	0.354	0.335
200	0.0	**-**	**0.019**	-	0.281	-	0.028	-	0.300	-	0.001	-	0.200
	0.1	**0.998**	**0**	0.324	0.276	1	0.031	0.993	0.223	0.001	0.001	0.264	0.193
	0.2	**1**	**0**	0.293	0.278	1	0.023	0.404	0.274	0.001	0	0.221	0.195
	0.3	**1**	**0**	0.287	0.279	1	0.012	0.323	0.290	0.001	0	0.209	0.196
	0.4	**1**	**0**	0.283	0.280	0.005	0.090	0.307	0.295	0	0.001	0.203	0.198

Table 17: c and f values for multivariate uniform distribution, $\lambda = 0.1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.001**	-	0.001	-	0.001	-	0	-	0	-	0
	0.1	**0.976**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.2	**0.986**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.3	**1**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.4	**1**	**0**	0.912	0	0.003	0.001	1	0	1	0	1	0
20	0.0	**-**	**0.004**	-	0.001	-	0.004	-	0.001	-	0	-	0
	0.1	**0.958**	**0**	1	0	1	0.004	1	0.001	1	0	1	0
	0.2	**0.970**	**0**	1	0	1	0.003	1	0.001	0.939	0	1	0.001
	0.3	**1**	**0**	0.751	0	1	0.003	1	0	0.790	0	0.879	0
	0.4	**1**	**0**	0.010	0.001	0.007	0.003	1	0.001	0.980	0	0.745	0.001
40	0.0	**-**	**0.015**	-	0.010	-	0.017	-	0.012	-	0	-	0.004
	0.1	**0.970**	**0.001**	1	0.006	1	0.014	1	0.011	0.846	0	1	0.006
	0.2	**0.996**	**0**	0.798	0.004	1	0.011	1	0.009	0.001	0	0.870	0.009
	0.3	**1**	**0**	0.016	0.008	1	0.006	1	0.007	0.002	0	0.026	0.004
	0.4	**1**	**0**	0.011	0.009	0.052	0.003	1	0.005	0.164	0	0.062	0.005
100	0.0	**-**	**0.024**	-	0.198	-	0.027	-	0.400	-	0.001	-	0.333
	0.1	**0.972**	**0**	0.419	0.178	1	0.026	1	0.333	0.003	0.001	0.997	0.303
	0.2	**0.992**	**0**	0.251	0.186	1	0.022	1	0.250	0.001	0	0.408	0.319
	0.3	**1**	**0**	0.219	0.191	1	0.011	1	0.143	0.001	0	0.362	0.319
	0.4	**1**	**0**	0.204	0.192	0.866	0.004	0.697	0.202	0.001	0.001	0.347	0.329
200	0.0	**-**	**0.019**	-	0.281	-	0.028	-	0.300	-	0.001	-	0.200
	0.1	**0.986**	**0**	0.327	0.277	1	0.032	1	0.222	0.001	0	0.257	0.194
	0.2	**0.994**	**0**	0.300	0.278	1	0.027	0.908	0.148	0.001	0	0.221	0.195
	0.3	**0.998**	**0**	0.290	0.278	1	0.014	0.768	0.099	0.001	0.001	0.207	0.197
	0.4	**1**	**0**	0.284	0.280	1	0.004	0.611	0.093	0.001	0.001	0.203	0.198

Table 18: c and f values for multivariate uniform distribution, $\lambda = 1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.001**	-	0.001	-	0.001	-	0	-	0	-	0
	0.1	**0.970**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.2	**0.982**	**0**	1	0	1	0.001	1	0	1	0	1	0
	0.3	**1**	**0**	1	0	1	0.001	1	0	1	0	0.998	0
	0.4	**1**	**0**	0.878	0	0.003	0.001	1	0	1	0	1	0
20	0.0	**-**	**0.005**	-	0.001	-	0.004	-	0.001	-	0	-	0
	0.1	**0.980**	**0.001**	1	0	1	0.004	1	0.001	1	0	1	0
	0.2	**0.974**	**0**	1	0	1	0.003	1	0.001	0.952	0	1	0.001
	0.3	**1**	**0**	0.789	0	1	0.003	1	0	0.810	0	0.886	0.001
	0.4	**1**	**0**	0.008	0.001	0.008	0.003	1	0.001	0.980	0	0.710	0.001
40	0.0	**-**	**0.015**	-	0.010	-	0.018	-	0.013	-	0	-	0.005
	0.1	**0.972**	**0**	1	0.006	1	0.014	1	0.011	0.845	0	1	0.007
	0.2	**0.982**	**0**	0.777	0.005	1	0.011	1	0.009	0.001	0	0.892	0.009
	0.3	**1**	**0**	0.016	0.007	1	0.007	1	0.007	0.004	0	0.024	0.004
	0.4	**1**	**0**	0.011	0.009	0.053	0.003	1	0.005	0.136	0	0.072	0.005
100	0.0	**-**	**0.025**	-	0.196	-	0.027	-	0.400	-	0.001	-	0.333
	0.1	**0.968**	**0**	0.418	0.178	1	0.025	1	0.333	0.004	0	0.995	0.297
	0.2	**0.996**	**0**	0.247	0.185	1	0.021	1	0.250	0.001	0	0.394	0.309
	0.3	**1**	**0**	0.218	0.188	1	0.011	1	0.143	0.001	0	0.364	0.324
	0.4	**1**	**0**	0.205	0.190	0.859	0.004	0.700	0.200	0.001	0	0.338	0.319
200	0.0	**-**	**0.018**	-	0.281	-	0.028	-	0.300	-	0.001	-	0.200
	0.1	**0.986**	**0**	0.324	0.276	1	0.032	1	0.222	0.001	0	0.262	0.193
	0.2	**0.984**	**0**	0.295	0.278	1	0.027	0.903	0.149	0.001	0	0.221	0.195
	0.3	**1**	**0**	0.288	0.279	1	0.014	0.749	0.108	0.001	0.001	0.208	0.197
	0.4	**1**	**0**	0.284	0.279	1	0.004	0.602	0.098	0.001	0.001	0.203	0.198

Table 19: c and f values for multivariate t_4–Student distribution, $\lambda = 1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.051**	-	0.125	-	0.198	-	0.129	-	0.106	-	0.122
	0.1	**1**	**0.046**	1	0.104	1	0.173	1	0.104	1	0.104	1	0.122
	0.2	**1**	**0.042**	1	0.086	1	0.150	1	0.079	1	0.101	0.995	0.124
	0.3	**1**	**0.041**	1	0.070	0.208	0.135	1	0.055	1	0.095	0.994	0.126
	0.4	**1**	**0.041**	0.279	0.097	0.187	0.160	0.338	0.092	0.998	0.074	0.650	0.099
20	0.0	**-**	**0.084**	-	0.162	-	0.223	-	0.158	-	0.126	-	0.171
	0.1	**0.998**	**0.074**	1	0.144	1	0.195	1	0.133	1	0.127	1	0.178
	0.2	**1**	**0.075**	1	0.125	1	0.167	1	0.105	0.874	0.124	0.950	0.181
	0.3	**1**	**0.072**	0.339	0.134	0.267	0.156	0.965	0.079	0.924	0.116	0.805	0.154
	0.4	**1**	**0.071**	0.158	0.145	0.220	0.185	0.157	0.143	1	0.086	1	0.129
40	0.0	**-**	**0.080**	-	0.251	-	0.242	-	0.241	-	0.152	-	0.356
	0.1	**1**	**0.073**	1	0.230	1	0.212	1	0.214	0.632	0.145	1	0.287
	0.2	**1**	**0.070**	0.380	0.226	1	0.179	1	0.179	0.176	0.135	0.590	0.306
	0.3	**1**	**0.063**	0.270	0.230	0.398	0.164	0.319	0.215	0.159	0.136	0.380	0.318
	0.4	**1**	**0.056**	0.250	0.233	0.257	0.196	0.240	0.223	0.794	0.094	0.762	0.176
100	0.0	**-**	**0.012**	-	0.307	-	0.261	-	0.400	-	0.130	-	0.400
	0.1	**0.998**	**0.006**	0.451	0.291	1	0.223	1	0.333	0.160	0.120	0.886	0.346
	0.2	**1**	**0.005**	0.349	0.292	1	0.181	0.780	0.305	0.131	0.118	0.463	0.383
	0.3	**1**	**0.005**	0.322	0.294	0.897	0.139	0.429	0.388	0.120	0.115	0.427	0.388
	0.4	**1**	**0**	0.310	0.297	0.290	0.223	0.412	0.392	0.120	0.117	0.410	0.390
200	0.0	**-**	**0.010**	-	0.272	-	0.274	-	0.300	-	0.039	-	0.200
	0.1	**1**	**0**	0.310	0.267	1	0.232	0.761	0.249	0.027	0.037	0.236	0.196
	0.2	**0.998**	**0**	0.283	0.268	1	0.180	0.327	0.293	0.029	0.036	0.210	0.198
	0.3	**1**	**0**	0.277	0.268	0.998	0.128	0.311	0.295	0.029	0.036	0.205	0.198
	0.4	**1**	**0**	0.273	0.269	0.329	0.212	0.305	0.297	0.033	0.035	0.203	0.198

Table 20: c and f values for multivariate t_4–Student distribution, $\lambda = 0.1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.051**	-	0.125	-	0.198	-	0.128	-	0.105	-	0.122
	0.1	**0.998**	**0.043**	1	0.105	1	0.174	1	0.104	1	0.104	1	0.123
	0.2	**1**	**0.043**	1	0.087	1	0.151	1	0.080	1	0.101	1	0.125
	0.3	**1**	**0.041**	1	0.070	1	0.128	1	0.054	1	0.095	1	0.126
	0.4	**1**	**0.045**	0.881	0.063	0.175	0.143	1	0.028	1	0.074	1	0.010
20	0.0	**-**	**0.084**	-	0.162	-	0.222	-	0.157	-	0.127	-	0.170
	0.1	**0.994**	**0.078**	1	0.143	1	0.195	1	0.133	1	0.127	1	0.176
	0.2	**0.998**	**0.075**	1	0.126	1	0.170	1	0.105	0.904	0.125	1	0.187
	0.3	**1**	**0.074**	0.399	0.131	1	0.141	1	0.076	0.971	0.117	0.985	0.160
	0.4	**1**	**0.071**	0.160	0.146	0.226	0.163	1	0.042	1	0.085	1	0.129
40	0.0	**-**	**0.081**	-	0.252	-	0.243	-	0.243	-	0.153	-	0.354
	0.1	**0.994**	**0.074**	1	0.229	1	0.211	1	0.212	0.701	0.146	1	0.285
	0.2	**1**	**0.070**	0.376	0.225	1	0.183	1	0.178	0.181	0.134	0.876	0.281
	0.3	**1**	**0.062**	0.270	0.230	1	0.148	1	0.135	0.170	0.135	0.388	0.319
	0.4	**1**	**0.054**	0.251	0.235	0.309	0.168	1	0.076	0.860	0.090	0.805	0.162
100	0.0	**-**	**0.012**	-	0.307	-	0.260	-	0.400	-	0.129	-	0.400
	0.1	**0.990**	**0.006**	0.452	0.290	1	0.224	1	0.333	0.160	0.120	0.980	0.335
	0.2	**1**	**0.005**	0.349	0.292	1	0.183	1	0.250	0.129	0.117	0.464	0.383
	0.3	**1**	**0.005**	0.325	0.294	1	0.136	1	0.143	0.123	0.117	0.427	0.387
	0.4	**1**	**0**	0.312	0.297	0.568	0.140	1	0.018	0.119	0.117	0.410	0.390
200	0.0	**-**	**0.009**	-	0.271	-	0.273	-	0.300	-	0.039	-	0.200
	0.1	**0.968**	**0.001**	0.303	0.268	1	0.231	0.997	0.223	0.026	0.037	0.233	0.196
	0.2	**0.982**	**0.001**	0.286	0.267	1	0.179	0.908	0.148	0.027	0.035	0.211	0.197
	0.3	**1**	**0**	0.276	0.267	1	0.125	1	0.011	0.030	0.035	0.204	0.198
	0.4	**1**	**0**	0.272	0.270	0.795	0.081	0.749	0.010	0.031	0.034	0.203	0.198

Table 21: c and f values for multivariate t_4–Student distribution, $\lambda = 1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.051**	-	0.123	-	0.197	-	0.127	-	0.103	-	0.120
	0.1	**1**	**0.043**	1	0.104	1	0.172	1	0.104	1	0.103	1	0.122
	0.2	**1**	**0.044**	1	0.087	1	0.151	1	0.080	1	0.101	1	0.124
	0.3	**1**	**0.044**	1	0.072	1	0.130	1	0.055	1	0.097	1	0.129
	0.4	**1**	**0.042**	0.890	0.062	0.177	0.142	1	0.029	1	0.074	1	0.109
20	0.0	**-**	**0.084**	-	0.162	-	0.221	-	0.157	-	0.126	-	0.169
	0.1	**1**	**0.077**	1	0.143	1	0.195	1	0.133	1	0.128	1	0.176
	0.2	**1**	**0.076**	1	0.126	1	0.170	1	0.106	0.888	0.125	1	0.187
	0.3	**1**	**0.074**	0.402	0.132	1	0.142	1	0.075	0.980	0.118	0.982	0.160
	0.4	**1**	**0.071**	0.163	0.144	0.224	0.161	1	0.044	1	0.084	1	0.129
40	0.0	**-**	**0.081**	-	0.250	-	0.241	-	0.242	-	0.152	-	0.349
	0.1	**0.994**	**0.070**	1	0.229	1	0.211	1	0.212	0.707	0.147	1	0.282
	0.2	**1**	**0.069**	0.372	0.226	1	0.182	1	0.178	0.178	0.135	0.890	0.281
	0.3	**1**	**0.063**	0.270	0.231	1	0.149	1	0.135	0.166	0.136	0.384	0.319
	0.4	**1**	**0.057**	0.251	0.234	0.311	0.167	1	0.075	0.835	0.093	0.802	0.167
100	0.0	**-**	**0.012**	-	0.306	-	0.260	-	0.400	-	0.130	-	0.400
	0.1	**0.994**	**0.006**	0.459	0.290	1	0.224	1	0.333	0.159	0.118	0.983	0.335
	0.2	**1**	**0.005**	0.349	0.292	1	0.182	1	0.250	0.133	0.118	0.462	0.384
	0.3	**1**	**0.005**	0.322	0.296	1	0.137	1	0.143	0.123	0.118	0.429	0.386
	0.4	**1**	**0**	0.309	0.298	0.561	0.140	1	0.017	0.120	0.117	0.410	0.391
200	0.0	**-**	**0.009**	-	0.271	-	0.274	-	0.300	-	0.039	-	0.200
	0.1	**0.968**	**0**	0.304	0.267	1	0.232	1	0.222	0.028	0.037	0.232	0.197
	0.2	**0.988**	**0**	0.282	0.269	1	0.180	0.918	0.145	0.029	0.037	0.209	0.198
	0.3	**0.998**	**0**	0.277	0.268	1	0.127	1	0.012	0.029	0.034	0.207	0.197
	0.4	**1**	**0**	0.272	0.270	0.795	0.080	0.749	0.010	0.032	0.034	0.202	0.199

Table 22: c and f values for multivariate gamma distribution, $\lambda = 1$ and $\delta = 3$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.070**	-	0.128	-	0.198	-	0.132	-	0.095	-	0.114
	0.1	**0.965**	**0.056**	1	0.106	1	0.169	1	0.106	1	0.097	1	0.117
	0.2	**1**	**0.049**	1	0.084	1	0.131	1	0.077	1	0.093	0.976	0.116
	0.3	**1**	**0.051**	1	0.064	0.226	0.123	1	0.048	0.998	0.087	0.932	0.113
	0.4	**1**	**0.047**	0.178	0.087	0.186	0.148	0.230	0.085	0.996	0.067	0.475	0.086
20	0.0	**-**	**0.117**	-	0.154	-	0.211	-	0.149	-	0.101	-	0.150
	0.1	**0.984**	**0.101**	1	0.131	1	0.176	1	0.121	1	0.103	1	0.156
	0.2	**1**	**0.092**	1	0.111	1	0.146	1	0.093	0.538	0.089	0.902	0.155
	0.3	**1**	**0.091**	0.284	0.115	0.255	0.141	0.995	0.061	0.865	0.096	0.744	0.132
	0.4	**1**	**0.082**	0.137	0.121	0.206	0.162	0.134	0.118	1	0.071	0.994	0.121
40	0.0	**-**	**0.099**	-	0.227	-	0.218	-	0.221	-	0.106	-	0.309
	0.1	**1**	**0.069**	1	0.202	1	0.180	1	0.192	0.468	0.094	1	0.274
	0.2	**1**	**0.059**	0.307	0.194	1	0.144	1	0.157	0.116	0.085	0.540	0.265
	0.3	**1**	**0.048**	0.230	0.195	0.356	0.131	0.317	0.179	0.103	0.085	0.309	0.264
	0.4	**1**	**0.027**	0.217	0.200	0.240	0.160	0.207	0.191	0.652	0.074	0.612	0.188
100	0.0	**-**	**0.023**	-	0.287	-	0.221	-	0.400	-	0.072	-	0.398
	0.1	**1**	**0**	0.409	0.273	1	0.179	1	0.333	0.086	0.060	0.910	0.342
	0.2	**1**	**0**	0.318	0.275	1	0.136	0.923	0.269	0.068	0.061	0.443	0.387
	0.3	**1**	**0**	0.300	0.277	0.872	0.097	0.424	0.390	0.066	0.061	0.418	0.390
	0.4	**1**	**0**	0.289	0.278	0.273	0.168	0.410	0.393	0.064	0.062	0.406	0.392
200	0.0	**-**	**0.017**	-	0.271	-	0.227	-	0.300	-	0.016	-	0.200
	0.1	**0.998**	**0**	0.306	0.268	1	0.182	0.891	0.234	0.014	0.014	0.242	0.195
	0.2	**0.998**	**0**	0.283	0.269	1	0.136	0.329	0.293	0.014	0.014	0.213	0.197
	0.3	**1**	**0**	0.276	0.270	0.998	0.088	0.311	0.296	0.013	0.014	0.207	0.197
	0.4	**1**	**0**	0.273	0.271	0.367	0.126	0.305	0.296	0.014	0.014	0.203	0.198

Table 23: c and f values for multivariate gamma distribution, $\lambda = 0.1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.074**	-	0.131	-	0.200	-	0.135	-	0.098	-	0.117
	0.1	**0.997**	**0.052**	1	0.105	1	0.169	1	0.105	1	0.095	1	0.116
	0.2	**1**	**0.053**	1	0.086	1	0.142	1	0.078	1	0.095	1	0.120
	0.3	**1**	**0.051**	1	0.064	1	0.116	1	0.047	1	0.088	0.998	0.118
	0.4	**1**	**0.049**	0.856	0.055	0.175	0.138	1	0.019	1	0.068	1	0.104
20	0.0	**-**	**0.118**	-	0.155	-	0.212	-	0.150	-	0.103	-	0.151
	0.1	**1**	**0.102**	1	0.132	1	0.177	1	0.121	1	0.105	1	0.155
	0.2	**0.996**	**0.099**	1	0.112	1	0.148	1	0.092	0.635	0.092	1	0.167
	0.3	**1**	**0.093**	0.353	0.113	1	0.119	1	0.059	0.964	0.097	0.947	0.137
	0.4	**1**	**0.081**	0.140	0.123	0.206	0.146	1	0.028	1	0.071	1	0.123
40	0.0	**-**	**0.097**	-	0.225	-	0.217	-	0.218	-	0.106	-	0.307
	0.1	**0.988**	**0.078**	1	0.202	1	0.182	1	0.191	0.534	0.096	1	0.269
	0.2	**1**	**0.057**	0.321	0.193	1	0.149	1	0.157	0.114	0.086	0.869	0.249
	0.3	**1**	**0.052**	0.230	0.199	1	0.114	1	0.114	0.101	0.086	0.317	0.268
	0.4	**1**	**0.029**	0.215	0.200	0.280	0.133	1	0.058	0.704	0.074	0.637	0.184
100	0.0	**-**	**0.023**	-	0.285	-	0.222	-	0.400	-	0.072	-	0.400
	0.1	**0.988**	**0**	0.408	0.273	1	0.181	1	0.333	0.088	0.063	0.979	0.335
	0.2	**0.996**	**0**	0.316	0.276	1	0.139	1	0.250	0.070	0.062	0.446	0.387
	0.3	**1**	**0**	0.298	0.278	1	0.095	1	0.143	0.065	0.062	0.417	0.389
	0.4	**1**	**0**	0.289	0.280	0.543	0.094	1	0.003	0.064	0.062	0.405	0.393
200	0.0	**-**	**0.016**	-	0.271	-	0.228	-	0.300	-	0.016	-	0.200
	0.1	**0.990**	**0**	0.301	0.268	1	0.184	1	0.222	0.014	0.014	0.242	0.195
	0.2	**0.986**	**0**	0.283	0.269	1	0.136	1	0.125	0.014	0.014	0.215	0.196
	0.3	**1**	**0**	0.277	0.268	1	0.089	1	0	0.014	0.014	0.205	0.198
	0.4	**0.998**	**0**	0.272	0.272	0.786	0.048	0.750	0	0.014	0.014	0.202	0.199

Table 24: c and f values for multivariate gamma distribution, $\lambda = 1$ and $\delta = 7$.

p	α	**Skew**		Fast–MCD		OGK		O–H		Kur1		Kur2	
		c	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**	**c**	**f**
10	0.0	**-**	**0.071**	-	0.128	-	0.198	-	0.132	-	0.095	-	0.114
	0.1	**0.995**	**0.055**	1	0.106	1	0.171	1	0.106	1	0.096	1	0.117
	0.2	**0.996**	**0.053**	1	0.084	1	0.141	1	0.077	1	0.095	1	0.119
	0.3	**1**	**0.049**	1	0.064	1	0.115	1	0.047	1	0.088	1	0.117
	0.4	**1**	**0.048**	0.866	0.052	0.174	0.137	1	0.019	1	0.067	1	0.103
20	0.0	**-**	**0.119**	-	0.153	-	0.210	-	0.149	-	0.102	-	0.148
	0.1	**0.980**	**0.095**	1	0.132	1	0.178	1	0.122	1	0.105	1	0.156
	0.2	**0.996**	**0.096**	1	0.110	1	0.148	1	0.092	0.637	0.092	1	0.165
	0.3	**1**	**0.095**	0.353	0.114	1	0.119	1	0.061	0.944	0.098	0.947	0.138
	0.4	**1**	**0.083**	0.137	0.123	0.206	0.146	1	0.029	1	0.071	1	0.123
40	0.0	**-**	**0.104**	-	0.226	-	0.218	-	0.221	-	0.106	-	0.307
	0.1	**0.990**	**0.077**	1	0.204	1	0.183	1	0.193	0.517	0.093	1	0.273
	0.2	**0.994**	**0.062**	0.312	0.192	1	0.148	1	0.156	0.112	0.085	0.862	0.248
	0.3	**1**	**0.045**	0.228	0.198	1	0.113	1	0.113	0.101	0.085	0.314	0.267
	0.4	**1**	**0.031**	0.213	0.200	0.278	0.134	1	0.059	0.640	0.076	0.614	0.190
100	0.0	**-**	**0.022**	-	0.285	-	0.220	-	0.400	-	0.071	-	0.398
	0.1	**0.986**	**0**	0.416	0.272	1	0.180	1	0.333	0.088	0.063	0.980	0.335
	0.2	**0.996**	**0**	0.316	0.276	1	0.139	1	0.250	0.068	0.061	0.444	0.387
	0.3	**1**	**0**	0.300	0.278	1	0.095	1	0.143	0.065	0.061	0.418	0.390
	0.4	**1**	**0**	0.289	0.280	0.549	0.094	1	0.003	0.064	0.063	0.404	0.392
200	0.0	**-**	**0.017**	-	0.271	-	0.227	-	0.300	-	0.016	-	0.200
	0.1	**0.998**	**0**	0.299	0.268	1	0.184	1	0.222	0.017	0.014	0.245	0.195
	0.2	**0.990**	**0**	0.283	0.268	1	0.136	1	0.125	0.015	0.014	0.212	0.197
	0.3	**1**	**0**	0.279	0.268	1	0.088	1	0	0.013	0.014	0.207	0.197
	0.4	**1**	**0**	0.274	0.270	0.795	0.047	0.750	0	0.014	0.014	0.202	0.199

Table 25: μ and Σ error limits estimation given n and p for multivariate $N(\mathbf{0}, I)$ distribution.

		$\widehat{EL}_{\mu,boot}$					$\widehat{EL}_{\Sigma,boot}$				
	p	**10**	**20**	**40**	**100**	**200**	**10**	**20**	**40**	**100**	**200**
n	**50**	0,758	0,953	1,211	1,742	2,314	2.135	3,607	6,555	15,399	29,970
	100	0,552	0,669	0,861	1,230	1,638	1.431	2,421	4,508	10,682	20,890
	200	0,389	0,482	0,602	0,861	1,155	1.001	1,710	3,147	7,463	14,605
	300	0,309	0,390	0,498	0,709	0,934	0.810	1,381	2,554	6,049	11,856
	350	0,287	0,356	0,462	0,656	0,875	0.737	1,287	2,362	5,587	10,980
	400	0,265	0,338	0,444	0,612	0,826	0.697	1,193	2,206	5,225	10,263
	450	0,262	0,315	0,400	0,583	0,773	0.657	1,126	2,072	4,918	9,661
	500	0,237	0,298	0,383	0,549	0,736	0.622	1,078	1,972	4,663	9,157
	600	0,2178	0,276	0,356	0,500	0,669	0.559	0,973	1,791	4,254	8,359
	700	0,210	0,253	0,329	0,461	0,620	0.524	0,901	1,659	3,932	7,732
	800	0,195	0,243	0,302	0,431	0,577	0.486	0,837	1,555	3,682	7,230
	900	0,181	0,231	0,285	0,406	0,546	0.463	0,797	1,457	3,475	6,803
	1000	0,169	0,215	0,274	0,388	0,520	0.437	0,746	1,381	3,287	6,461
	2000	0,125	0,151	0,191	0,275	0,366	0.306	0,531	0,976	2,323	4,567
	5000	0,078	0,094	0,121	0,173	0,231	0.192	0,333	0,616	1,468	2,882