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Could the Colombian economy grow faster? How it would be possible?

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Abstract

This paper presents an economic growth model based on the positive externalities generated by the accumulations of physical and human capital. Such externalities imply, at the macroeconomic level, increasing returns to scale. The model helps to better understand the Colombian economic growth process from 2005-2019, and make conditional forecasts. One of the big obstacles in Colombia to have higher growth rates of the per capita product in the long term is everything that is slowing down a higher human capital growth rate and a greater creation of externalities derived from human capital, that is, everything that is hindering improvements in coverage and quality of the educational process.

***Key Words**: Economic growth, externalities, increasing returns to scale, human capital, production function, growth accounting.*

***JEL Codes**: O11, O33, O41, O47, O54.*

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I. Introduction

Is economic growth spontaneous in a decentralized society, i.e., a society with a market economy and autonomous agents whose decisions are motivated by their own interests? At present there is no consensus answer among economists to such a question, nor has there been such a consensus in the history of economic thought over the last 150 years.

To give two important examples of a negative answer we can mention the two main economic growth models of traditional neoclassical theory: Cass-Koopmans-Ramsey and Solow-Swan (Wickens, 2011, chaps. 2 and 3). Both models predict the long-run tendency for per worker (and per capita) output to stagnate, even in market economies, in the absence of persistent increases in the scale factor (or technical factor or "total factor productivity") of the aggregate production function. And, following these approaches, if such increases were to occur, it would be due to something exogenous, i.e., unexplained.

On the other hand, Schumpeter's disciples (and, more generally, those of the Austrian school) continued to believe, as Schumpeter himself believed, that long-run economic growth is inherent to the functioning of capitalism (Aghion and Howitt, 1992, and 2009 [Ch. 4]).

The revival of economic growth theory, in the mid-1980s of the last century (with Arrow [1962], Nelson and Phelps [1966], and Uzawa [1965] as forerunners) materialized in the designs and applications of models in which the increase in the technical factor of the aggregate production function ceases to be exogenous: it becomes endogenous based on some hypothesis about its change over time. These are the so-called endogenous growth models (Jones, 2002) or "semi-endogenous", as Jones (2021) prefers to call them. However, some of these models are not, by their nature or structure, specific to the case of the decentralized society. Two of the most famous ones are Lucas' (1988: human capital accumulation as an engine of growth) and Romer's (1990: R&D as another engine: Jones [2002, Ch. 5]). Such models are relatively easy to adapt to the case of a centralized economy (without markets or autonomous agents) in which the only agent making the decisions is the mythical social planner: he would decide the socially optimal fraction of the labor force to be taken out of production for teaching-learning or to specialize in R&D; and what would follow thereafter is long-run economic growth.

But there is a model of endogenous growth that, besides being simple, has an essential characteristic: it cannot be described or solved using the metaphor of the centralized society commanded by the social planner. This is the model based on positive externalities originating in individual decisions. In this paper we present an algebraic and numerical version of such a model and contrast its predictions with some features of the Colombian economy for the period 2005-2019 through exercises of comparative statics and the so-called "growth accounting" (one of these with the author's assumptions, and the other with those of the Colombian National Bureau of Statistics, *DANE*).

Having said the above, it should be clarified that in two of the following sections it is assumed that the degree of resources misallocation between sectors of the economy and between firms in each sector is constant. This topic is extremely important to understand why the gap between the Colombia´s per capita GDP or other Latin American countries and that of the United States is greater than would be expected when comparing the differences between the average per worker capital of any Latin American country and the same variable from the United States (Restuccia, 2013). And it would be mandatory to include levels and changes in the degree of resource misallocation in a full analysis of Colombian economic growth over a much longer period, say, 30 or more years. But it seems to us that a change in such a degree that might have occurred between 2005 and 2019 is not of great importance in explaining the average annual rate of GDP growth in this period[2]. For this reason, and also to avoid excessive complications in the model to be used, this degree will be considered constant and its effects (we suppose so) would be implicit in the magnitude of the elasticities of GDP with respect to each production factor.

Four sections follow: sections II and III present, one, the model, and, another, the main statistical figures of the Colombian case 2005-2019 used to calibrate it, and the numerical results of two comparative statics exercises. Section IV contains two growth accounting exercises to discuss DANE's estimates of the total factor productivity (TFP) growth rate and present other estimates compatible with what is stated in section III. Section V discusses the issue of the human capital evolution in Colombia between 2005 and 2019, and section VI consists of a summary, some final comments and the conclusions (including an answer to the questions in the title). The Annex seeks to clarify matters relating to human capital remuneration, its optimal level, and the savings supporting its increase.

II. The Model

The model presented below can be considered as a member of the first generation of endogenous growth models. The model is about a closed economy and generates a real total ouput growth rate that is higher than the population growth rate in the long run without assuming an exogenous increase in TFP. The reason for that is the existence of positive externalities of physical and human capital. In this sense, this model follows by the path started by Romer (1986) and developed by Stiglitz (1989) and Klenow and Rodríguez-Clare (2005). Bosi *et al*. (2021) present a broad version of the model and a summary of the main works that have contributed to its development. Posada (1993), González *et al*. (1999) and Gaviria (2007) made applications of this model to the Colombian case, and Arteaga (2011) applies it through econometric estimations for a group of 60 economies.

[2] According to Arellano *et al*., 2021, in Colombia, between 2009 and 2015, there were moments or periods of (likely) drops in aggregate productivity associated with greater distortions generating misallocation of resources, and other moments or periods in which there were generated the opposite effects.

Before presenting the model, it should be clarified that what is understood by a steady state path is that path of endogenous variables in which the following conditions are met: agents optimize; their forecasts are fulfilled; markets are in equilibrium, and both aggregate output and the levels of the variables that are macroeconomic values associated with it grow at a single rate, the same for all these variables (the case of a balanced growth). This means that the steady state path has an attraction power (because on other paths someone doesn't optimize) so we call it the long run.

Structural Form

The structural form of the model is the following set of equations:

$$(1)\;\; Y_t = A_t K_t^{\alpha} H_t^{\beta} L_t^{1-(\alpha+\beta)};\;\; 0 < \alpha, \beta < 1$$

where Y, A, K, H, L stand for the total output, the technology factor (A = TFP), the "physical" capital (tangible and intangible), a human capital index[3] and the total labor force. Equation 1 is a clone (with aggregate variables) of what would be the production function of a representative firm characterized, among other things, by constant returns to scale (the sum of its exponents is equal to 1).

$$(2)\;\; A_t = aK_t^{\gamma} H_t^{\eta};\;\; 0 < \gamma, \eta < 1$$

We assume that equation 2 holds only for the case of the aggregate production function, not for the firm´s function. This equation expresses the specific feature of this model: the hypothesis of dependence of A on physical and human capital. That is, it serves to express in a simple way various observations and conjectures about the generation of positive externalities for society thanks to the accumulation processes of both capitals by various means (widely specified in the aforementioned literature) such as their influences and leverage (some philanthropic, others involuntary) to the quality of education, to technological advance and to improvements of the physical infrastructure and the legal system, etc.

Replacing equation 2 in equation 1 results in:

$$(1.a)\; Y_t = aK_t^{\alpha+\gamma} H_t^{\beta+\eta} L_t^{1-(\alpha+\beta)}$$

In all that follows a is taken to be an exogenous factor and, until otherwise clarified, constant.

The remaining equations of the structural form are the accumulation laws and the assumptions about the origin of investments in physical and human capital:

$$(3)\; K_{t+1} = I_t + (1-\delta_K)K_t;\;\; 0 < \delta_K < 1$$

[3] We can consider human capital as usual: a stock associated with the number of years of education of the workforce and the economic returns to their education.

$$(4)\ H_{t+1} = J_t + (1 - \delta_H)H_t;\ \ 0 < \delta_K, \delta_H < 1$$

$$(5)\ \ I_t = S_t\ ;$$

$$(6)\ S_t = S(r, \dots);\ \ \frac{\partial S}{\partial r} > 0;$$

$$(7)\ \ J_t = \rho e^{\mu(1+lnK_t)};\ \rho, \mu > 0$$

Where $I, \delta_K, J, \delta_H, S, r$ stand for investment in physical capital, physical capital depreciation rate, investment in human capital, human capital depreciation rate, savings and (real) interest rate.

Equation 6 expresses the positive dependence of savings on the interest rate, clarifying that other unspecified factors also affect savings; this function is compatible with what is discussed below with respect to consumption. The interest rate is flexible and thus plays an important role in the equilibrium between savings and investment, but its steady state level is unique and will be determined later. Equation 7 is a hypothesis of a positive association between society's investment in human capital and the increase in physical capital. The specific form of function 7 has the advantage that numerical simulation of the steady state does not require making guesses about the parameter ρ´s magnitudes. And it has another advantage: it makes unnecessary to propose an assumption about savings to finance human capital accumulation.

This last issue should be clarified: this model (or, in precise terms, the version of this model presented in the central body of this paper) does not make explicit the existence of savings to finance human capital accumulation, nor does it present an analysis of the optimal level of human capital or, therefore, of the savings compatible with such an optimum. The main reason for this omission is that the Colombian National Accounts includes spending on education as part of household and government consumption, so an application of the model in line with this data would be hampered by consider the financing of investment in human capital as a component of aggregate savings. In this model, then, education expenditures are part of household consumption and, moreover, as will be seen below, are implicitly considered to be associated with total consumption of the representative family by means of a fixed proportion. The Annex presents the alternative case, namely the existence of savings to finance human capital accumulation and the optimal human capital. It follows from what is presented in the Annex that the model can be extended with this issue but that would not change the other equations or the main results of the model.

<u>The Steady State: Distribution and Output Growth, Interest and Savings Rates</u>

From equations 1 and 1.a it follows that:

$$\frac{\partial Y}{\partial L} = [1-(\alpha+\beta)]\frac{Y}{L}$$

Assuming that the marginal productivity of labor is equal to *w*, which is the average real wage, it follows that the income (or output or GDP) share of labor (before including the labor income associated with the holding of human capital) is:

$$(8)\ \ w\frac{L}{Y} = 1-(\alpha+\beta)$$

In a steady state with persistent increases in *A* the real wage also increases. The determination of its growth rate in the steady state path is explained later. From equation 1 it follows that:

$$(9)\ \left(\frac{\partial Y}{\partial K}\right)_{(1)} = \alpha\frac{Y}{K} \Rightarrow \left(\frac{\partial Y}{\partial K}\right)_{(1)}\frac{K}{Y} = \ \alpha;$$

$$(10)\ \left(\frac{\partial Y}{\partial H}\right)_{(1)} = \beta\frac{Y}{H} \Rightarrow \left(\frac{\partial Y}{\partial H}\right)_{(1)}\frac{H}{Y} = \ \beta;$$

Therefore:

$$\alpha+\beta+[1-(\alpha+\beta)] = 1$$

It will become clear later that conditions 9 and 10 are the relevant ones for individual decisions.

But an implication of equation 1.a is as follows:

$$(11)\ \left(\frac{\partial Y}{\partial K}\right)_{(1.a)} \cdot\ \frac{K}{Y} = \alpha+\gamma;\ \ (12)\left(\frac{\partial Y}{\partial H}\right)_{(1.a)} \cdot\ \frac{H}{Y} = \beta+\eta;\ \ \frac{\partial Y}{\partial L}\frac{L}{Y} = 1-(\alpha+\beta)$$

Therefore:

$$(13)\ \ \left(\frac{\partial Y}{\partial K}\right)_{(1.a)} \cdot\ \frac{K}{Y} + \left(\frac{\partial Y}{\partial H}\right)_{(1.a)} \cdot\ \frac{H}{Y} + \frac{\partial Y}{\partial L}\frac{L}{Y} + \ \frac{\partial Y}{\partial L}\frac{L}{Y} > 1,$$

For the aggregate analysis (and with simplified national accounts) we consider the following:

$$(14)\quad \frac{Physical\ capital\ gross\ income + labor\ income}{Y} =$$

$$\frac{(r+\delta_K)K}{Y} + \frac{\Pi H}{Y} + \frac{wL}{Y} = 1$$

Where Π is the human capital gross return rate (or the "premium" per unit of human capital received, on average, by workers with human capital, in addition to the average real wage). It follows from the comparison between conditions 13 and 14 that:

$$(15)\quad \left(\frac{\partial Y}{\partial K}\right)_{(1.a)} \neq (r + \delta_K);\ \left(\frac{\partial Y}{\partial H}\right)_{(1.a)} \neq \Pi$$

The recognition of inequalities, according to condition 15, is the way of expressing that in the decentralized economy characterized by the externalities considered in this model, the effective levels of physical and human capital could be suboptimal (in the absence of government intervention or private philanthropy) since individual decisions on capital accumulation are based on conditions 9 and 10.

Since the (physical) capital return rate is constant in the steady state path (provided that the real wage growth rate is equal to the *A* growth rate), it can be deduced that the *K/Y* ratio is also constant, which implies that the capital growth rate is equal to that of output (a rate that will be denoted g):

$$\frac{K_{t+1} - K_t}{K_t} = \frac{Y_{t+1} - Y_t}{Y_t} \equiv g$$

On the other hand, equation 7 implies the following:

$$(7.a)\ \frac{J_{t+1} - J_t}{J_t} = \mu \frac{K_{t+1} - K_t}{K_t}$$

But, according to 4:

$$\frac{H_{t+1} - H_t}{H_t} = \frac{J_t}{H_t} - \delta_H$$

In what follows it is assumed that, in the steady state, the human capital growth rate is constant and equal to that of physical capital, so $\mu = 1$.

Therefore:

$$(16)\quad \frac{J_{t+1} - J_t}{J_t} = \frac{K_{t+1} - K_t}{K_t} = \frac{H_{t+1} - H_t}{H_t} = g$$

And from equations 16 and 1.a it follows that in the steady state:

$$g = (\alpha + \gamma)g + (\beta + \eta)g + [1 - (\alpha + \beta)]\frac{L_{t+1} - L_t}{L_t}$$

$$\Rightarrow\ (17)\ g = \left(\frac{1 - \alpha - \beta}{1 - \alpha - \gamma - \beta - \eta}\right) g_L;\ where{:}\ g_L \equiv \frac{L_{t+1} - L_t}{L_t}$$

Whereas the TFP growth rate, according to equation 2, is:

$$(18)\ g_A \equiv \frac{A_{t+1} - A_t}{A_t} = (\gamma + \eta)\ g = (\gamma + \eta)\left(\frac{1 - \alpha - \beta}{1 - \alpha - \gamma - \beta - \eta}\right) g_L$$

On the other hand, since the steady state capital/output ratio is constant, we can denote it as follows:

$$\frac{K_{ss}}{Y_{ss}} = \chi$$

Furthermore:

$$I_t = S_t, but: I_t = K_{t+1} - K_t + \delta_K K_t$$

$$\Rightarrow \frac{K_{t+1} - K_t}{K_t} + \delta_K = \frac{S_t}{Y_t}\frac{Y_{ss}}{K_{ss}} = \frac{s}{\chi}, where\ s \equiv \frac{S_t}{Y_t}$$

$$Therefore:$$

$$(19)\ \ s = (g + \delta_K)\chi$$

The interpretation of equation 19 is as follows: for each output growth rate, and given the capital depreciation rate and the other parameters (to be made explicit later), there is one and only one savings rate, s, compatible with the output growth rate along the steady state path.

The Determination of Interest Rate and Capital/Output Ratio.

It follows from the above that the aggregate household consumption rate is equal to that of output in the steady state path. We will assume, for simplicity, that this rate is equal to the sum of the growth rates of the number of households and of consumption per household, c, and that the average number of persons per household is constant, so that:

$$(20)\ \ \frac{c_{t+1} - c_t}{c_t} + g_p = \frac{C_{t+1} - C_t}{C_t} = g$$

With g_p being the number of families´s growth rate, and equal to that of the population.

In turn, the consumption growth rate is deduced from the optimization program described in the next paragraphs.

The representative family has one objective: to maximize the following welfare function:

$$\Omega = \mathrm{E}\sum_{t=0}^{T}\left(\frac{1}{1+\theta}\right)^t u(c_t)\,;$$

$$\mathrm{E}: Mathematical\ Expectation\ Operator;\ \ \theta: subjective\ descount\ rate;\ \ 0 < \theta < 1$$

And a usual assumption is made: that the utility function is *CRRA* type:

$$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma};\ \ \sigma > 0$$

The Euler equation corresponding to the maximization of Ω (subject to the budget constraints of each period and the inter-temporal budget constraint), that is, the equation describing the optimal level of consumption in a period t and its steady state path, is (see, e.g., Wickens, 2011, Ch. 4):

$$(21)\quad \left(\frac{1}{1+\theta}\right)\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}(1+r_{t+1}) = 1$$

From 21 it follows that:

$$(21.a)\quad \frac{c_{t+1}-c_t}{c_t} \cong \frac{r-\theta}{\sigma}$$

Therefore:

$$\frac{c_{t+1}-c_t}{c_t} + g_p \cong \frac{r-\theta}{\sigma} + g_p$$

$$\Rightarrow g = \frac{r-\theta}{\sigma} + g_p$$

$$\Rightarrow$$

$$(22)\ \ r = \theta + \sigma\left(g - g_p\right)$$

Two important conclusions for the model are drawn from this exercise: a) consumption and hence savings, depend on the interest rate[4], and b) the interest rate is endogenous, and its steady state level is determined by equation 22.

On the other hand, suppose that there is a representative firm *i* whose product is:

$$y_i = a_i k_i^{\alpha} h_i^{\beta} l_i^{1-(\alpha+\beta)} \ \Rightarrow \ \frac{\partial y_i}{\partial k_i} = \alpha \frac{y_i}{k_i}$$

Its optimal capital is deduced from the following condition:

[4] This is clear with equation 21.a: according to this, any increase in the gap between the interest rate and the discount rate (θ) induces a fall in present consumption and then (after reviewing the future consumption plan), increases its growth rate (to put it graphically: the intercept falls and the slope increases). Therefore, present savings would rise if the interest rate increases, the rest remaining constant, as expressed in equation 6.

$$\frac{\partial y_i}{\partial k_i} - \delta_K = r \Rightarrow \alpha \frac{y_i}{k_i} = r + \delta_K \Rightarrow \frac{k_i}{y_i} = \frac{\alpha}{r + \delta_K}$$

Therefore, the macroeconomic result is:

$$(23) \quad \frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n} y_i} = \frac{K}{Y} = \frac{\alpha}{r + \delta_K}$$

A Final Form of the Model

An "(almost) reduced form" of the model for output growth in the steady state path is a set of five equations already justified:

$$(I)\ g = \left(\frac{1-\alpha-\beta}{1-\alpha-\gamma-\beta-\eta}\right) g_L; \ \ \frac{1-\alpha-\beta}{1-\alpha-\gamma-\beta-\eta} > 1 \Rightarrow g > g_L$$

$$(II) \quad g_A = (\gamma+\eta)\left(\frac{1-\alpha-\beta}{1-\alpha-\gamma-\beta-\eta}\right) g_L; \ \ 0 < \gamma+\eta < 1 \ \Rightarrow \ g_A < g$$

$$(III) \quad s = (g + \delta_K)\frac{K}{Y};$$

$$\text{(IV)} \quad \frac{K}{Y} = \frac{\alpha}{r+\delta_K};$$

$$(V) \quad r = \theta + \sigma(g - g_p)$$

One could further reduce the model, but the previous presentation is more intuitive and didactic. In any case, what is stated is the following for the steady state: 1) the output and A growth rates depend on the 4 parameters of the aggregate production function $(\alpha, \beta, \gamma, \eta)$ and on an exogenous variable: the labor force growth rate. The effects of increases in each of these 4 parameters on output growth are positive up to a certain point, but beyond that they become negative. Given such parameters, the relationships between: a) the GDP growth rate and that of the labor force, and b) the A growth rate and that of the labor force are positives and linear (as in the model with the growth engine associated with R&D: Jones 2021). 2) The savings rate, the capital/output ratio and the interest rate are endogenous, depending on parameters $(\alpha, \beta, \gamma, \eta, \delta_K, \theta, \sigma)$ and on the labor force and population growth rates. Likewise, the relationships between these 3 variables and the parameters are nonlinear.

In particular, replacing V and IV in III results in the savings rate as follows:

$$(VI)\ s = \frac{\alpha(g+\delta_K)}{\theta + \sigma(g - g_p) + \delta_K}; \ \ therefore\text{:}\ \frac{ds}{dg} \gtreqless 0$$

At this point it becomes clear that this model has several features in common with the Cass-Koopmans-Ramsey model. The three differences are the inclusion of human capital, the externalities and, by these, the absence of an exogenous technical change.

III. A Numerical Illustration of the Model and the Colombian Case (2005-2019)

The most recent version of the Colombian National Accounts System has quarterly frequency figures available since 2005. The data for the period 2005-2019 ("pre-pandemic") can be used for a numerical illustration of the model and, therefore, for some remarks on the determinants of the Colombian economic growth, and on the possibilities and conditions for higher future growth rates, and upon the savings rate and (physical) capital real return rate compatible with that.

Table 1 shows the main statistics associated with the Colombian economic growth (2005-2019) corresponding to the model´s variables.

Table 1. Colombia. Some Principal Variables Related to Economic Growth 2005-2019

Variable	Average annual growth rate	Average level
Population	0,0122	
EAP	0,0190	
GDP (constant prices)	0,0397	
GDP/Population	0,0269	
GDP//EAP	0,0200	
Gross Investmen/GDP (constant prices)		0,215

Source: a) GDP and Investment: DANE, National Accounts; b) EAP (Economically Active Population): Households´s Survey (GEIH): DANE; c) Population: Census and Projections: DANE, and author´s estimations.

The EAP (economically active population) variable is our approximation to the L variable. Table 2 presents the numerical values assigned to the model parameters in order to replicate the Table 1 numbers[5].

[5] Arteaga (2011) estimated with econometric methods the parameter corresponding to the externality of human capital (a parameter equivalent to η); her estimate is in the range (0.84; 1.31). Gaviria (2007) used a value of 0.27 for the equivalent parameter in the calibration of the basic scenario of his economic growth exercises. Why we assume such a low value (0.131: Table 2)? Because higher values were incompatible with the replication of the macroeconomic series (Table 3). This may indicate deficiencies in our model, but it can also be said that econometric exercises lacking all the restrictions imposed by theoretical macroeconomic models have the overestimation bias risk, as seems to be the case in the Arteaga´s estimation. On the other hand, Davies (2002)

Table 2. Baseline Scenery´s Parameters								
α	β	γ	δ_K $Annual\ rate$	η	σ	θ $Annual\ rate$	g_L $Annual\ rate$	g_p $Annual\ rate$
0,47	0,2	0,041	0,045	0,131	3	0,058	0,019	0,0122

Table 3 presents the main results of the model in the baseline scenario. These results correspond to the model´s steady state path.

Table 3. Baseline Scenery Main Results				
g	g_A	r	K/Y	s
0,0397	0,0068	0,1404	2,5344	0,2146

Looking at the results for the baseline scenario, is striking the apparently exaggerated magnitude of the capital real return rate (for that the magnitude of the discount rate, θ, is unusually high). But the following should be known: according to the model, the portion of GDP that is not remuneration to labor is remuneration to the owners of physical capital (and land). Now, according to the National Accounts at current prices for the years 2015-2017, remuneration for capital (and land) ownership, plus a fraction estimated by the author for capital income included in the so-called "mixed income" (from capital and labor), is 47%; the complement, 53%, being what would correspond to remunerations to skilled and unskilled labor, including in these the other part of the mixed income. Therefore, r, which in the model is "the interest rate", must now be interpreted as the implicit weighted average of multiple return rates on capital and land ownership (including rents and leases) in different activities and types of businesses, incorporating various risk premiums, and before deducting direct taxes[6].

How important was the "surplus" from externalities? A way to answer this question is the following: compare the difference between the GDP observed at the last quarter of 2019 with the GDP that would have been recorded at that time if the technology factor A had remained constant at its 2005 level (that is, if the elasticities γ, η were 0, and assuming the same evolutions of physical and human capital and EAP) for the cases with or without externalities. Table 4 presents an estimate of this surplus.

considered that human capital externalities in Canada would probably be in the range (0.06: 0.08) plus or minus a standard deviation which, according to him, would be considerable.

[6] This multiplicity of rates may be, to a large extent, a signal of a socially inappropriate resource allocation (misallocation), as stated by Banerjee and Duflo (2005),

Table 4. Externalities Surplus		
	2005-IV	2019-IV
A	5,288	5,288
K	1530559,116 (1)	2529754,780
H	2,272 (2)	2,599
EAP	19562,512 (3)	25165,200
Estimated GDP	13115,350 (4)	185351,910 Without externalities (5)
Observed GDP	13115,350	223430,810
Surplus = (Observed GDP2019-IV – Estimated GDP2019-IV without externalities)/ Observed GDP2019-IV = 17%		
(1) and (4): Billions (constant pesos; 2015 prices); see table 8 notes for capital stock calculations; (2) Index; see table 8; (3) thousands; see table 8 for additional explanations; (5) estimation based on equation 1 with the same magnitude for A than represented in this table. α, β: the same of table 2; γ, η = 0. Author´s estimations.		

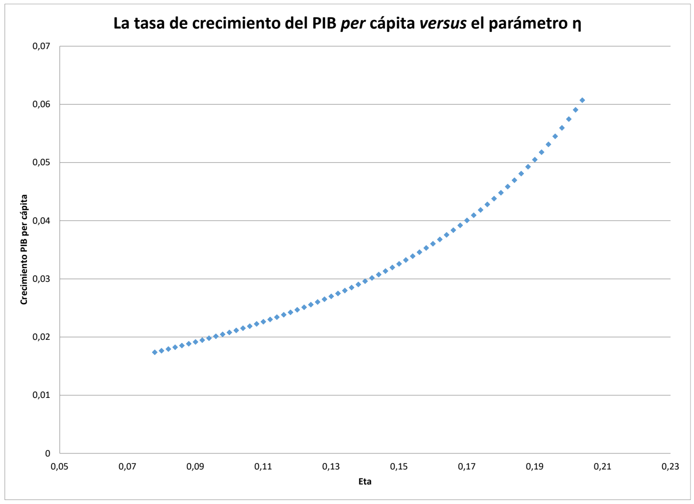
Table 5 reports a simulation results based on increasing the η parameter by 3 points (that is, assuming that its magnitude is 0.161, which means an increase of 23% with respect to its value in the baseline scenario), leaving other parameters unchanged.

Table 5. Alternative Scenery Main Results A Higher η Value: $\eta = 0,161$				
g	g_A	r	K/Y	s
0,0489	0,0098	0,168	2,203	0,207

The main results of such an increase are an uptick in the output growth rate (a raise of 23%), in g_A (which increases by 45%) and in the capital return rate (an uptick of 20%), while the capital/output ratio and the savings rate decrease.

Figure 1 shows the path of the per capita GDP growth rate in the face of successive increases in the parameter η in a certain range, with the other parameters remaining constant and equal to those in Table 2.

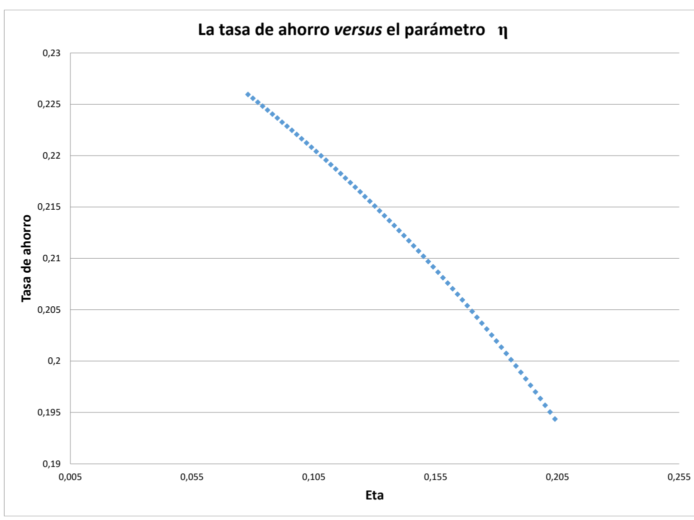
Figure 1. GDP Growth Rate (vertical axis) vs. η Parameter



Author's estimates

Figure 2 shows the savings rate fall in the face of increases in the parameter η remaining constant the other parameters (i.e., with their numbers as shown in Table 2). The reason for this inverse relationship is the magnitude for α parameter: lower than that of σ parameter (see equation VI and table 2).

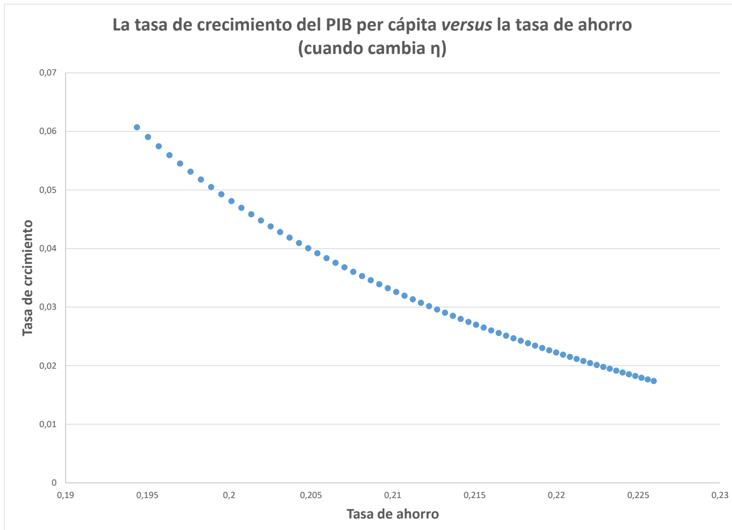
Figure 2. Savings Rate (vertical axis) vs. η Parameter



Author's estimates

So, in the face of variations in η parameter, a negative correlation would be observed between GDP growth rates (total, per worker and per capita) and the savings rates (Figure 3).

Figure 3. Per Capita GDP Growth Rates (vertical axis) vs. Savings Rates (In face of η Parameter Variations)

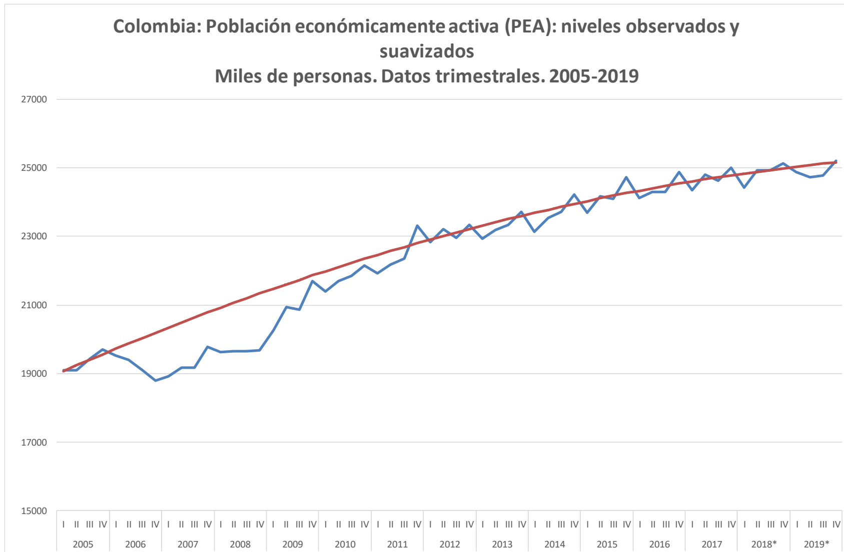


Author's estimates

What does it mean to raise the η parameter? This could mean improving the scope and quality of the human capital formation process so that, on average, an additional unit of human capital can generate a higher output for society (beyond of the increases for the incumbent firm and workers), and a lower savings rate. Sections V and VI will discuss this issue more extensively.

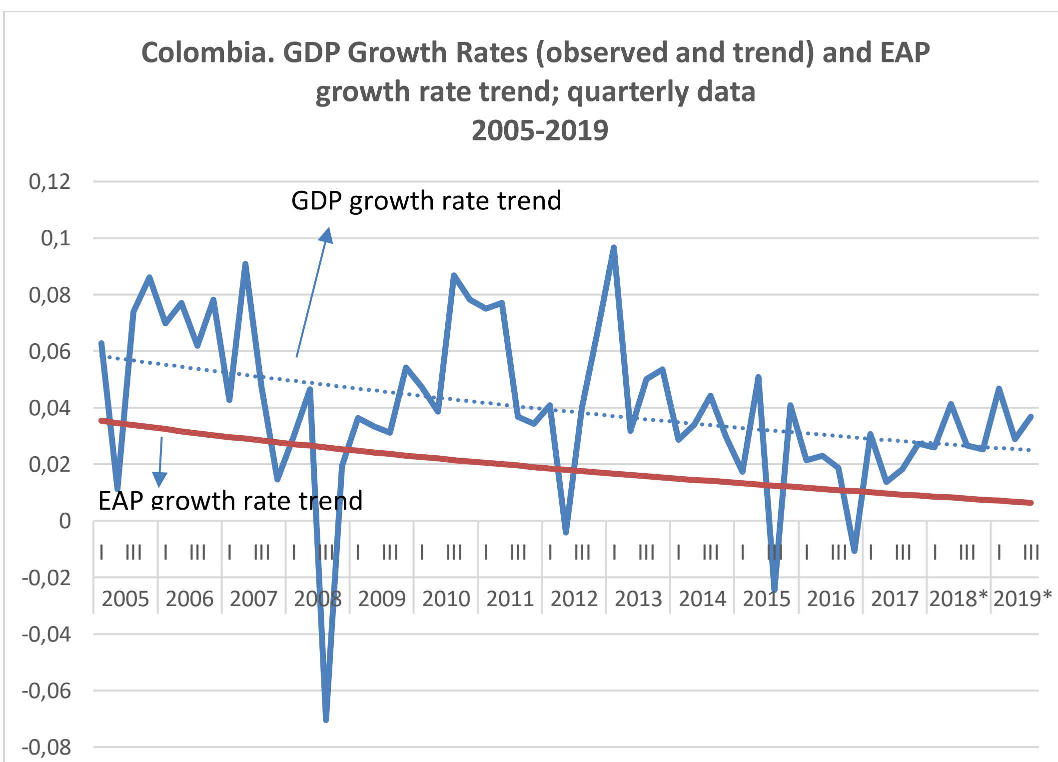
Additionally, the model predicts a positive linear relationship between the GDP and the labor force growth rates (see equation I). If we assume something that seems reasonable, namely, the EAP growth rate is the same as of the L factor, it turns out that such a prediction holds for the pre-pandemic Colombian case, comparing the trend of GDP growth rates with the trend of EAP growth rates (Figures 4 and 5).

Figure 4. Colombia. EAP (Economically Active Population): Observed and Trend Numbers. (Thousands. Quarterly Data). 2005 – 2019



Source: DANE: Labor Market Data (GEIH), and author´s estimation.

Figure 5



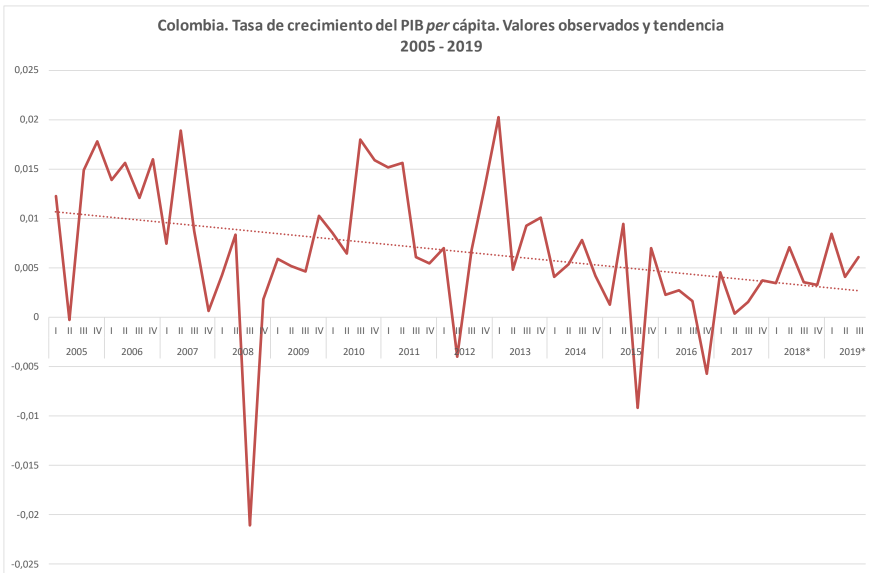
Source: DANE (National Accounts, Labor Market Data (GEIH), and author´s estimates.

In addition, the per capita GDP growth rate showed a decreasing trend (Figure 6). But the observed GDP growth rate per worker (that is, per EAP member) did not have a significant upward or downward trend between 2005 and 2019 (Figure 7): it oscillated (irregularly) along a line almost parallel to the time axis.

Figure 6. Colombia. Per Capita GDP Growth Rate. Observed Data and Trend

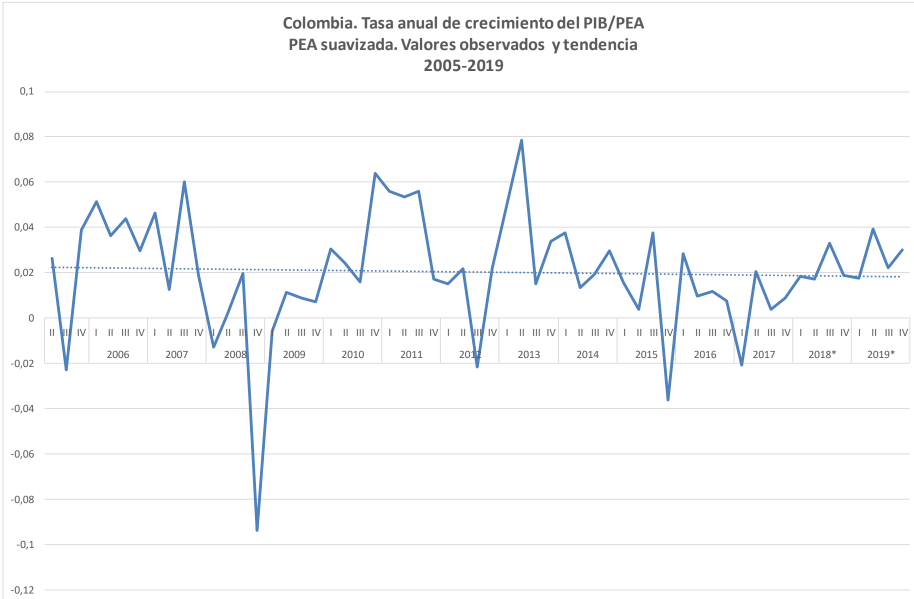
Quarterly Rates

2005-2019



Source: DANE (National Accounts and Demography); author´s estimates.

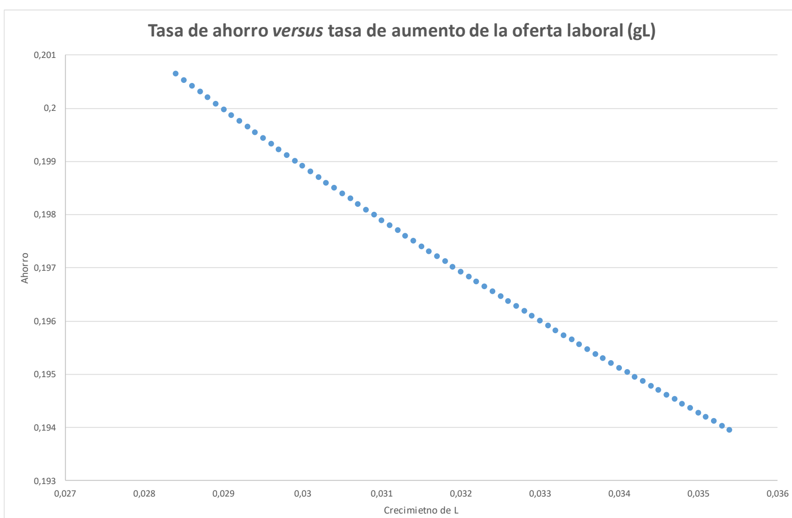
Figure 7. Colombia. Per worker (EAP member) GDP Growth Rate. Observed Data and Trend
Annual Rate
2005 – 2019



Source: DANE (National Accounts, and Labor Market Data).

On the other hand, according to the model, an increase in the labor force growth rate, with all other exogenous factors remaining constant, generates an uptick in the GDP growth rate, and this, in turn, leads to a fall in the savings rate. Figure 8 shows this result (with the values of the other parameters as shown in Table 2).

Figure 8. Savings Rate (vertical axis) vs. Labor Force Growth Rate

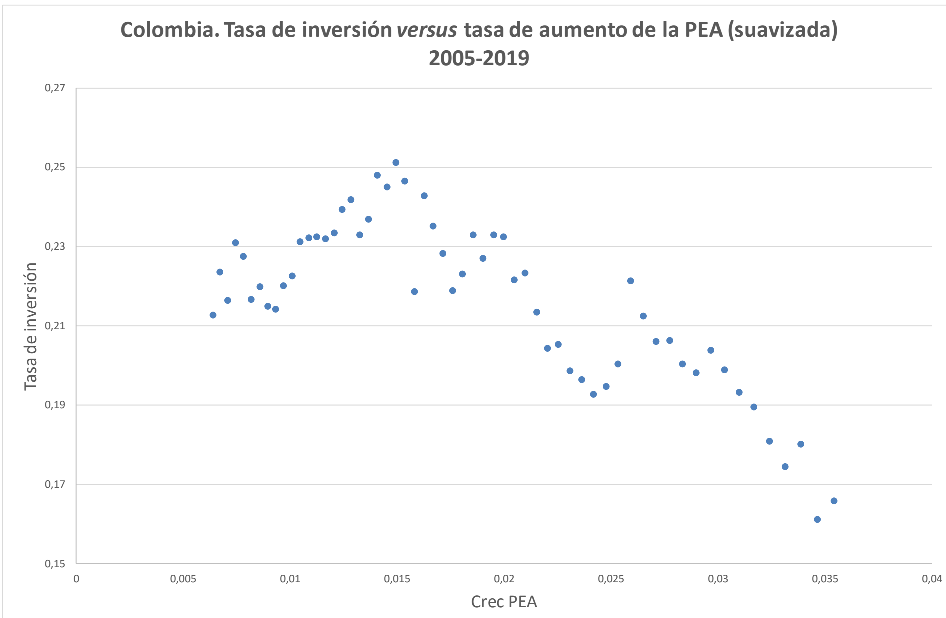


Author´s estimates

Figure 9 shows that in the Colombian case 2005-2019 the relationship between the EAP growth rate (which is our approximation to g_L) and the investment rate seems negative and non-linear.

Figure 9. Colombia. Investment Rate (I/Y) vs. EAP Growth Rate Trend

2005- 2019



Source: DANE (National Accounts and GEIH).

IV. Exercises in Colombian “Growth Accounting”

The previous section contrasted some model results for a steady state path with the Colombian statistical evidence. In the present section we abandon the scenarios in which variables are simulated along steady state paths; we do that in order to deduce certain implications of some facts of the 2005-19 period that do not correspond to what, according to the model, is a steady state.

The practical way to achieve the objective proposed in the previous paragraph is to start from the "growth accounting" exercise carried out by the DANE´s National Accounts Section. This was done for each year of the 2005-2020 period but we restrict it to 2005-2019 to exclude the first year of the pandemic.

Table 6 is almost a copy of the table presented by DANE. It is not an exact copy because it does not take into account the 2020 figures and because it presents the averages for 2005-2019 of the DANE exercise numbers.

Table 6. TFP Growth Rate according to DANE 2005-2019

	Gross Aggregate Value (GAV) Annual Growth Rate (1)	Labor Services Contribution to GAV Growth Rate (2)	Capital Services Contribution to GAV Growth Rate (3)	Factors´s Contribution (4) = (2) + (3)	TFP Annual Growth Rate (5) = (1) – (4)
Averages for 2005-19	0,0383	0,0162	0,0234	0,0396	-0,0013
Source: DANE; National Accounts.					

With the figures in Tables 1 and 2 and with our own estimates of: (a) the plausible ranges in which the growth rates of (physical) capital and total labor should have been located, and (b) the plausible ranges for the elasticities of gross value added (GVA) with respect to capital and labor, we can reconstruct the results of Table 6 and describe as simply as possible, with a Cobb-Douglas production function, the relationship between GVA and the factors of production coherent with these results. Table 7 presents this exercise.

Table 7. Exercises to make explicit an aggregate production function compatible with DANE´s estimates. 2005-2019

Annual Capital Growth Rate Feasible Range			Annual Labor (EAP) Growth Rate Feasible Range		
Maximum	Minimum	Average	Maximum	Minimum	Average
0,0365	0,0345	0,0355	0,023	0,0195	0,02125
Capital-Elasticity of GAV Feasible Range			Labor-Elasticity of GAV Feasible Range		
Maximum	Minimum	Average	Maximum	Minimum	Average
0,91	0,4	0,655	0,92	0,57	0,745
Capital Contribution			Labor Contribution		
Maximum	Minimum	Average	Maximum	Minimum	Average
0,0332	0,0138	0,0235	0,0212	0,0111	0,0161
Total Factors Contribution					
Maximum	Minimum	Average	TFP Growth Rate = GAV Growth Rate – Total Factors contribution		
0,0544	0,0249	0,0396	Maximum	Minimum	Average
Source: Author´s Estimations.			0,0134	-0, 0160	-0,0013

The results from those exercises can be synthesized in a simple way with the following Cobb-Douglas function:

$$VAB = TFP \cdot K^{0,655} \cdot EAP^{0,745}$$

The sum of the exponents is 1.4, so, according to DANE the aggregate (implied) production function has increasing returns to scale. Jones (2021) would say that, in this case, the measure of increasing returns to scale is 0.4, higher than he assumed in his growth accounting exercise for the U.S. case (1953-2007) which was 0.33, although he cited two papers with higher magnitudes for other cases (the U.S. between 1880 and 1920, and post-World War II Germany),

This finding does not depend on assuming that the production function is of the Cobb-Douglas type. With any other function we would also have found that the DANE estimates imply increasing (and large) returns to scale.

An exercise analogous to the above was done in several respects, but considering equation 1.a. The exercise is depicted after Table 8.

Table 8 presents the growth rates of physical capital, human capital and the variable with which we approximate the labor force: the EAP.

Table 8. Physical and Human Capital, and Labor. Annual Growth Rates (Averages) 2005-2019

Capital (K)	Human Capital (H)	Labor Force (L)
0,0356	0,0097	0,0190

Sources: K: Author´s estimates based on Gross Investment (National Accounts, DANE), supposing capital at the end of 2004 equal to 12 times the GDP of 2005 (first quarter), and a depreciation rate equal to 0.045 per year.
B) *H*: Original Series: Human Capital Index from *Penn World Table 10*, University of Groningen.
C) *L*: Original Series: EAP (Labor Market Data [GEIH], DANE). The author took the series' trend (see: Figure 4).

Table 9 records the calculation of TFP growth considering the Table 8 values. The basis for the calculations in Table 9 is equation 1.a. From this one we derive the following:

$$\frac{a_{t+1}-a_t}{a_t} = \frac{Y_{t+1}-Y_t}{Y_t} - \left\{(\alpha+\gamma)\left(\frac{K_{t+1}-K_t}{K_t}\right) + (\beta+\eta)\left(\frac{H_{t+1}-H_t}{H_t}\right) + [1-(\alpha+\beta)]\left(\frac{L_{t+1}-L_t}{L_t}\right)\right\}$$

Therefore:

$$(1.b)\ \ P_a \approx P_Y - [(\alpha+\gamma)P_K + (\beta+\eta)P_H + [1-(\alpha+\beta)]P_L]$$

In equation 1.b, the letter P indicates the average for 2005-2019 of the annual growth rates of the variable whose name is the subscript of P. Therefore, in this exercise we understand by TFP growth rate what in equation 1. b is P_a. Table 9 uses the values for the parameters $\alpha, \beta, \gamma, \eta$ presented in Table 2.

Table 9. TFP Annual Growth Rate (According to 1.b equation). 2005-2019				
P_Y (1)	$(\alpha+\gamma)P_K$	$(\beta+\eta)P_H$	$[1-(\alpha+\beta)]P_L$	P_a
0,0397 0,0383	0,0182	0,0032	0,0063	0,0120 0,0107
(1) P_Y has two alternative values: The highest is related to GDP (demand side; market prices); the lower is related to GAV (supply side; factors costs); so P_a has two alternative values. Source: see previous tables. Author´s estimates.				

According to the above, the aggregate production function (for GDP and for GVA) is:

$$Y = PTF_Y \cdot K^{0{,}511} \cdot H^{0{,}331} \cdot EAP^{0{,}33}$$

$$VAB = PTF_{VAB} \cdot K^{0{,}511} \cdot H^{0{,}331} \cdot EAP^{0{,}33}$$

The sum of the exponents is 1.172. The returns to scale are increasing but substantially lower than those of the function we derived from the DANE´s estimates. Thus, it is not surprising that our estimate of the annual growth rate for TFP associated with GVA, 1.07%, is higher than the one estimated by DANE (which, as we seen, was negative). Even so, its magnitude is in the range that we estimated when the exercise of reproducing the DANE figures was carried out (Table 7).

Moreover, the discrepancy between our estimate and that of DANE does not necessarily mean that the latter is wrong or that its error is large. Perhaps part of the problem is the use of a human capital index that underestimates its growth (according to Table 8, the average rate of increase of this index was only 1% per year between 2005 and 2019). If it had grown, on average, at the same rate as estimated for physical capital, 3.5% per year (as in a steady state path), the average annual rate of increase in TFP (considering GVA) would have been only 0.2%, a figure close to that of DANE; and DANE's calculation would be replicated if the true average annual increase in human capital had been 4.58%. However, as discussed in the following section, official statistics and economists' analysis of the Colombian case allow us to reject the hypothesis of human capital growing as fast as physical capital between 2005 and 2019. Probably its growth rate between 2005 and 2019 was in the range 1% - 2.5%. With the average of this range, 1.8%, the average TFP increase rate (similarly to how it was calculated and reported in Table 9) for the case of GVA would have been 0.79% per year.

V. The Case of Colombian Human Capital

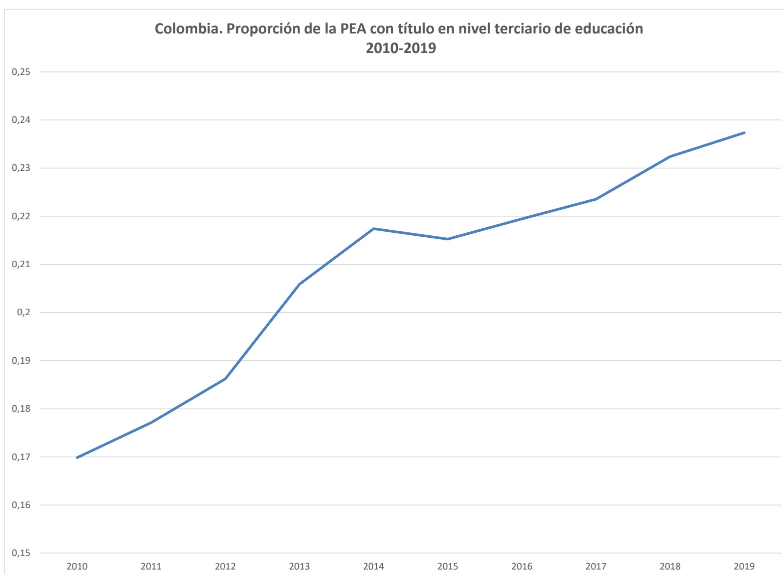
In the previous section we used a measure of human capital for Colombia: the index of such variable included in the Penn World Table database (version 10; source cited in Table 8). This index was constructed considering both the years of schooling of the labor force and the labor income associated with the level of schooling. And as observed, the average per year increase rate of this index during the period 2005-2019 was much lower than those of the EAP, physical capital and total GDP.

This would indicate that the progress in the coverage of the public and private educational apparatus since the end of the 20th century does not seem to have been sufficiently rapid, after having been so in previous decades, as Ramírez and Téllez, 2006, and Aponte, 2015, have argued. Based on recent official data, this assessment can be considered correct for our case: that of 2005 - 2019. Figures 10 and 11 were constructed with this data and allow us to affirm that the coverage of tertiary education (technical and technological, and undergraduate and graduate university levels) is still low and its progress is not made at a steady pace; on the contrary, it has declined.

Figure 10

Colombia. EAP´s Share with Tertiary Education Level

2005-2019



Source: labor market (GEIH), DANE. Author's calculations.

Figure 11

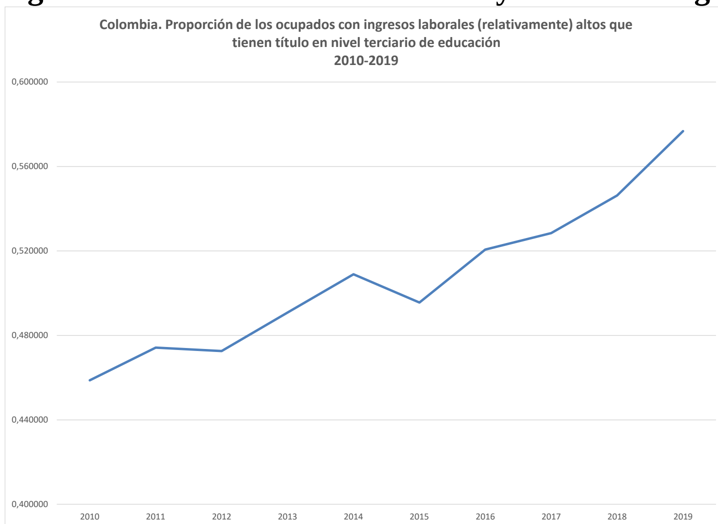
Colombia. Annual Growth Rate of EAP Members with

Tertiary Education Degrees. 2005-2019

Source: labor market (GEIH), DANE. Author's calculations.

And, furthermore, the quality of education is not high, and it has been even worse in the cases of primary and secondary education offered by the public system (see: Barrera and Gaviria, 2003, Barrera *et al*., 2012, Ayala, 2015, Galvis, 2015, Melo *et al*., 2016, and Barrera *et al*. 2020). But, according to Figure 13, it is not evident that tertiary education is of a much higher quality than primary and secondary education, if this could be judged by the differences in labor income, which have not been large between 2010 and 2019, as shown in Figure 13.

Figure 12
Colombia. Proportion of employed people with (relatively) high labor income and with a tertiary education degree-2019



Note: (relatively) high income means (using DANE's criteria) this: more than 50% of the minimum monthly wage in force in the corresponding year. Source: labor market (GEIH), DANE. Author's calculations.

What Figure 12 shows is not necessarily an indication of poor quality of tertiary education. But if we give credit to the studies finding low quality of education at the primary and secondary levels in Colombia, and bearing in mind the greater inequalities in labor income for Colombian households as a whole compared to the United States and Western Europe, we can assume that the quality of tertiary level education is not high in Colombia and for that, therefore, the firms in the labor market are not willing to grant a very high premium to those who accredit a degree at that level (and only for the simple fact of accrediting it).

Taking into account what has been discussed in this section, it does not seem feasible to consider that the human capital index used in the previous section implies an exaggerated underestimation of what has happened in Colombia in this regard.

VI. Summary, final comments and conclusions

In this paper we present a model of long-term economic growth derived from the positive externalities generated by physical and human capital accumulations. Such externalities imply, at the macroeconomic level (but not at the level of individual firms), increasing returns to scale. The model has the following advantages: a) it makes easy to establish an explicit relationship between the general source of economic growth and the basic nature or characteristics of a decentralized economy (markets and autonomous agents); b) it provides a plausible numerical illustration of the hypothetical externalities of human and physical capitals; and c) it would also make it easy to establish the relationship between the general source of economic growth and specific sources such as education, international trade, etc.

And, in particular, it is a model that allows us to ignore a specific growth engine that does not operate or is not important in emerging economies: that of innovation motivated by the search for patent protection and carried out by those endowed with very high and very specific human capital: those who work in laboratories and research and development centers seeking (and succeeding) in expanding the scientific-technological frontier. In other words, the externalities model makes the analysis of emerging economies more relevant and easier. Finally, this model makes it possible to describe in a simple way some (but few) channels that relate economic growth to the factorial distribution of income.

The model helps a better interpretation of the Colombian economic growth process from 2005-2019 in several ways, namely: (a) per capita GDP tends to grow spontaneously over long periods provided that no huge obstacles or disincentives are erected to the accumulation of physical and human capital; (b) the observed falls in the economically active population (EAP) growth rate generate, in turn, falls in the total GDP and GDP per capita growth rates (Table 10 illustrates the latter)[7] ; 3) what is probably one of the main obstacles in Colombia to have higher per capita GDP growth rates of, say, 3.2% per year or more in the long term (its average was 2.7% per year between 2005 and 2019), is everything that is slowing down a higher human capital growth rate and a greater creation of externalities derived from human capital, that is, in more concrete terms, everything that is hindering improvements in coverage and quality of the educational process; 4) Colombian savings and investment rates are not high but, according to the model, their current levels would not be a bottleneck for a highest economic growth if it were achieved through higher human capital externalities.

Table 10 is included to clarify the previous paragraph. This table presents the total GDP and per capita GDP growth rates in the long run, if the elasticity of GDP to human capital, $\beta+\eta$, were not the one estimated for 2005-2019, 0.331, but a higher one (in the range 0.332 - 0.38), assuming the same for everything else contemplated in Table 2 but under two demographic sceneries, one of them similar to that of the last 15 years and another, apparently much more likely, with a lower population (and, therefore, EAP) growth rate.

[7] But the GDP/EAP growth rate showed no fall in the Colombian case 2005-2019, contradicting the model's prediction. On the other hand, the population growth rate has been falling almost continuously since 1951 (but, for statistical reasons, the implicit rates in the annual population series show an increase from 2015 to 2019, no doubt because of the need to match population estimates previous years with census figures). In any case, the fall in the rate of increase of the EAP is explained by that of the population.

Table 10. Future Economic Growth Rates (Long Term). Conditional Forecasts (Annual Rates)				
$\beta + \eta$	g $g_p = 0,0122$	$g - g_p$	g $g_p = 0,0085$	$g - g_p$
0,332	0,0399	0,0277	0,028	0,0194
0,34	0,0421	0,0299	0,0295	0,0209
0,348	0,0445	0,0323	0,0311	0,0226
0,356	0,0471	0,0349	0,033	0,0245
0,364	0,0502	0,0380	0,0351	0,0266
0,372	0,0536	0,0414	0,0375	0,029
0,38	0,0575	0,0453	0,0403	0,0317

Notes: The first column refers to the elasticity of GDP to human capital.
g, g_p are the GDP and Population per year growth rates.
In the second scenery, the growth rates of the labor force and of the population are 70% of those of the first scenery. Author´s estimations.

The foregoing assumes that the accumulation of physical capital will have sufficient incentives to maintain the speed that was observed, on average, between 2005 and 2019. Otherwise, an excess supply of skilled labor would be created, and so that would end up frustrating the process of economic growth (for the Colombian case, see: Uribe 1993).

Finally, it seems possible to state that exercises to estimate parameters of the aggregate production function with a partial equilibrium or single equation approach (i.e. without subjecting such estimates to consistency checking with the magnitudes of the capital/output ratio, investment rate, rate of return on capital and factor distribution of income) run the risk of producing results with substantial distortions. This risk is minimized by giving importance to the set of basic macroeconomic relationships. In our case, the growth accounting exercise was subjected to such discipline, i.e., it was done with a production function whose parameters had magnitudes that were previously calculated: in the framework of a macroeconomic model with a hypothetical steady state path.

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Annex. Savings to finance human capital, human capital remuneration, and its optimal level.

Savings financing human capital accumulation is equal to: $H_{t+1} - H_t + \delta_H H_t$. Therefore, the savings rate with this destination, which we will call s_H, is:

$$(A.1)\quad s_{H,t} = \frac{H_{t+1} - H_t}{Y_t} + \frac{\delta_H H_t}{Y_t} = \left(\frac{H_{t+1} - H_t}{H_t} + \delta_H\right)\frac{H_t}{Y_t}$$

Where δ_H is the human capital depreciation rate.

On the other hand, equation 1 (from the main text) implies that:

$$(A.2)\ \frac{\partial Y}{\partial H} = \beta \frac{Y}{H} \Rightarrow \frac{H}{Y} = \frac{\beta}{\partial Y/\partial H}$$

Let us express the optimality condition thus:

$$\frac{\partial Y}{\partial H} = \pi + \delta_H = \Pi$$

In such a case, the optimal level of human capital per unit of output is:

$$(A.3)\ \frac{H}{Y} = \frac{\beta}{\pi + \delta_H}$$

But what is π? In the present context π cannot be understood as the payment per unit of labor time of a skilled worker. It can only be understood as an additional payment or premium (net of the human capital depreciation rate): a payment per unit of labor time and per unit of human capital. This means that the (real) income per period of a skilled worker (i.e., of higher than average qualification than that of the EAP) is equal to the average wage plus πH/N, where N is the number of such workers. The average real income of a skilled worker in the steady state, according to this model, increases by increases in the average real wage and by increases in human capital per skilled worker (H/N), with π remaining constant.

Since β and δ_H, are constant, and assuming, based on the previous paraFigure, that π is constant, then the optimal level of $\frac{H}{Y}$ is constant, so in the steady state trajectory will have that:

$$\frac{H_{t+1} - H_t}{H_t} = \frac{Y_{t+1} - Y_t}{Y_t} = g; \;\; s_H = (g + \delta_H) \; \frac{\beta}{\pi + \delta_H}$$

In the decentralized economy a characteristic of the steady state path (with perfect information and foresight, mobile resources, and flexible price markets) is the following equilibrium:

$$(A.4) \quad \frac{\partial Y}{\partial K} - \delta_K = \frac{\partial Y}{\partial H} - \delta_H$$

And from equation 1 it follows that:

$$(A.5) \quad \frac{\partial Y}{\partial K} = \frac{\alpha Y}{K}; \frac{\partial Y}{\partial H} = \frac{\beta Y}{H}$$

Therefore, A.5 in A.4 implies that:

$$\frac{\alpha Y}{K} - \frac{\beta Y}{H} = \delta_K - \delta_H \;\; \Rightarrow \; \frac{\alpha}{K} - \frac{\beta}{H} = \frac{\delta_K - \delta_H}{Y}$$

Since: $\frac{\delta_K - \delta_H}{Y} \approx 0$, then the optimal human capital/physical capital ratio without considering externalities (which would correspond to the decentralized equilibrium without government intervention or private philanthropy) is:

$$(A.6) \quad \frac{H}{K} = \frac{\beta}{\alpha}$$

Repeating the previous procedure, but taking into account the externalities (that is, considering equation 1.a of the main text), it turns out that the social optimum (Pareto's) is:

$$(A.7) \quad \frac{H}{K} = \frac{\beta + \eta}{\alpha + \gamma}$$

With the values in table 2 it can be deduced that the optimal level of human capital per unit of physical capital is 0.425 in a decentralized equilibrium and the optimal social level is 0.601, which means that the socially optimal human capital per unit of physical capital would be 41.4% higher than that which would result from the decentralized equilibrium (without state intervention or philanthropy).