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A note on allocation policy in two-parallel–series and two-series–parallel systems with respect to likelihood ratio order

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ABSTRACT

In a recent article of Laniado and Lillo (2014), the authors obtained some comparison results on component allocation policies in two-parallel–series and two-series–parallel systems with two types of components. They proposed a conjecture that those results can be extended to likelihood ratio order. In this note, we provide an affirmative answer to their conjecture.

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1. Introduction

How to allocate components to make a system work efficiently is an interesting problem in engineering and system safety. Such a problem was initially studied by Boland et al. (1988, 1992a), thenceforth many researchers have worked in this field and a lot of interesting results have been established. For instance, Boland et al. (1992b) and Shaked and Shanthikumar (1992) studied the allocation problem for series and parallel systems in terms of the usual stochastic order, whereas Singh and Misra (1994), Romera et al. (2004), Valdés and Zequeira (2006), and Li et al. (2011) used other stochastic orders. El-Neweihi et al. (1986) considered the allocation problem in parallel–series and series–parallel systems. Those systems can be regarded as generalization of the usual parallel systems and series systems. Recently, Laniado and Lillo (2014) studied the problem of component allocation in two-parallel–series system (Fig. 1) and two-series–parallel system (Fig. 2), (from now on (2PSS) and (2SPS), respectively). They obtained some stochastic comparison results for these systems in terms of hazard rate order and reversed hazard rate order. They conjectured that the results for 2PSS in terms of hazard rate order and reversed hazard rate order remain true in terms of likelihood ratio order.

In this note, we provide an affirmative answer to their conjecture. Moreover, we enhance their comparison result about 2SPS to likelihood ratio order.

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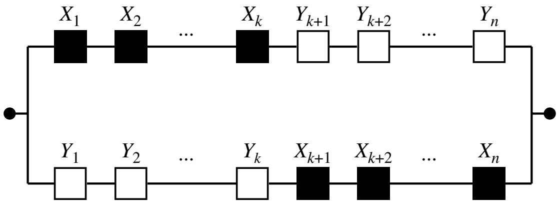

Fig. 1. Two-parallel–series system $\mathbb{S}_k$.

2. Main results

In order to state the main results, we need the following definitions of some stochastic orders.

Let X and Y be two positive absolutely continuous random variables with distribution functions F and G, and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Let f and g be their density functions, $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$ be their hazard rate functions, and $\tilde{r}_X = f/F$ and $\tilde{r}_Y = g/G$ be their reversed hazard rate functions, respectively.

We say X is smaller than Y

- in *usual stochastic order* (denoted as $X \le_{st} Y$) if, $\bar{F}(t) \le \bar{G}(t)$ for all $t \in [0, \infty)$,
- in *hazard rate order* (denoted as $X \le_{hr} Y$) if, $r_X(t) \ge r_Y(t)$ for all $t \in [0, \infty)$,
- in *reversed hazard rate order* (denoted as $X \le_{rhr} Y$) if, $\tilde{r}_X(t) \le \tilde{r}_Y(t)$ for all $t \in (0, \infty)$, and
- in *likelihood ratio order* (denoted as $X \le_{lr} Y$) if, the ratio $g(t)/f(t)$ is non-decreasing for $t \in (0, \infty)$.

It is well known that likelihood ratio order implies hazard rate order and reversed hazard rate order, while both of these orders imply usual stochastic order. Extensive theoretical treatments and variety of applications of these orders can be found in Shaked and Shanthikumar (2007).

Two positive absolutely continuous random variables X and Y are said to follow a *proportional hazard rate model* (PHR) if, there exists some $\alpha > 0$, such that, $\bar{G}(t) = \bar{F}^{\alpha}(t)$ for all $t > 0$. The two variables X and Y are said to follow a *proportional reversed hazard rate model* (PRHR) if, there exists some $\alpha > 0$, such that, $G(t) = F^{\alpha}(t)$ for all $t > 0$. Valdés and Zequeira (2003), Kochar and Xu (2007a,b) and Finkelstein (2008) studied PHR in system reliability analysis. For examples and applications of PRHR models in the study of reliability of systems, see Gupta et al. (1998), Di Crescenzo (2000) and Gupta and Gupta (2007).

Given two vectors $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ and let $a_{[1]} \ge a_{[2]} \ge \cdots \ge a_{[n]}$ and $b_{[1]} \ge b_{[2]} \ge \cdots \ge b_{[n]}$ be the decreasing arrangements of the components of the two vectors, then the vector $\boldsymbol{a}$ is said to be *majorized* by the vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \prec \boldsymbol{b}$) if and only if, $\sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}$ and $\sum_{i=1}^{k} a_{[i]} \le \sum_{i=1}^{k} b_{[i]}$ for all $k = 1, 2 \dots, n-1$. For extensive and comprehensive details on the majorization order, we refer the reader to the book of Marshall and Olkin (2011).

Consider a (2PSS) with two different types of components. The system consists of $2n$ components with n components of type I and other n components of type II. Denote X_i, $i = 1, \dots, n$, as the lifetimes of the n components of type I, and Y_i, $i = 1, \dots, n$, as the lifetimes of the other n components of type II. Clearly, X_i, $i = 1, \dots, n$, are i.i.d. and so do Y_i, $i = 1, \dots, n$. One practical situation that can be modeled by such a system is, for instance, components of type I are purchased from provider A and components of type II are purchased from provider B. Denote by $\mathbb{S}_k$ the system which has k components of type I and $(n-k)$ components of type II in the up-series. Fig. 1 is a diagram of this system.

Suppose X_i, $i = 1, \dots, n$, has the same distribution as a random variable X, and Y_i, $i = 1, \dots, n$, has the same distribution as a random variable Y. Denote by $S_k(X, Y)$ the lifetime of the system $\mathbb{S}_k$. Sometimes, we denote $S_k(X, Y)$ by S_k for convenience.

Clearly, $S_k \underset{\text{st}}{=} S_{n-k}$, and

$$S_k = \left(\bigwedge_{i=1}^{k} X_i \wedge \bigwedge_{i=k+1}^{n} Y_i\right) \bigvee \left(\bigwedge_{i=1}^{k} Y_i \wedge \bigwedge_{i=k+1}^{n} X_i\right), \quad \text{for } 1 \le k \le n-1,$$

$$S_0 \underset{\text{st}}{=} S_n = \left(\bigwedge_{i=1}^{n} Y_i\right) \bigvee \left(\bigwedge_{i=1}^{n} X_i\right)$$

where $\bigwedge$, $\bigvee$ are the minimum and maximum operators, respectively.

The number of possible different allocations of the $2n$ components in the system is $(n+1)/2$ if n is odd, or $(n+2)/2$ if n is even. A natural question is: what is the optimal configuration of these components in order to improve the reliability of the system?

In a recent paper of Laniado and Lillo (2014), the authors compared S_k in terms of hazard rate and reversed hazard rate. They showed that, under PHR model, when $0 \le k_1 \le k_2 \le n/2$, $S_{k_1} \ge_{hr} S_{k_2}$ and $S_{k_1} \ge_{rhr} S_{k_2}$. Based on their computer simulations, they conjectured that, under the same condition, $S_{k_1} \ge_{lr} S_{k_2}$ would also hold.

In this note, we provide an affirmative answer to their conjecture. To prove the conjecture, we need the following result in Dykstra et al. (1997). We state it here as a lemma.

Lemma 2.1 (*Theorem 3.1, Dykstra et al., 1997*). *Let X_1 and X_2 be two independent exponential random variables with hazard rates λ_1 and λ_2, respectively. Let Y_1 and Y_2 be another set of independent exponential random variables with respective hazard rates γ_1 and γ_2. Let $X_{(2)} = \max\{X_1, X_2\}$ and $Y_{(2)} = \max\{Y_1, Y_2\}$. Then $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$ implies $X_{(2)} \geq_{lr} Y_{(2)}$.*

The following theorem is an affirmative answer to the conjecture proposed by Laniado and Lillo (2014).

Theorem 2.2. *Suppose X and Y follow PHR model, then for every $0 \leq k_1 \leq k_2 \leq n/2$,*

$$S_{k_1}(X, Y) \geq_{lr} S_{k_2}(X, Y).$$

Proof. At first, we assume X follows an exponential distribution with hazard rate λ and Y follows an exponential distribution with hazard rate γ. By symmetry, we can assume $\lambda \geq \gamma$, since otherwise, we can relabel X and Y. Denote U_1, U_2 respectively as the lifetimes of the up-series and the lower-series in the system $\mathbb{S}_{k_1}$. Denote V_1, V_2 respectively as the lifetimes of the up-series and the lower-series in the system $\mathbb{S}_{k_2}$. We have, $S_{k_1} = \max\{U_1, U_2\}$, and $S_{k_2} = \max\{V_1, V_2\}$. As we can easily see, U_1 follows an exponential distribution with hazard rate $\lambda k_1 + \gamma(n - k_1)$, and U_2 follows an exponential distribution with hazard rate $\lambda(n - k_1) + \gamma k_1$. Similarly, V_1 follows an exponential distribution with hazard rate $\lambda k_2 + \gamma(n - k_2)$, and V_2 follows an exponential distribution with hazard rate $\lambda(n - k_2) + \gamma k_2$. The condition $0 \leq k_1 \leq k_2 \leq n/2$ implies

$$(\lambda k_1 + \gamma(n - k_1),\ \lambda(n - k_1) + \gamma k_1) \succ (\lambda k_2 + \gamma(n - k_2),\ \lambda(n - k_2) + \gamma k_2).$$

By Lemma 2.1, $S_{k_1} \geq_{lr} S_{k_2}$.

Now consider the general situation in which X and Y satisfy a PHR model, i.e., $\bar{G}(t) = \bar{F}^{\alpha}(t)$ for some positive α, here $\bar{F}(t)$ and $\bar{G}(t)$ are the survival functions of X and Y, respectively. If $\alpha > 1$, then $\bar{F}(t) = \bar{G}^{1/\alpha}(t)$. Thus, by relabeling X and Y, we can assume $\alpha \leq 1$. Again, denote U_1, U_2 respectively as the lifetimes of the up-series and the lower-series of the system $\mathbb{S}_{k_1}$, and similar to V_1, V_2.

Let H^{-1} be the right inverse of $H(t) = -\log \bar{F}(t)$. As in the proof of Theorem 2.6 in Da et al. (2010), the variable $X' = H(X)$ is exponential with hazard rate 1, and $Y' = H(Y)$ is exponential with hazard rate α. Then, $U_1' = H(U_1)$ follows an exponential distribution with hazard rate $k_1 + \alpha(n - k_1)$, and $U_2' = H(U_2)$ follows an exponential distribution with hazard rate $(n - k_1) + \alpha k_1$. Similarly, V_1' and V_2' are exponentially distributed with hazard rates $k_2 + \alpha(n - k_2)$ and $(n - k_2) + \alpha k_2$, respectively. $0 \leq k_1 \leq k_2 \leq n/2$ and $\alpha \leq 1$ imply

$$(k_1 + \alpha(n - k_1),\ (n - k_1) + \alpha k_1) \succ (k_2 + \alpha(n - k_2),\ (n - k_2) + \alpha k_2).$$

Thus, by Lemma 2.1, $\max\{U_1', U_2'\} \geq_{lr} \max\{V_1', V_2'\}$.

By Theorem 1. C. 8. in Shaked and Shanthikumar (2007), the likelihood ratio order is closed under the increasing transformation. Also, it is self-evident that an increasing transformation keeps the $\bigwedge, \bigvee$ orders. Note the function H^{-1} is increasing, thus we have,

$$S_{k_1} = \max\{U_1, U_2\} = H^{-1}(\max\{U_1', U_2'\}) \geq_{lr} H^{-1}(\max\{V_1', V_2'\}) = \max\{V_1, V_2\} = S_{k_2}. \quad \square$$

As pointed out by Shaked in Shaked (2013), 'stochastic comparison of orders statistics from a collection of independent heterogeneous exponential random variables, and a stochastic comparison of orders statistics from a collection of independent heterogeneous random variables with proportional hazard rates, are, in some respects, essentially the same comparison'. In fact, with the same idea in the proof of the above theorem, we can extend Lemma 2.1 to PHR model.

Lemma 2.3. *Let X_1, X_2 be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, 2$, and Y_1, Y_2 be independent random variables with Y_i having survival function $\bar{F}^{\gamma_i}$, $i = 1, 2$. Let $X_{(2)} = \max\{X_1, X_2\}$ and $Y_{(2)} = \max\{Y_1, Y_2\}$. Then, $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$ implies $X_{(2)} \geq_{lr} Y_{(2)}$.*

The following lemma will be used later on.

Lemma 2.4. *Let*

$$R(u) = \frac{\lambda_1 u^{\lambda_1} + \lambda_2 u^{\lambda_2} - (\lambda_1 + \lambda_2) u^{\lambda_1 + \lambda_2}}{\gamma_1 u^{\gamma_1} + \gamma_2 u^{\gamma_2} - (\gamma_1 + \gamma_2) u^{\gamma_1 + \gamma_2}}, \quad 0 < u \leq 1.$$

Then, when $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$, the function $R(u)$ is decreasing.

Proof. Let $\bar{F}(t)$ be the survival function of a positive random variable. Consider the function

$$R(t) = \frac{\lambda_1 \bar{F}^{\lambda_1} + \lambda_2 \bar{F}^{\lambda_2} - (\lambda_1 + \lambda_2) \bar{F}^{\lambda_1 + \lambda_2}}{\gamma_1 \bar{F}^{\gamma_1} + \gamma_2 \bar{F}^{\gamma_2} - (\gamma_1 + \gamma_2) \bar{F}^{\gamma_1 + \gamma_2}}.$$

From Lemma 2.3, we know when $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$, the function $R(t)$ is increasing with respect to t. Denote $u(t) = \bar{F}(t)$. From the arbitrary of $\bar{F}(t)$, we can say, for any decreasing function $u(t)$ with $0 \leq u(t) \leq 1$,

$$R(u(t)) = \frac{\lambda_1 u^{\lambda_1} + \lambda_2 u^{\lambda_2} - (\lambda_1 + \lambda_2) u^{\lambda_1 + \lambda_2}}{\gamma_1 u^{\gamma_1} + \gamma_2 u^{\gamma_2} - (\gamma_1 + \gamma_2) u^{\gamma_1 + \gamma_2}},$$

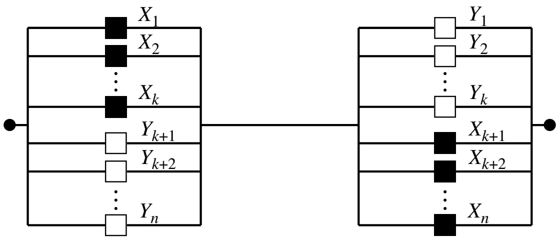

Fig. 2. Two-series–parallel system $\mathbb{T}_k$.

is an increasing function with respect to t. So, $dR(u(t))/dt \geq 0$. Since

$$\frac{dR(u(t))}{dt} = \frac{dR}{du}\frac{du}{dt},$$

therefore, $dR(u(t))/dt \geq 0$ and $du/dt \leq 0$ imply $dR/du \leq 0$. That is, the function $R(u)$ is decreasing with respect to u. □

Now, we consider stochastic comparison of component allocation policies in a 2SPS. A 2SPS is formed by two subsystems connected in series, where each subsystem is formed by n components in parallel. Again, we assume there are two kinds of components, each has n components, and components of the same type are identical. Suppose a component of type I has lifetime as a random variable X, and a component of type II has lifetime as a random variable Y. Denote $\mathbb{T}_k$ as the system when the components are allocated in such a way that one of the parallel subsystems has k components of type I and $n-k$ of type II. The system is represented in Fig. 2.

Denote $T_k(X, Y)$ as the lifetime of $\mathbb{T}_k$. Clearly, we have,

$$T_k(X,Y) = \left(\bigvee_{i=1}^{k} X_i \vee \bigvee_{i=k+1}^{n} Y_i\right) \wedge \left(\bigvee_{i=1}^{k} Y_i \vee \bigvee_{i=k+1}^{n} X_i\right), \quad \text{for } 1 \leq k \leq n-1,$$

$$T_0(X,Y) \underset{\text{st}}{=} T_n(X,Y) = \left(\bigvee_{i=1}^{n} Y_i\right) \bigwedge \left(\bigvee_{i=1}^{n} X_i\right).$$

In Laniado and Lillo (2014), it is demonstrated that when X, Y follow a RPHR model; i.e., $G(t) = F^{\alpha}(t)$, for some positive α and for all $t > 0$, then, $T_{k_1}(X, Y) \leq_{hr} T_{k_2}(X, Y)$, for $0 \leq k_1 \leq k_2 \leq n/2$. Our next theorem enhances this result to likelihood ratio order.

Theorem 2.5. *Suppose positive continuous random variable X and Y follow RPHR model, then, for every* $0 \leq k_1 \leq k_2 \leq n/2$,

$$T_{k_1}(X,Y) \leq_{lr} T_{k_2}(X,Y).$$

Proof. Let the distribution function of X be $F(t)$ and that of Y be $G(t)$, respectively. X and Y follow RPHR model which indicates there exists some positive α, such that $G(t) = F^{\alpha}(t)$. Assume $0 < \alpha \leq 1$. If $\alpha > 1$, then by changing the label of X and Y, the resulting α will be less than 1. Denote $T_k(X, Y)$ as T_k for simplicity. As we can see, the survival function of T_k is

$$\begin{aligned}\bar{F}_{T_k}(t) &= \{1 - F^k(t)G^{(n-k)}(t)\}\{1 - F^{n-k}(t)G^k(t)\} \\ &= 1 - F^k(t)G^{(n-k)}(t) - F^{n-k}(t)G^k(t) + F^n(t)G^n(t) \\ &= 1 - F^{k+\alpha(n-k)}(t) - F^{(n-k)+\alpha k}(t) + F^{(1+\alpha)n}(t).\end{aligned}$$

Thus, the density function of T_k is

$$f_k(t) = \frac{f(t)}{F(t)}\left[\mu_1 F^{\mu_1}(t) + \mu_2 F^{\mu_2}(t) - (\mu_1+\mu_2)F^{\mu_1+\mu_2}(t)\right],$$

where $\mu_1 = k + \alpha(n-k)$, and $\mu_2 = (n-k) + \alpha k$. And so, the ratio of the density functions of T_{k_1} over T_{k_2} is given by

$$R(t) = \frac{f_{k_1}(t)}{f_{k_2}(t)} = \frac{\lambda_1 F^{\lambda_1}(t) + \lambda_2 F^{\lambda_2}(t) - (\lambda_1+\lambda_2)F^{\lambda_1+\lambda_2}(t)}{\gamma_1 F^{\gamma_1}(t) + \gamma_2 F^{\gamma_2}(t) - (\gamma_1+\gamma_2)F^{\gamma_1+\gamma_2}(t)}$$

where $\lambda_1 = k_1 + \alpha(n-k_1)$, $\lambda_2 = (n-k_1) + \alpha k_1$, $\gamma_1 = k_2 + \alpha(n-k_2)$, and $\gamma_2 = (n-k_2) + \alpha k_2$. $0 \leq k_1 \leq k_2 \leq n/2$ implies $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$. By Lemma 2.4, $dR/dF \leq 0$. Since $dF/dt \geq 0$, we obtain $dR(t)/dt \leq 0$, which means $T_{k_1}(X,Y) \leq_{lr} T_{k_2}(X,Y)$. □

It is worth pointing out that Theorem 2.5 improves the (if part) of Proposition 2 in Brito et al. (2011) by considering the likelihood rate order instead of the reversed hazard rate order. In addition, Theorem 2.5 considers a more general situation for $n \geq 2$.

3. Conclusions

This note considers stochastic comparisons for the allocation of components in two-parallel–series systems (2PSS) and in two-series–parallel systems (2SPS). The results from a recent paper of Laniado and Lillo (2014) indicate that, for a two-parallel–series system, the reliability of the system improves, in terms of hazard rate order and of reversed rate order, if the components are allocated by unbalancing as much as possible the two-series subsystems. Hence, the optimal situation would be to allocate the components of the same type in the same line. While for a two-series–parallel system, the reliability of the system will be improved, in terms of hazard rate, when both subsystems are arranged to be more similar. This paper enhances their conclusions in terms of likelihood ratio order. We believe that the results presented in this note can be extended to some more general situations, for instance, the situations when there are more than two types of components, and the situations when the lifetimes of those components follow in general different distributions. Stochastic comparison of more general two-parallel–series systems and more general two-series–parallel systems merits further investigation.

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